# Approximating higher algebra by derived algebra

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# 1 minute summary

## Theory

My work develops:

- A framework for producing obstruction theories, which is
  - General, intended to be easily applicable to new situations;
  - Conceptual, with good formal properties;
- Some of the algebra necessary for applications.

## Applications

In a motivating case, this gives:

- New tools for working over Lubin-Tate spectra;
- Application: new $\mathbb{E}_\infty$ complex orientations at heights $n \leq 2$.

# Some background

## Definition

A *periodic complex orientation* of $R$ is a multiplicative map $MUP \to R$.

## Big question

If $R$ is $\mathbb{E}_\infty$, can $MUP \to R$ be made $\mathbb{E}_\infty$?

## Example

Todd genus: $MUP \to KU$. Can this be made $\mathbb{E}_\infty$? Theorem (B.): Yes.

# Some background (cont.)

## Arithmetic fracture

Problem splits into primes: enough for $MUP \to KU_p^\wedge$ to be made $\mathbb{E}_\infty$.

## Lubin-Tate spectra

The spectrum $KU_p^\wedge$ is a Lubin-Tate spectrum. These are:

- Associated to finite height $n$ formal groups over perfect fields;
  - $KU_p^\wedge$ comes from $\widehat{\mathbb{G}}_m$ over $\mathbb{F}_p$, this is height $n = 1$;
- $\mathbb{E}_\infty$ ring spectra by the Goerss-Hopkins-Miller theorem.

# Towards the general framework

## $\mathbb{H}_\infty$ orientations (Ando, Ando-Hopkins-Strickland, Zhu)

Can start by studying $\mathbb{H}_\infty$ orientations $MUP \to E$. These

- Exist for any $E$;
- Can be classified with formal group theory.

## Input

- $\mathbb{H}_\infty$ orientations $\Leftrightarrow$ $E_*MUP \to E_*$ respecting $E$-power operations;
- Classification amounts to describing power operations on $E_*MUP$.

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## Dream

Use $E$-power operations to say $\mathbb{E}_\infty$ things, not just $\mathbb{H}_\infty$ things.

## Focus of work

- Develop a framework that makes it easy to realize dreams like this;
- Understand relevant algebra, and apply to $E$-theory and more.

# Algebraic approximations

## A common recipe

- Input: a homotopy theory $\mathcal{M}$ with free objects $\mathcal{P} \subset \mathcal{M}$;
- Output: an algebraic category approximating the category $\mathcal{M}$.

This is made easy using *algebraic theories*.

## Construction

- Idea: make $\mathrm{h}\mathcal{P}$ the free objects of the algebraic category;
- Get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ = presheaves $X$ with $X(\coprod_i P_i) \simeq \prod_i X(P_i)$.

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## Lots of examples

- For $(G\text{-})\mathbb{A}_\infty$ ring $A$, get $\mathrm{L}\mathcal{M}\mathrm{od}_A^{\mathrm{free}} \rightsquigarrow \mathrm{L}\mathcal{M}\mathrm{od}_{A_*}$;
- Have $\mathcal{C}\mathrm{Alg}_{H\mathbb{F}_p}^{\mathrm{free}} \rightsquigarrow \{\mathbb{F}_{p*}\text{-rings with Dyer-Lashof operations}\}$;
- Have $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}} \rightsquigarrow \{\text{Objects with } E\text{-power operations}\}$.

# Homotopy theory of models

## Models of an algebraic theory $\mathcal{P}$

- Object of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ can be a presheaf $X\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}\mathrm{pd}_\infty$;
- $\mathcal{P}$ itself can be an $\infty$-category.

## Quillen's homotopical algebra

- For $\mathcal{P}$ a 1-category, $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ = homotopy theory of s$\mathcal{S}$et-models;
- Cohomology intrinsic to $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$: classic Quillen cohomology.

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## Refinement of dream

For $\mathcal{P} \subset \mathcal{M}$ free objects, produce obstruction theories $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} \Rightarrow \mathcal{M}$.

# Higher homotopy theories

## A problem

In our examples of free objects $\mathcal{P} \subset \mathcal{M}$:

- $\mathcal{P}$ is an algebraic theory before truncating;
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Intuition: need operations indexed over higher-dimensional shapes.

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## Defn: resolution theories (following Hopkins-Lurie, Pstrągowski)

- $\mathcal{P}$ such that for $P \in \mathcal{P}$, tensor $S^1 \otimes P = \operatorname{colim}_{x \in S^1} P$ exists in $\mathcal{P}$;
- $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ = models $X$ such that $X(S^1 \otimes P) \simeq X(P)^{S^1}$.

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## Metatheorem I (B.)

For given $\mathcal{P} \subset \mathcal{M}$, and anything like them, we have $\mathcal{M} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$.

# Back to applications

## Metatheorem II (B.).

The context

$$\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \leftrightsquigarrow \mathcal{M}\mathrm{odel}_{\mathcal{P}} \leftrightsquigarrow \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$$

lets you easily build obstruction theories etc. of the form

$$\{\text{Algebra of } \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}\} \Rightarrow \{\text{Homotopy of } \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}\}.$$

## Basic example

Have obstruction theory "$H^{p-q}_{\mathrm{h}\mathcal{P}/\pi B}(\pi A; \pi B^{S^q}) \Rightarrow \pi_q \operatorname{Map}_{\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}}(A, B)$".

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## Ex: apply to $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}} \simeq \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}}$ for Lubin-Tate $E$

- General features of $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}}$ fit into a nice algebraic story, and Quillen cohomology here can be made more explicit;
- Using Rezk's work on power operations, get bounds on $H^*_{\mathrm{h}\mathcal{P}}$;
- Vanishing obstruction groups at heights $\leq 2$ give $\mathbb{E}_\infty$ $MUP \to E$.