Section 5.6, #20 $\quad x^3y'' + \alpha x y' + \beta y = 0$ ①

$$y'' + \frac{\alpha}{x^2}y' + \frac{\beta}{x^3}y = 0.$$

$xp(x) = \alpha/x$, and $x^2q(x) = \beta/x$. Since both $xp(x)$ and $x^2q(x)$ are not analytic at $x=0$, $x=0$ is irregular.

We will attempt to find a solution of the form

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$; so $y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$

Thus,

$$L[y] = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r+1} + \alpha\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \beta\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=1}^{\infty} a_{n-1}(n-1+r)(n-2+r)x^{n+r}$$

$$+ \alpha\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \beta\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= [\alpha a_0 r + \beta a_0]x^r$$

$$+ \sum_{n=1}^{\infty}\left([a_{n-1}(n-1+r)(n-2+r) + a_n[(n+r)\alpha + \beta]\right)x^{n+r} = 0$$

$\therefore\ r = -\beta/\alpha$. For $n \geq 1$

$$a_n[(n+r)\alpha + \beta] = -a_{n-1}(n-1+r)(n-2+r)$$

or $\quad a_n = \dfrac{-a_{n-1}(n-1+r)(n-2+r)}{(n+r)\alpha + \beta}, \quad n \geq 1$

Let $r = -\beta/\alpha$. Then,

$$a_n = -a_{n-1}\frac{(n-1-\beta/\alpha)(n-2-\beta/\alpha)}{(n-\beta/\alpha)\alpha + \beta}$$

$$= -a_{n-1}\frac{(n-1-\beta/\alpha)(n-2-\beta/\alpha)}{n\alpha} \qquad (*)$$

Let $a_0 = 1$.

Let $\beta/\alpha = -1$, then

$$a_n = -a_{n-1}\frac{(n-1+1)(n-2+1)}{n\alpha} \qquad (*)(*)$$

$$a_1 = -\frac{1\cdot 0}{1\cdot\alpha} = 0.$$

From (*) $a_2 = a_3 = \cdots = 0$. Thus,

$$y_1(x) = 1 \cdot x^{-\beta/\alpha} = x$$

Let $\alpha/\beta = 0$. Then,

$$a_n = -a_{n-1}\frac{(n-1)(n-2)}{n\alpha}$$

$$a_1 = -\frac{a_0 \cdot 0 \cdot (-1)}{1 \cdot \alpha} = 0, \quad a_2 = a_3 = \cdots = 0$$

Then

$$y_1(x) = 1 \cdot x^{-\beta/\alpha} = 1$$

Let $\alpha/\beta = 1$, then

$$a_n = -\frac{a_{n-1}(n-2)(n-3)}{n\alpha}$$

$$a_1 = -\frac{a_0(-1)(-2)}{1 \cdot \alpha} = -\frac{2}{\alpha}$$

$$a_2 = -\frac{a_1 \cdot 0}{2 \cdot \alpha} = 0$$

Thus, $a_3 = a_4 = \cdots = 0$. Then

$$y_1(x) = x^{-1}\left(1 - \frac{2x}{\alpha}\right)$$

Because of the factor $n-1-\beta/\alpha$ in (*), we see that if α/β is any positive integer, then when $n = 1+\beta/\alpha$, $a_n = 0$. From (*) $a_{n+1} = a_{n+2} = \cdots = 0$, and we get a ~~polynomial~~ solution of the form $x^{-\beta/\alpha} P_{n-1}(x)$,

where P_{n-1} is a polynomial of degree $n-1$.

We apply the ratio test to determine convergence if $n \neq -1, 0, 1, 2, \ldots$ We want to calculate, by (*),

$$\lim_{n\to\infty} \frac{a_n x^{n+r}}{a_{n-1}x^{n+r-1}} = \lim_{n\to\infty} -\frac{x(n-1-\beta/\alpha)(n-2-\beta/\alpha)}{n\alpha} = \infty.$$

Thus, the series converges for no value of x (except possibly for $x=0$).

Section 5.7 #6 $\quad x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0.$

Let $y = x^{-1/2}V(x)$. By the product rule,

$$y' = -\frac{1}{2}x^{-3/2}V(x) + x^{-1/2}V'(x),$$

$$y'' = +\frac{3}{4}x^{-5/2}V(x) - \frac{1}{2}x^{-3/2}V'(x) - \frac{1}{2}x^{-3/2}V'(x) + x^{-1/2}V''(x)$$

$$= \frac{3}{4}x^{-5/2}V(x) - x^{-3/2}V'(x) + x^{-1/2}V''(x).$$

Thus,

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = x^2\left[\frac{3}{4}x^{-5/2}\cancel{V(x)} - \cancel{x^{-3/2}V'(x)} + x^{-1/2}V''(x)\right]$$

$$+ x\left[-\frac{1}{2}x^{-3/2}\cancel{V(x)} + \cancel{x^{-1/2}V'(x)}\right] + (x^2 - \cancel{1/4})x^{-1/2}V(x)$$

$$= x^{3/2}V''(x) + x^{3/2}V(x) = 0$$

$$\therefore \quad V''(x) + V(x) = 0 \quad (\text{if } x \neq 0)$$

Constant coefficients $r^2 + 1 = 0$ is characteristic eq. $\pm i$ are the characteristic roots. Thus,

$$V(x) = c_1 \cos x + c_2 \sin x$$

$$\therefore \quad y(x) = x^{-1/2}(c_1 \cos x + c_2 \sin x)$$

is the general solution of Bessel's d.e. of order 1/2.

Section 5.7, #11 Bessel's d.e. of order 1 is

$$x^2y''+xy'+(x^2-1)y=0$$

The indicial roots are ± 1. We first write our potential 2nd solution as $y=\sum_{n=0}^{\infty} a_n(r)x^{n+r}$ and then specialize by setting $r=-1$. For Bessel's d.e. of order p, we showed in class that

$$L[y]= a_0(r)(r^2-p^2)x^r + a_1(r)(r+1+p)(r+1-p)x^{r+1}$$
$$+\sum_{n=2}^{\infty}\left[(n+r+p)(n+r-p)a_n(r)+a_{n-2}(r)\right]x^{n+r}. \qquad (1)$$

If $r=\pm p$, then $a_0(r)$ is arbitrary when we begin to equate coefficients to 0. In particular if $p=1$ and $r=-1$, $a_0(-1)$ is arbitrary. For simplicity, we set $a_0(-1)=1$. Second, we see that

$$a_1(r)(r+1+p)(r+1-p)=0.$$

For the values of p and r that we chose, $a_1(r)\equiv 0$. As we showed in class, for $n\ge 2$, $p=1$, we have the recurrence relation below. We calculate a few values of $a_n(r)$, $n\ge 2$, in order to determine a general formula for $a_n(r)$.

$$a_n(r)=\frac{-a_{n-2}(r)}{(n+r-1)(n+r-2)} \qquad (2)$$

$$a_2(r)=\frac{-a_0}{(r+1)(r+3)},\quad a_4(r)=\frac{-a_2(r)}{(r+3)(r+5)}=\frac{a_0}{(r+1)(r+3)(r+3)(r+5)}$$

$$a_6(r)=\frac{-a_4(r)}{(r+5)(r+7)}=-\frac{a_0}{(r+1)(r+3)(r+5)(r+3)(r+5)(r+7)}$$

In general,

$$a_{2n}(r)=\frac{(-1)^n a_0}{(r+1)(r+3)\cdots(r+2n-1)(r+3)(r+5)\cdots(r+2n+1)}$$

$$=\frac{(-1)^n}{(r+1)(r+3)^2\cdots(r+2n-1)^2(r+2n+1)},\ \text{as } a_0=1 \quad (3)$$

For odd n, since $a_1(r)=0$, we see from (2) that $a_{2n+1}(r)=0$ for $n\ge 0$.

We now follow the authors of our textbook on pages 292-294 to use the above recurrence relation (2) and formula (3) to determine the coefficients $c_n(r)$. From (19),

$$c_n(r_2) = \frac{d}{dr}\left[(r-r_2)a_n(r)\right]_{r=r_2}, \quad n \geq 1.$$

Since $a_{2n+1}(r) = 0$, it follows from above that $c_{2n+1}(-1)=0$. Thus, replace n by $2n$ above, and set $r_2 = -1$. Thus,

$$c_{2n}(-1) = \frac{d}{dr}\left[(r+1)a_{2n}(r)\right]_{r=-1}. \qquad (4)$$

We also need to calculate a from (20). Thus, since $N = 1-(-1) = 2$,

$$a = \lim_{r\to -1}(r+1)a_2(r) = \lim_{r\to-1}(r+1)\cdot\frac{-1}{(r+1)(r+3)} = -\frac{1}{2}.$$

We now apply the formula (2.5) in the Bessel function blurb on the website for $p=0$, or from a class lecture, i.e., if

$$f(x) = (x+\alpha_1)^{\beta_1}(x+\alpha_2)^{\beta_2}\cdots(x+\alpha_m)^{\beta_m},$$

then

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x+\alpha_1} + \frac{\beta_2}{x+\alpha_2} + \cdots + \frac{\beta_m}{x+\alpha_m}.$$

Return to (3) and (4). We want to calculate $\frac{d}{dr}[(r+1)a_{2n}(r)] = \frac{d}{dr}A_{2n}(r)$, say. Thus,

$$\beta_1 = -2,\ \beta_2 = -2,\ \ldots,\ \beta_{n-1} = -2,\ \beta_n = -1;$$
$$\alpha_1 = 3,\ \alpha_2 = 5,\ \ldots,\ \alpha_n = 2n+1.$$

Hence,

$$\frac{A_{2n}'(r)}{A_{2n}(r)} = -\frac{2}{r+3} - \frac{2}{r+5} - \cdots - \frac{2}{r+2n-1} - \frac{1}{r+2n+1}.$$

Thus, for $n \geq 2$,

$$\begin{aligned} c_{2n}(-1) &= \frac{d}{dr}\left[A_{2n}(r)\right]_{r=-1} \\ &= \left[\frac{-2}{2} - \frac{2}{4} - \cdots - \frac{2}{2n-2} - \frac{1}{2n}\right]A_{2n}(-1) \\ &= -(H_n + H_{n-1})\cdot\frac{(-1)^n}{2^{2n-1}(n-1)!\,n!} \qquad (5) \end{aligned}$$

by (3). ~~Thus~~, Note that $c_2 = -1/2$. Thus, (5) holds for $n=1$ as well. Hence,

$$y_2(x) = -\frac{1}{2}J_1(x)\log x + x^{-1}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(H_n + H_{n-1})x^{2n}}{(n-1)!\,n!\,2^{2n-1}}\right)$$