Math 441, Solutions, HW Set # 10

<u>Section 5.5, #9</u> $x^2y'' - x(x+3)y' + (x+3)y = 0$

$$y'' - \frac{x+3}{x}y' + \frac{x+3}{x^2}y = 0$$

$xp(x) = -\frac{x+3}{x}\cdot x = -(x+3)$ is analytic at $x=0$. $p_0 = -3$

$x^2q(x) = x+3$ is analytic at $x=0$, $q_0 = 3$.

$\therefore\quad r(r-1) - 3r + 3 = r^2 - 4r + 3 = (r-3)(r-1) = 0$

$\therefore\ r_1 = 3,\ r_2 = 1$ are indicial roots.

We find the solution for the larger root 3. Let $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$

$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$

$$L[y] = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r+1}$$
$$-3\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} + 3\sum_{n=0}^{\infty} a_n x^{n+r}$$

In the 2nd & 4th series replace n by $n-1$. Thus,

$$L[y] = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=1}^{\infty} a_{n-1}(n-1+r)x^{n+r}$$
$$-3\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} + 3\sum_{n=0}^{\infty} a_n x^{n+r}$$
$$= a_0[r(r-1) - 3r + 3]x^r$$
$$+ \sum_{n=1}^{\infty}\left(a_n[(n+r)(n+r-1) - 3(n+r) + 3] + a_{n-1}[-(n-1+r)+1]\right)x^{n+r} = 0$$

Thus, $r(r-1) - 3r + 3 = r^2 - 4r + 3 = (r-3)(r-1) = 0$.

$\therefore\ r = 3, 1$, as we calculated above. For $n \geq 1$

$$a_n[(n+r)^2 - 4(n+r) + 3] + a_{n-1}(-n-r+2) = 0$$

$$\therefore\quad a_n = \frac{a_{n-1}(n+r-2)}{(n+r-3)(n+r-1)},\quad n \geq 1.$$

Let $a_0 = 1$, $r = 3$.

$$a_1 = \frac{a_0(1+3-2)}{1\cdot 3} = \frac{2}{3}$$

$$a_2 = \frac{a_1 \cdot 3}{2 \cdot 4} = \frac{2 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{2}{2 \cdot 4}, \quad a_3 = \frac{a_2 \cdot 4}{3 \cdot 5} = \frac{2 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{2}{2 \cdot 3 \cdot 5}$$

It appears that

$$a_n = \frac{2}{n!\,(n+2)}, \quad n \geq 1. \qquad (*)$$

We will prove this by induction. Assume the formula and use our recurrence relation. Thus,

$$a_{n+1} = \frac{a_n (n+2)}{(n+1)(n+3)} = \frac{2\cancel{(n+2)}}{n!\,\cancel{(n+2)}(n+1)(n+3)} = \frac{2}{(n+1)!\,(n+3)}.$$

Thus, we have proved $(*)$, and

$$y_1(x) = 2x^3 \sum_{n=0}^{\infty} \frac{x^n}{(n)!\,(n+2)}. \qquad (**)$$

(Note that $(*)$ holds for $n = 0$ as well.)

Note that $x = 0$ is the only singular point, thus, $(**)$ converges for $0 \leq |x| < \infty$.

Sect. 5.5 #16 $\quad x^2y''+xy'+(x^2-1)y=0$

$$y''+\frac{1}{x}y'+\left(1-\frac{1}{x^2}\right)y=0$$

$xp(x)=1$, $x^2q(x)=-1+x^2$ Both are analytic. Thus, 0 is a regular singular pt. The indicial equation is

$$r(r-1)+r-1=r^2-1=0 \Rightarrow r=\pm 1.$$

Let $y=\sum_{n=0}^{\infty}a_nx^{n+r}$. Then

$$L[y]=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r}+\sum_{n=0}^{\infty}a_n(n+r)x^{n+r}$$
$$+\sum_{n=0}^{\infty}a_nx^{n+r+2}-\sum_{n=0}^{\infty}a_nx^{n+r}$$
$$=\sum_{n=0}^{\infty}a_n[(n+r)(n+r-1)+(n+r)-1]x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

For $n=0$,
$$a_0[r(r-1)+r-1]=a_0[r^2-1]=0\Rightarrow r=\pm 1.$$

For $n=1$,
$$a_1[(r+1)r+(r+1)-1]=a_1[(r+1)^2-1]=0$$

If either $r=1$ or $r=-1$, we see that $a_1=0$. For $n\ge 2$

$$a_n=\frac{-a_{n-2}}{(n+r)^2-1} \qquad (*)$$

If $r=1$, $\quad a_n=\frac{-a_{n-2}}{(n+1)^2-1}=\frac{-a_{n-2}}{n^2+2n}=\frac{-a_{n-2}}{n(n+2)}$.

Since $a_1=0$, we see $a_3=0, a_5=0, \dots$, i.e. $a_{2n+1}=0$

$$a_2=\frac{-a_0}{2\cdot 4}=\frac{-1}{2\cdot 4} \quad \text{if } a_0=1$$

$$a_4=\frac{-a_2}{4\cdot 6}=\frac{1}{2\cdot 4^2\cdot 6},\quad a_6=\frac{-a_4}{6\cdot 8}=\frac{-1}{2\cdot 4^2\cdot 6^2\cdot 8}=\frac{-2}{2\cdot 4\cdot 6\cdot 2\cdot 4\cdot 6\cdot 8}$$

We see that
$$a_{2n}=\frac{(-1)^n}{2^{2n}\,n!\,(n+1)!},\quad n\ge 0$$

Thus,
$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n}\,n!\,(n+1)!}$$

The authors multiply this solution by 1/2 to get the solution

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n} n!\,(n+1)!},$$

which is the Bessel function of order 1. We use the ratio test to check for convergence.

$$\lim_{n\to\infty} \frac{(-1)^{n+1} x^{2n+3}\, 2^{2n}\, n!\,(n+1)!}{2^{2n+2}(n+1)!\,(n+2)!\,(-1)^n x^{2n+1}}$$

$$= \lim_{n\to\infty} \frac{(-1)\,x^2}{2^2\,(n+2)(n+1)} = 0 \quad \text{for all } x.$$

Thus, $J_1(x)$ converges for $0 \le |x| < \infty$.

Suppose $r = -1$. Then, from (*),

$$a_n = \frac{-a_{n-2}}{(n-1)^2 - 1} = \frac{-a_{n-2}}{n(n-2)}$$

We see that $a_3 = a_5 = \cdots = a_{2n+1} = 0$ as before. But for $n = 2$, a_2 is undefined. Thus, we cannot find a solution for $r = -1$ of the form

$$x^{-1}\sum_{n=0}^{\infty} b_n x^n.$$