Approximating higher algebra by derived algebra

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Künneth spectral sequence

Fix: R an $\mathbb{E}_\infty$-ring spectrum, $M, N \in \mathcal{M}\mathrm{od}_R$. Then:

$$\pi_{*+q}(M \otimes_R N) \Leftarrow \mathrm{Tor}^{R_*}_{p+q}(M_*, N_{*-p}).$$

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Idea of proof

(i) Choose resolution $M \leftarrow F_\bullet$ with $\pi_* M \leftarrow \pi_* F_\bullet$ a free resolution;

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Downsides to approach

(i) If we replace $\mathcal{M}\mathrm{od}_R$ with more exotic (unstable etc.) categories, working directly with resolutions becomes difficult;

(ii) Extra properties (multiplicativity, naturality, ...) need to be checked by hand, and some of these are difficult;

(iii) No direct relation between $\mathcal{M}\mathrm{od}_R$ and $\mathcal{D}(R_*)$.

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Goal of talk

To explain a framework for building obstruction theories, spectral sequences, etc. which:

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Outline of talk

§1 Motivation: more on $\mathcal{M}\mathrm{od}_R$;

§2 Well-behaved infinitary algebraic theories;

§3 Description of the general framework;

§4 Some example output;

§5 An application to chromatic homotopy theory.

Understanding $\pi_*\colon \mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_{R_*}$

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Definition

(i) A finitary *algebraic theory* is a small $\mathcal{C}$ with finite coproducts;

(ii) A *model* of $\mathcal{C}$ is a presheaf X such that $X(C_1 \coprod \cdots \coprod C_n) \simeq X(C_1) \times \cdots \times X(C_n)$.

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If $\mathcal{M}\mathrm{od}_{R_*}^{\mathrm{fgf}}$ = sums and shifts of R_*, then

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New interpretation of π_*

$$\begin{array}{ccccc}
\mathcal{M}\mathrm{od}_R & & \xrightarrow{\quad\pi_*\quad} & & \mathcal{M}\mathrm{od}_{R_*} \\
\downarrow h & & & & \simeq\downarrow h \\
\mathrm{Psh}^\times(\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}}) & \xrightarrow{\pi_0} & \mathrm{Psh}^\times(\mathrm{Ho}(\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}}), \mathcal{S}\mathrm{et}) & \xrightarrow{=} & \mathrm{Psh}^\times(\mathcal{M}\mathrm{od}_{R_*}^{\mathrm{fgf}}, \mathcal{S}\mathrm{et})
\end{array}$$

Closer look at $\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}}$

Observations

(i) The inclusion $h\colon \mathcal{M}\mathrm{od}_R \to \mathrm{Psh}^\times(\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}})$ is not an equivalence:

(ii) If $X = h(M)$, then $X(\Sigma F) \simeq \Omega X(F)$.

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Theorem (Hopkins-Lurie)

This is a full characterization: $\mathcal{M}\mathrm{od}_R \simeq \mathrm{Psh}^{\times,\Omega}(\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}})$.

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General philosophy

Looking at $\mathcal{M}\mathrm{od}_R \subsetneq \mathrm{Psh}^\times(\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}})$ lets us relate
$\mathcal{M}\mathrm{od}_R = \mathrm{Psh}^{\times,\Omega}(\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}})$ with $\mathcal{D}^+(R_*) = \mathrm{Psh}^\times(\mathrm{Ho}(\mathcal{M}\mathrm{od}_R^{\mathrm{fgf}}))$.

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Definition

A *Mal'cev theory* is $\mathcal{P}$ such that

(i) $\mathcal{P}$ has small coproducts;

(ii) ... κ ....

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(iv) Why herds: if $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, then $\mathcal{P}/X$ still a Mal'cev theory.

Mal'cev theories II

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Examples

(i) $\mathcal{A}$ cocomplete abelian category, $\mathcal{P} \subset \mathcal{A}$ enough projectives, then

- $\mathcal{A} \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$;
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(ii) T a monad on $\mathrm{Psh}^{\Pi}(\mathcal{P})$ preserving geometric realizations, $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ objects free on $P \in \mathcal{P}$, then
- $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi}(T\mathcal{P})$.

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Good setting for Quillen cohomology and left-derived functors.

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Definition: Quillen's cohomology

$\Lambda \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$, $M \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{A}\mathrm{b})$, define

$$\mathcal{H}^n_{\mathcal{P}}(\Lambda; M) = \mathrm{Map}(\Lambda; B^n M), \qquad \pi_k \mathcal{H}^n_{\mathcal{P}}(\Lambda; M) = H^{n-k}_{\mathcal{P}}(\Lambda; M).$$

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Definition: Left-derived functors

Fix $F\colon \mathrm{Psh}^{\Pi}(\mathcal{P}', \mathcal{S}\mathrm{et}) \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$, set f = restriction to $\mathcal{P}'$. Define

$$\mathbb{L}F = f_!\colon \mathrm{Psh}^{\Pi}(\mathcal{P}') \to \mathrm{Psh}^{\Pi}(\mathcal{P}).$$

$f_!$ = left Kan extension of f = "apply F to free resolution".

Resolution theories

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Related work (finitary pointed setting)

(i) Hopkins-Lurie: uses particular stable theory to study Brauer groups of Lubin-Tate spectra;

(ii) Pstrągowski: reinterprets Blanc-Dwyer-Goerss obstruction theory for Π-algebras, and has work generalizing in different directions in the stable case.

Resolution theories II

Theorem (B.)

For many $\mathcal{M}$, one can find (possibly many) $\mathcal{P} \subset \mathcal{M}$ so $\mathcal{M} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$.

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Basic examples

(i) R an $\mathbb{A}_\infty$ ring spectrum: Have $\mathrm{L}\mathcal{M}\mathrm{od}_R \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}})$;

(ii) R an $\mathbb{E}_\infty$ ring, $\mathcal{O}$ operad in $\mathcal{M}\mathrm{od}_R$: Have $\mathcal{A}\mathrm{lg}_{\mathcal{O}} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}^{\mathrm{free}})$.

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Harder example

(iii) R an $\mathbb{E}_2$ ring, $I \subset R_0$ f.g. ideal, $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)} = I$-complete R-modules: Have $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}})$.

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This requires infinitary theories: $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}}$ not generated by compact objects in $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$. Also $\mathcal{A}\mathrm{lg}_{\mathcal{O}}^{\mathrm{Cpl}(I)} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}^{\mathrm{Cpl}(I),\mathrm{free}})$.

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Realizing the general philosophy

(i) Given $\mathcal{M}$, find $\mathcal{P} \subset \mathcal{M}$ such that $\mathcal{M} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$;

(ii) Using $\mathcal{M} \subset \mathrm{Psh}^{\Pi}(\mathcal{P})$, get new filtrations of questions about $\mathcal{M}$;

(iii) Filtration quotients are questions in $\mathrm{Psh}^{\Pi}(\mathrm{Ho}(\mathcal{P}))$.

Example I: Mapping spaces

Setup

Goal: for $X, Y \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, produce maps $X \to Y$. Fix:

(i) $\phi\colon \pi_0 X \to \pi_0 Y$ in $\mathrm{Psh}^{\Pi}(\mathrm{Ho}(\mathcal{P}), \mathcal{S}\mathrm{et})$;

(ii) $\mathrm{Map}^{\phi}(X, Y) = f$ with $\pi_0 f = \phi$.

Postnikov tower in $\mathrm{Psh}^{\Pi}(\mathcal{P})$: get $\mathrm{Map}^{\phi}(X, Y) = \lim_n \mathrm{Map}^{\phi}(X, Y_{\leq n})$.

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Theorem (B.)

There are canonical fiber sequences.

$$\mathrm{Map}^{\phi}(X, Y_{\leq n}) \to \mathrm{Map}^{\phi}(X, Y_{\leq n-1}) \to \mathcal{H}^{n+1}_{\mathrm{Ho}(\mathcal{P})/\pi_0 Y}(\pi_0 X; \pi_0 Y^{S^n}).$$

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Variant

For $\mathcal{P}$ stable, have mapping spectrum $\mathbf{Map}(X, Y)$, and

$$\pi_{-q}\mathbf{Map}(X, Y) \Leftarrow \mathrm{Ext}^{p+q}_{\mathrm{Ho}(\mathcal{P})}(\pi_0 X; \pi_0 Y[p])$$

Example II: Left derived functors

Setup

(i) $\mathcal{P}'$ and $\mathcal{P}$ with $\mathcal{P}$ stable;

(ii) $F\colon \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}') \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ preserving geometric realizations;

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Comments

(i) Involves studying $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$: comes from an explicit localization $L\colon \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p})$;

(ii) Has good monoidal properties.

Concrete application

Setup

(i) E = height h Lubin-Tate spectrum;

(ii) $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}} = K(h)$-local commutative E-algebras;

(iii) Then $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}})$;

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Cohomology of $\widehat{\mathbb{T}}$-algebras

(i) Can identify $\mathcal{A}\mathrm{b}(\mathcal{A}\mathrm{lg}_{\widehat{\mathbb{T}}}/B_\star) \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\Delta_B}$ for an associative ring Δ_B;

(ii) If $M_\star \in \mathrm{L}\mathcal{M}\mathrm{od}_{\Delta_B}$ projective over $B_\star$: length h Koszul resolution;

(ii) $A_\star \in \mathcal{A}\mathrm{lg}_{\widehat{\mathbb{T}}}/B_\star$ "smooth": $H^*_{\widehat{\mathbb{T}}/B_\star}(A_\star; M_\star) = \mathrm{Ext}^*_{\Delta_A}(\Omega_{A_\star|E_\star}, M_\star)$.

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Example where obstructions vanish

If $h \leq 2$, then $\mathcal{C}\mathrm{Alg}(MUP, E) \to \mathbb{H}_\infty(MUP, E)$ surjects.