From power operations to $\mathbb{E}_\infty$ maps

William Balderrama

University of Illinois at Urbana-Champaign

September 17, 2020

Background: complex orientations

Fix E a homotopy ring spectrum, or multiplicative cohomology theory.

Definition

A (periodic) complex orientation of E is a class

$$u \in \widetilde{E}^0 BU(1) = \widetilde{E}^0 \mathbb{CP}^\infty$$

such that under $j\colon S^2 \subset BU(1)$,

$$j^*(u) \in \widetilde{E}^0 S^2 = \pi_2 E$$

is invertible.

Background: complex orientations

Fix E a homotopy ring spectrum, or multiplicative cohomology theory.

Definition

A (periodic) complex orientation of E is a class

$$u \in \widetilde{E}^0 BU(1) = \widetilde{E}^0 \mathbb{CP}^\infty$$

such that under $j \colon S^2 \subset BU(1)$,

$$j^*(u) \in \widetilde{E}^0 S^2 = \pi_2 E$$

is invertible.

Example: $E = KU$

$\pi_* KU = \mathbb{Z}[\beta^{\pm 1}]$ with $|\beta| = 2$. Canonical bundle $\mathcal{L} \to BU(1)$ gives

$$u = 1 - \mathcal{L} \in \widetilde{KU}^0 BU(1).$$

Then

$$j^*(u) = -\beta \in \pi_2 KU.$$

Chern classes

What does a complex orientation $u \in E^0BU(1)$ give you?

Facts

- $E^0BU(1) \cong E^0[[u]]$;
- In general $E^0BU(n) \cong E^0[[c_1, \ldots, c_n]]$.

Chern classes

What does a complex orientation $u \in E^0 BU(1)$ give you?

Facts

- $E^0 BU(1) \cong E^0[[u]]$;
- In general $E^0 BU(n) \cong E^0[[c_1, \ldots, c_n]]$.

Chern classes

- Fix $\xi \to X$ complex vector bundle;
- Classified by some $p\colon X \to BU(n)$;
- Get Chern classes $c_i(\xi) = p^*(c_i) \in E^0 X$.

So a complex orientation lets us define invariants of vector bundles.

Formal group laws

Tensor products

- Have $p\colon BU(1)\times BU(1)\to BU(1)$ classifying tensor product;
- Have $E^0(BU(1)\times BU(1)) = E^0[[u_1,u_2]]$, so $p^*(u)=\Gamma(u_1,u_2)$;
- In general $c_1(\mathcal{L}_1\otimes\mathcal{L}_2)=\Gamma(c_1(\mathcal{L}_1),c_1(\mathcal{L}_2))$.

Formal group laws

Tensor products

- Have $p\colon BU(1) \times BU(1) \to BU(1)$ classifying tensor product;
- Have $E^0(BU(1) \times BU(1)) = E^0[[u_1, u_2]]$, so $p^*(u) = \Gamma(u_1, u_2)$;
- In general $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = \Gamma(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2))$.

Formal group laws

Power series Γ satisfies

$$\Gamma(u_1, u_2) = \Gamma(u_2, u_1), \qquad \Gamma(u_1, \Gamma(u_2, u_3)) = \Gamma(\Gamma(u_1, u_2), u_3).$$

These are called formal group laws.

Formal group laws

Tensor products

- Have $p\colon BU(1) \times BU(1) \to BU(1)$ classifying tensor product;
- Have $E^0(BU(1) \times BU(1)) = E^0[[u_1, u_2]]$, so $p^*(u) = \Gamma(u_1, u_2)$;
- In general $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = \Gamma(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2))$.

Formal group laws

Power series Γ satisfies

$$\Gamma(u_1, u_2) = \Gamma(u_2, u_1), \qquad \Gamma(u_1, \Gamma(u_2, u_3)) = \Gamma(\Gamma(u_1, u_2), u_3).$$

These are called formal group laws.

Philosophy of chromatic homotopy theory

There is a close correspondence

$$\{\text{Stable homotopy theory}\} \leftrightsquigarrow \{\text{Geometry of formal group laws}\}.$$

Quillen's theorem

Quillen's theorem

- There is a universal complex oriented spectrum MUP:

$$\{\text{Complex orientations of } E\} \cong \{\text{Multiplicative maps } MUP \to E\}$$

- Formal group law over $\pi_0 MUP$ is universal:

$$\{\text{Formal group laws over ring R}\} \cong \{\text{Ring maps } \pi_0 MUP \to R\}.$$

Quillen's theorem

Quillen's theorem

- There is a universal complex oriented spectrum MUP:

$$\{\text{Complex orientations of } E\} \cong \{\text{Multiplicative maps } MUP \to E\}$$

- Formal group law over $\pi_0 MUP$ is universal:

$$\{\text{Formal group laws over ring R}\} \cong \{\text{Ring maps } \pi_0 MUP \to R\}.$$

Formal groups

- Have coproduct $E^0BU(1) \to E^0BU(1) \mathbin{\widehat{\otimes}} E^0BU(1)$;
- Description without $E^0BU(1) \cong E^0[[u]]$ gives "formal group" $\mathbb{G}_E$;
- Complex orientation $u \in E^0BU(1)$ is a "coordinate", on $\mathbb{G}_E$ and

$$\{\text{Coordinates on } \mathbb{G}_E\} \cong \{\text{Complex orientations } MUP \to E\}$$

Highly structured orientations

$\mathbb{E}_\infty$ rings

MUP is a $\mathbb{E}_\infty$-ring:

- $MUP^{\otimes n} \to MUP$ refines to $P_n \colon MUP^{\otimes n}_{\mathrm{h}\Sigma_n} \to MUP$;
- Maps P_n satisfy compatibilities: "$P_m \circ P_n \simeq P_{nm}$";
- Homotopies witnessing compatibilites satisfy compatibilities, etc.

Just the first two say MUP is an $\mathbb{H}_\infty$ ring.

Highly structured orientations

$\mathbb{E}_\infty$ rings

MUP is a $\mathbb{E}_\infty$-ring:

- $MUP^{\otimes n} \to MUP$ refines to $P_n \colon MUP^{\otimes n}_{\mathrm{h}\Sigma_n} \to MUP$;
- Maps P_n satisfy compatibilities: "$P_m \circ P_n \simeq P_{nm}$";
- Homotopies witnessing compatibilites satisfy compatibilities, etc.

Just the first two say MUP is an $\mathbb{H}_\infty$ ring.

Big question

Say E is $\mathbb{E}_\infty$. Can we make $MUP \to E$ a map of $\mathbb{E}_\infty$ rings?

Goal

Describe a positive answer for a number of E.

Lubin-Tate spectra

Cohomology theories from formal group laws

- Given fgl Γ on ring R_0, classified by $\pi_0 MUP \to R_0$, so can define

$$R_0 X = R_0 \otimes_{\pi_0 MUP} MUP_0 X.$$

- Landweber identifies class of "Landweber exact" formal group laws where this is a homology theory.
- Example: have $KU_0 X = KU_0 \otimes_{\pi_0 MUP} MUP_0 X$ (Conner-Floyd).

Lubin-Tate spectra

Cohomology theories from formal group laws

- Given fgl Γ on ring R_0, classified by $\pi_0 MUP \to R_0$, so can define

$$R_0 X = R_0 \otimes_{\pi_0 MUP} MUP_0 X.$$

- Landweber identifies class of "Landweber exact" formal group laws where this is a homology theory.
- Example: have $KU_0 X = KU_0 \otimes_{\pi_0 MUP} MUP_0 X$ (Conner-Floyd).

Lubin-Tate spectra

Morava singles out formal group laws constructed by Lubin-Tate:

- Fgl Γ over char. p perfect field κ has a "height" $1 \leq n \leq \infty$;
- For $n < \infty$, have universal lift Γ_u to complete local ring E_0;
- Γ_u is Landweber exact, so get E with $\pi_0 E = E_0$.

Lubin-Tate spectra

Cohomology theories from formal group laws

- Given fgl Γ on ring R_0, classified by $\pi_0 MUP \to R_0$, so can define

$$R_0 X = R_0 \otimes_{\pi_0 MUP} MUP_0 X.$$

- Landweber identifies class of "Landweber exact" formal group laws where this is a homology theory.
- Example: have $KU_0 X = KU_0 \otimes_{\pi_0 MUP} MUP_0 X$ (Conner-Floyd).

Lubin-Tate spectra

Morava singles out formal group laws constructed by Lubin-Tate:

- Fgl Γ over char. p perfect field κ has a "height" $1 \leq n \leq \infty$;
- For $n < \infty$, have universal lift Γ_u to complete local ring E_0;
- Γ_u is Landweber exact, so get E with $\pi_0 E = E_0$.

Goerss-Hopkins-Miller theorem

Lubin-Tate spectra have canonical $\mathbb{E}_\infty$ ring structures.

$\mathbb{H}_\infty$ orientations

Before studying $\mathbb{E}_\infty$ orientations, study $\mathbb{H}_\infty$ orientations.

$\mathbb{H}_\infty$ orientations

- $\mathbb{H}_\infty$ orientation: $MUP \to E$ giving homotopy commutative

$$\begin{array}{ccc} MUP^{\otimes n}_{\mathrm{h}\Sigma_n} & \longrightarrow & E^{\otimes n}_{\mathrm{h}\Sigma_n} \\ \downarrow & & \downarrow \\ MUP & \longrightarrow & E \end{array} .$$

- Cohomological condition: asks for equality in $E^0(MUP^{\otimes n}_{\mathrm{h}\Sigma_n})$.

$\mathbb{H}_\infty$ orientations

Before studying $\mathbb{E}_\infty$ orientations, study $\mathbb{H}_\infty$ orientations.

$\mathbb{H}_\infty$ orientations

- $\mathbb{H}_\infty$ orientation: $MUP \to E$ giving homotopy commutative

$$\begin{array}{ccc} MUP^{\otimes n}_{h\Sigma_n} & \longrightarrow & E^{\otimes n}_{h\Sigma_n} \\ \downarrow & & \downarrow \\ MUP & \longrightarrow & E \end{array}.$$

- Cohomological condition: asks for equality in $E^0(MUP^{\otimes n}_{h\Sigma_n})$.

Theorems of Ando, Ando-Hopkins-Strickland, Zhu

Let $\mathbb{G}$ = formal group of E, deformation of $\mathbb{G}_0$ over $\kappa = E_0/\mathfrak{m}$.

- $\mathbb{H}_\infty(MUP, E) \subset \mathrm{Coord}(\mathbb{G})$ describable with formal group data;
- Composite $\mathbb{H}_\infty(MUP, E) \to \mathrm{Coord}(\mathbb{G}) \to \mathrm{Coord}(\mathbb{G}_0)$ is iso.

Statement of theorem

Some known $\mathbb{E}_\infty$ orientations

- Walker: $\mathbb{E}_\infty(MU, KU_p) \to \mathbb{H}_\infty(MU, KU_p)$ iso, so in particular can get $\mathbb{E}_\infty$ orientation $MU \to KU$;
- Möllers: $\mathbb{E}_\infty(MU, R) = \mathbb{H}_\infty(MU, R)$ for $K(1)$-local even-periodic Landweber exact R (see also Hopkins-Lawson);
- Senger (unpublished): $\mathbb{E}_\infty(MU, E) \neq \emptyset$ for Lubin-Tate E at $n = 2$ with $\kappa = \overline{\mathbb{F}}_p$;
- Hahn-Yuan: $\mathbb{E}_\infty(MUP, E) \neq \emptyset$ for Lubin-Tate E at $n = 1$, $\kappa = \mathbb{F}_2$.

Statement of theorem

Some known $\mathbb{E}_\infty$ orientations

- Walker: $\mathbb{E}_\infty(MU, KU_p) \to \mathbb{H}_\infty(MU, KU_p)$ iso, so in particular can get $\mathbb{E}_\infty$ orientation $MU \to KU$;
- Möllers: $\mathbb{E}_\infty(MU, R) = \mathbb{H}_\infty(MU, R)$ for $K(1)$-local even-periodic Landweber exact R (see also Hopkins-Lawson);
- Senger (unpublished): $\mathbb{E}_\infty(MU, E) \neq \emptyset$ for Lubin-Tate E at $n = 2$ with $\kappa = \overline{\mathbb{F}}_p$;
- Hahn-Yuan: $\mathbb{E}_\infty(MUP, E) \neq \emptyset$ for Lubin-Tate E at $n = 1$, $\kappa = \mathbb{F}_2$.

Theorem (B.)

- $\mathbb{E}_\infty(MUP, R) = \mathbb{H}_\infty(MUP, R)$ for $K(1)$-local even-periodic R, so in particular can get $\mathbb{E}_\infty$ orientation $MUP \to KU$;
- Surj $\mathbb{E}_\infty(MUP, E) \twoheadrightarrow \mathbb{H}_\infty(MUP, E)$ for height 2 Lubin-Tate E.

Better yet: proof only really uses smoothness of $\pi_0(E \otimes MUP)$ over E_0.

Power operations

To study $\mathbb{H}_\infty$ orientations, Ando introduced E-power operations.

Power operations

To study $\mathbb{H}_\infty$ orientations, Ando introduced E-power operations.

Formal construction

- Have a category $\mathcal{C}\mathrm{Alg}_E^{\mathrm{Cpl}(\mathfrak{m})}$ of $\mathfrak{m}$-complete $\mathbb{E}_\infty$ E-algebras;
- E-power operations are natural transformations $\prod_i \pi_{r_i} \to \pi_s$;
- Have free functor $\widehat{\mathbb{P}}\colon \mathcal{M}\mathrm{od}_E \to \mathcal{C}\mathrm{Alg}_E^{\mathrm{Cpl}(\mathfrak{m})}$. By Yoneda:

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}},\mathcal{S}\mathrm{et})}\left(\prod \pi_{r_i}, \pi_s\right) = \pi_s \widehat{\mathbb{P}} \bigoplus \Sigma^{r_i} E;$$

Power operations

To study $\mathbb{H}_\infty$ orientations, Ando introduced E-power operations.

Formal construction

- Have a category $\mathcal{C}\mathrm{Alg}_E^{\mathrm{Cpl}(\mathfrak{m})}$ of $\mathfrak{m}$-complete $\mathbb{E}_\infty$ E-algebras;
- E-power operations are natural transformations $\prod_i \pi_{r_i} \to \pi_s$;
- Have free functor $\widehat{\mathbb{P}}\colon \mathcal{M}\mathrm{od}_E \to \mathcal{C}\mathrm{Alg}_E^{\mathrm{Cpl}(\mathfrak{m})}$. By Yoneda:

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}},\mathcal{S}\mathrm{et})}\left(\prod \pi_{r_i}, \pi_s\right) = \pi_s \widehat{\mathbb{P}} \bigoplus \Sigma^{r_i} E;$$

Fact

The following are equivalent:

- $\mathbb{H}_\infty$ orientations $MUP \to E$;
- $E_0^\wedge MUP \to E_0$ that respect power operations.

$\mathbb{H}_\infty$ is far from $\mathbb{E}_\infty$, but this suggests we should start by understanding power operations.

Algebraic theories

General recipe

- Input: homotopy theory $\mathcal{M}$ and collection $\mathcal{P} \subset \mathcal{M}$ of free objects;
- Output: homotopy groups $\pi_P M = \pi_0 \mathcal{M}(P, M)$ for $P \in \mathcal{M}$, and homotopy operations $\mathrm{Hom}(\pi_P, \pi_Q) = \pi_Q P$.

We can nicely package this using algebraic theories.

Algebraic theories

General recipe

- Input: homotopy theory $\mathcal{M}$ and collection $\mathcal{P} \subset \mathcal{M}$ of free objects;
- Output: homotopy groups $\pi_P M = \pi_0 \mathcal{M}(P, M)$ for $P \in \mathcal{M}$, and homotopy operations $\mathrm{Hom}(\pi_P, \pi_Q) = \pi_Q P$.

We can nicely package this using algebraic theories.

Definition

A (bounded Mal'cev algebraic) theory is a category $\mathcal{P}$ such that

- $\mathcal{P}$ has all small coproducts;
- Technical conditions: boundedness, Mal'cev.

Category of models $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is presheaves X with $X(\coprod_i P_i) = \prod_i X(P_i)$.

Algebraic theories

General recipe

- Input: homotopy theory $\mathcal{M}$ and collection $\mathcal{P} \subset \mathcal{M}$ of free objects;
- Output: homotopy groups $\pi_P M = \pi_0 \mathcal{M}(P, M)$ for $P \in \mathcal{M}$, and homotopy operations $\mathrm{Hom}(\pi_P, \pi_Q) = \pi_Q P$.

We can nicely package this using algebraic theories.

Definition

A (bounded Mal'cev algebraic) theory is a category $\mathcal{P}$ such that

- $\mathcal{P}$ has all small coproducts;
- Technical conditions: boundedness, Mal'cev.

Category of models $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is presheaves X with $X(\coprod_i P_i) = \prod_i X(P_i)$.

Idea

Theories encode algebraic structures defined using operations with set-indexed arities and "equational" or "universally quantified" axioms.

Algebraic theories (cont.)

General recipe (repackaged)

From $\mathcal{P} \subset \mathcal{M}$ free objects, get theory $\mathrm{h}\mathcal{P}$ and functor $\pi\colon \mathcal{M} \to \mathrm{Model}_{\mathrm{h}\mathcal{P}}$.

Algebraic theories (cont.)

General recipe (repackaged)

From $\mathcal{P} \subset \mathcal{M}$ free objects, get theory $\mathrm{h}\mathcal{P}$ and functor $\pi\colon \mathcal{M} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$.

Examples

- G a finite group, A a G-equivariant $\mathbb{A}_\infty$-ring, $\mathcal{P} = \mathrm{L}\mathcal{M}\mathrm{od}_A^{\mathrm{free}} = \{\bigoplus \Sigma^\alpha A \otimes (G/H)_+\}$, get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} = \mathrm{L}\mathcal{M}\mathrm{od}_{\underline{\pi}_\star A}$;
- R an $\mathbb{E}_\infty$-ring, $\mathcal{P}$ = free associative R-algebras, get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} =$ R_*-algebras (you can even add in G);
- $R \in \mathcal{C}\mathrm{Alg}_{H\mathbb{F}_p}$, $\mathcal{P} = \mathcal{C}\mathrm{Alg}_R^{\mathrm{free}}$, get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} =$ commutative R_*-rings with Dyer-Lashof operations;
- $\mathcal{P} = \mathcal{C}\mathrm{Alg}_{KU}^{\mathrm{Cpl}(p),\mathrm{free}}$, get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} =$ Ext-p-complete graded θ-rings
- Many more.

Homotopical algebra

Quillen's insight

For $\mathcal{P}$ a theory, there is a good homotopy theory of simplicial models.

Homotopical algebra

Quillen's insight

For $\mathcal{P}$ a theory, there is a good homotopy theory of simplicial models.

Facts

We can let a theory $\mathcal{P}$ be an ∞-category; models are $\mathcal{P}^{\mathrm{op}} \to \mathcal{G}\mathrm{pd}_\infty$.

- $\mathcal{M}\mathrm{odel}_\mathcal{P}$ freely adjoins geometric realizations to $\mathcal{P}$;
- For $\mathcal{P}$ a 1-category, $\mathcal{M}\mathrm{odel}_\mathcal{P}$ = Quillen's homotopy theory of simplicial set-valued models of $\mathcal{P}$.

Homotopical algebra

Quillen's insight

For $\mathcal{P}$ a theory, there is a good homotopy theory of simplicial models.

Facts

We can let a theory $\mathcal{P}$ be an ∞-category; models are $\mathcal{P}^{\mathrm{op}} \to \mathcal{G}\mathrm{pd}_\infty$.

- $\mathcal{M}\mathrm{odel}_\mathcal{P}$ freely adjoins geometric realizations to $\mathcal{P}$;
- For $\mathcal{P}$ a 1-category, $\mathcal{M}\mathrm{odel}_\mathcal{P}$ = Quillen's homotopy theory of simplicial set-valued models of $\mathcal{P}$.

Example usage

$\Lambda \in \mathcal{M}\mathrm{odel}_\mathcal{P}$ and $M \in \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_\mathcal{P})$, get Quillen cohomology groups

$$H^n_\mathcal{P}(\Lambda; M) = \pi_0 \operatorname{Map}_{\mathcal{M}\mathrm{odel}_\mathcal{P}}(\Lambda; B^n M).$$

What's the point of all this?

Metatheorem (B.)

For good $\mathcal{P} \subset \mathcal{M}$, one can build obstruction theories etc.

$$\{\text{Homotopical algebra of } \mathcal{M}\text{odel}_{\text{h}\mathcal{P}}\} \Rightarrow \{\text{Homotopy theory of } \mathcal{M}\}$$

What's the point of all this?

Metatheorem (B.)

For good $\mathcal{P} \subset \mathcal{M}$, one can build obstruction theories etc.

$$\{\text{Homotopical algebra of } \mathrm{Model}_{\mathrm{h}\mathcal{P}}\} \Rightarrow \{\text{Homotopy theory of } \mathcal{M}\}$$

Very brief idea

Study the context

$$\mathcal{M} \leftrightsquigarrow \mathrm{Model}_{\mathcal{P}} \leftrightsquigarrow \mathrm{Model}_{\mathrm{h}\mathcal{P}}$$

What's the point of all this?

Metatheorem (B.)

For good $\mathcal{P} \subset \mathcal{M}$, one can build obstruction theories etc.

$$\{\text{Homotopical algebra of } \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}\} \Rightarrow \{\text{Homotopy theory of } \mathcal{M}\}$$

Very brief idea

Study the context

$$\mathcal{M} \leftrightsquigarrow \mathcal{M}\mathrm{odel}_{\mathcal{P}} \leftrightsquigarrow \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$$

Particular case

Given $A, B \in \mathcal{M}$ and $\phi\colon \pi A \to \pi B$ in $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$;

- Obstructions in $H^{n+1}_{\mathrm{h}\mathcal{P}/\pi B}(\pi A; \pi B^{S^n})$ for $n \geq 1$ to realizing ϕ;
- Given a lift f, fringed $H^{p-q}_{\mathrm{h}\mathcal{P}/\pi B}(\pi A; \pi B^{S^p}) \Rightarrow \pi_q(\mathcal{M}(A,B), f)$.

Getting back on track

Where we're at

- Want $\mathbb{E}_\infty$ orientations $MUP \to E$, and have $\mathbb{H}_\infty$ orientations;
- $\mathbb{H}_\infty$ orientations give $E_*^\wedge MUP \to E_*$ in $\mathrm{Model}_{\mathcal{P}}$, $\mathcal{P} = \mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$
- Obstructions to lifting in $H^{m+1}_{\mathcal{P}/E_*}(E_*^\wedge MUP; \pi_* E^{S^m})$ for $n \geq 1$.

So we need to understand $\mathrm{Model}_{\mathcal{P}}$.

Getting back on track

Where we're at

- Want $\mathbb{E}_\infty$ orientations $MUP \to E$, and have $\mathbb{H}_\infty$ orientations;
- $\mathbb{H}_\infty$ orientations give $E_*^\wedge MUP \to E_*$ in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, $\mathcal{P} = \mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$
- Obstructions to lifting in $H^{m+1}_{\mathcal{P}/E_*}(E_*^\wedge MUP; \pi_* E^{S^m})$ for $n \geq 1$.

So we need to understand $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

Rezk's monad

- Rezk studies a monad $\mathbb{T}$ on $\mathcal{M}\mathrm{od}_{E_*}$ encoding E-power operations;
- $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \subset \mathcal{R}\mathrm{ing}_{\mathbb{T}}$ is a reflective subcategory of $\mathfrak{m}$-complete objects;
- In particular can identify $H^*_{\mathcal{P}} = H^*_{\mathbb{T}}$.

So we can work with $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$ instead.

Interlude: θ-rings

Definition: θ-rings

Ring R with $\theta\colon R \to R$ so $\psi(x) = x^p + p\theta(x)$ generically a ring map:

$$\theta(x+y) = \theta(x) + \theta(y) - p^{-1}((a+b)^p - a^p - b^p);$$
$$\theta(xy) = \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y);$$
$$\theta(1) = 1.$$

Relevance

In height 1, even objects of $\mathcal{R}\mathrm{ing}_{\mathbb{T}} = \theta$-rings under E_0.

Interlude: θ-rings

Definition: θ-rings

Ring R with $\theta\colon R \to R$ so $\psi(x) = x^p + p\theta(x)$ generically a ring map:

$$\theta(x+y) = \theta(x) + \theta(y) - p^{-1}((a+b)^p - a^p - b^p);$$
$$\theta(xy) = \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y);$$
$$\theta(1) = 1.$$

Relevance

In height 1, even objects of $\mathcal{R}\mathrm{ing}_{\mathbb{T}} = \theta$-rings under E_0.

Algebraic facts

- W = free θ-ring $= \mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing},\mathcal{S}\mathrm{et})}(U,U)$ with U = forgetful;
- Can identify $W \cong \mathbb{Z}[a, \theta a, \theta^{\circ 2}a, \ldots]$.

Interlude: θ-rings (cont.)

The additive bialgebra

- Have $\Gamma := \mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing},\mathcal{A}\mathrm{b})}(U, U) = \mathrm{Prim}(W) = \mathbb{Z}[\psi]$;
- Identity $\psi(xy) = \psi(x)\psi(y)$ encoded by coproduct $\Delta(\psi) = \psi \otimes \psi$;
- Coproduct equivalent to: $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma \to \mathcal{A}\mathrm{b}$ symmetric monoidal.

Interlude: θ-rings (cont.)

The additive bialgebra

- Have $\Gamma := \mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing},\mathcal{A}\mathrm{b})}(U,U) = \mathrm{Prim}(W) = \mathbb{Z}[\psi]$;
- Identity $\psi(xy) = \psi(x)\psi(y)$ encoded by coproduct $\Delta(\psi) = \psi \otimes \psi$;
- Coproduct equivalent to: $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma \to \mathcal{A}\mathrm{b}$ symmetric monoidal.

Remember $\psi(x) = x^p + p\theta(x)$.

The Wilkerson criterion

- If R is torsion-free, then ψ and θ determine each other;
- $\mathcal{R}\mathrm{ing}_\theta \to \mathrm{C}\mathcal{M}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma)$ is fully faithful on torsion-free objects, image characterized by the congruence $\psi(x) \equiv x^p \pmod{p}$;
- As W is torsion-free, congruence determines structure of $\theta\mathcal{R}\mathrm{ing}$.

Interlude: θ-rings (still cont.)

Why θ-rings are well-behaved

Quillen cohomology of θ-rings pleasant for two reasons:

1. $\theta\mathcal{R}\text{ing} \to \mathrm{C}\mathcal{R}\text{ing}$ both monadic and comonadic;
2. Free θ-ring W is (formally) smooth.

These fit into a general story of "smooth algebras over theories".

Interlude: θ-rings (still cont.)

Why θ-rings are well-behaved

Quillen cohomology of θ-rings pleasant for two reasons:

1. $\theta\mathcal{R}\text{ing} \to \mathcal{CR}\text{ing}$ both monadic and comonadic;
2. Free θ-ring W is (formally) smooth.

These fit into a general story of "smooth algebras over theories".

Consequences of (1)

- Have $\Delta := \mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}},\mathcal{S}\mathrm{et})}(Q, Q) = Q(W) = \mathbb{Z}[\theta]$;
- Have $\mathcal{A}\mathrm{b}(\theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}}) = \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$, and Q gives abelianization.

Interlude: θ-rings (still cont.)

Why θ-rings are well-behaved

Quillen cohomology of θ-rings pleasant for two reasons:

1. $\theta\mathcal{R}\mathrm{ing} \to \mathrm{C}\mathcal{R}\mathrm{ing}$ both monadic and comonadic;
2. Free θ-ring W is (formally) smooth.

These fit into a general story of "smooth algebras over theories".

Consequences of (1)

- Have $\Delta := \mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}},\mathcal{S}\mathrm{et})}(Q, Q) = Q(W) = \mathbb{Z}[\theta]$;
- Have $\mathcal{A}\mathrm{b}(\theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}}) = \mathrm{L}\mathcal{M}\mathrm{od}_{\Delta}$, and Q gives abelianization.

Consequence of (2)

For $A \in \theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}}$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\Delta}$, have

$$H^*_{\theta^{\mathrm{aug}}}(A; M) = \mathrm{Ext}^*_{\Delta}(\mathbb{L}Q(A); M).$$

Interlude: θ-rings (still cont.)

Why θ-rings are well-behaved

Quillen cohomology of θ-rings pleasant for two reasons:

1. $\theta\mathcal{R}\mathrm{ing} \to \mathrm{C}\mathcal{R}\mathrm{ing}$ both monadic and comonadic;
2. Free θ-ring W is (formally) smooth.

These fit into a general story of "smooth algebras over theories".

Consequences of (1)

- Have $\Delta := \mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}},\mathcal{S}\mathrm{et})}(Q,Q) = Q(W) = \mathbb{Z}[\theta]$;
- Have $\mathcal{A}\mathrm{b}(\theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}}) = \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$, and Q gives abelianization.

Consequence of (2)

For $A \in \theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}}$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$, have

$$H^*_{\theta^{\mathrm{aug}}}(A;M) = \mathrm{Ext}^*_\Delta(\mathbb{L}Q(A);M).$$

Generally $\mathcal{A}\mathrm{b}(\theta\mathcal{R}\mathrm{ing}/A) = \mathrm{L}\mathcal{M}\mathrm{od}_{A\otimes\Delta}$, $A\otimes\Delta = A\langle\theta\rangle/(\theta\cdot a = \psi(a)\cdot\theta)$.

Structure of $\mathbb{T}$

Fix E height n. Everything from $\theta\mathcal{R}$ing generalizes to $\mathcal{R}\text{ing}_{\mathbb{T}}$: have additive bialgebra Γ and algebra Δ acting on indecomposables.

Structure of $\mathbb{T}$

Fix E height n. Everything from $\theta\mathcal{R}\mathrm{ing}$ generalizes to $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$: have additive bialgebra Γ and algebra Δ acting on indecomposables.

Strickland's theorem

$\Gamma = \bigoplus_{m\geq 0}\Gamma[m]$ with $\operatorname{Spf}\Gamma[m]^\vee =$ rank p^m subgroups of $\mathbb{G}_E$.

All of Γ's structure is encoded by formal group theory.

Structure of $\mathbb{T}$

Fix E height n. Everything from $\theta\mathcal{R}\mathrm{ing}$ generalizes to $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$: have additive bialgebra Γ and algebra Δ acting on indecomposables.

Strickland's theorem

$\Gamma = \bigoplus_{m\geq 0}\Gamma[m]$ with $\operatorname{Spf}\Gamma[m]^\vee =$ rank p^m subgroups of $\mathbb{G}_E$.

All of Γ's structure is encoded by formal group theory.

Rezk's theorems

- T.f. objects of $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$ = t.f. objects of $\mathcal{R}\mathrm{ing}_{\Gamma} = \mathrm{CMon}(\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma})$ satisfying $Q(x) \equiv x^p \pmod p$ for certain $Q \in \Gamma$. Heuristic: $\mathbb{T}$-rings are like θ-rings, but with more additive operations and relations.

Structure of $\mathbb{T}$

Fix E height n. Everything from $\theta\mathcal{R}\mathrm{ing}$ generalizes to $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$: have additive bialgebra Γ and algebra Δ acting on indecomposables.

Strickland's theorem

$\Gamma = \bigoplus_{m\geq 0}\Gamma[m]$ with $\operatorname{Spf}\Gamma[m]^\vee$ = rank p^m subgroups of $\mathbb{G}_E$.

All of Γ's structure is encoded by formal group theory.

Rezk's theorems

- T.f. objects of $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$ = t.f. objects of $\mathcal{R}\mathrm{ing}_{\Gamma} = \mathrm{C}\mathcal{M}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma)$ satisfying $Q(x) \equiv x^p \pmod p$ for certain $Q \in \Gamma$. Heuristic: $\mathbb{T}$-rings are like θ-rings, but with more additive operations and relations.
- Δ is Koszul, and $H^m(\Delta) = 0$ for $m > n$. Thus every $M \in \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$ projective over E_* admits length n projective resolution.

Application

Theorem (B.)

Fix height $n \leq 2$, and $A, B \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ with $A_1 = 0 = B_1$ and A_0 "completed smooth" over E_0. Then

- Every $\mathbb{T}$-ring map $A_* \to B_*$ lifts to $A \to B$;
- $\pi_* \mathcal{C}\mathrm{Alg}_E(A, B)$ is algebraic, up to extensions at $n = 2$.

Application

Theorem (B.)

Fix height $n \leq 2$, and $A, B \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ with $A_1 = 0 = B_1$ and A_0 "completed smooth" over E_0. Then

- Every $\mathbb{T}$-ring map $A_* \to B_*$ lifts to $A \to B$;
- $\pi_* \mathcal{C}\mathrm{Alg}_E(A, B)$ is algebraic, up to extensions at $n = 2$.

Proof sketch

- Obstructions in $H^{k+1}_{\mathbb{T}/B_*}(A_*; \pi_* B^{S^k})$ for $k \geq 1$ to lifting;
- By algebraic story + smoothness, this is $\mathrm{Ext}^{k+1}_{A_0 \otimes \Delta}(\widehat{\Omega}_{A_*|E_*}; \pi_* \Omega^k B)$;
- Can upgrade Rezk's theorem to show group vanishes for $k \geq n$;
- $\mathrm{Ext}^2_{A_0 \otimes \Delta}(\widehat{\Omega}_{A_*|E_*}; \pi_* \Omega B) = 0$ as source is even and target is odd.

Application

Theorem (B.)

Fix height $n \leq 2$, and $A, B \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ with $A_1 = 0 = B_1$ and A_0 "completed smooth" over E_0. Then

- Every $\mathbb{T}$-ring map $A_* \to B_*$ lifts to $A \to B$;
- $\pi_*\mathcal{C}\mathrm{Alg}_E(A, B)$ is algebraic, up to extensions at $n = 2$.

Proof sketch

- Obstructions in $H^{k+1}_{\mathbb{T}/B_*}(A_*; \pi_* B^{S^k})$ for $k \geq 1$ to lifting;
- By algebraic story + smoothness, this is $\mathrm{Ext}^{k+1}_{A_0 \otimes \Delta}(\widehat{\Omega}_{A_*|E_*}; \pi_* \Omega^k B)$;
- Can upgrade Rezk's theorem to show group vanishes for $k \geq n$;
- $\mathrm{Ext}^2_{A_0 \otimes \Delta}(\widehat{\Omega}_{A_*|E_*}; \pi_* \Omega B) = 0$ as source is even and target is odd.

Corollary with $A = E \mathbin{\widehat{\otimes}} MUP$ and $B = E$

Every $\mathbb{H}_\infty$ orientation $MUP \to E$ refines to an $\mathbb{E}_\infty$ orientation.