# Approximating higher algebra by derived algebra

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# Intro

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Describe some obstruction theories and related algebra via an example.

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Describe some obstruction theories and related algebra via an example.

## Definition

A *complex orientation* of $R$ is a homotopy ring map $MU \to R$.

## Big question

If $R$ is $\mathbb{E}_\infty$, can $MU \to R$ be made $\mathbb{E}_\infty$?

# Highly structured orientations

Focus on $R = E$ a Lubin-Tate spectrum.

### Facts about $E$

- Different $E$ for every finite height formal group over perfect field.
- Turns out to be canonically $\mathbb{E}_\infty$ (Goerss-Hopkins-Miller).
  Problem: $\mathbb{E}_\infty$ structure not related to $MU$.

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### $\mathbb{H}_\infty$ orientations (Ando, Ando-Hopkins-Strickland, Zhu)

- Exist for any $E$;
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### $\mathbb{E}_\infty$ orientations

- Height $n = 1$: "$\mathbb{H}_\infty = \mathbb{E}_\infty$" (Walker, Möllers);
- Height $n = 2$: Understanding starts to break down.

# Highly structured orientations (cont.)

Extra complication: $E$ is even-periodic, so want $MUP$ orientations.

## What's known

- $\mathbb{H}_\infty$ orientations: just as well-understood.
- $\mathbb{E}_\infty$ orientations: known exist at $n = 1$, $p = 2$ (Hahn-Yuan).

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### Theorem (B.)

- $\mathbb{H}_\infty(MUP, R) = \mathbb{E}_\infty(MUP, R)$ for $K(1)$-local even-periodic $R$, so in particular can get $\mathbb{E}_\infty$ orientation $MUP \to KU$;
- Surj $\mathbb{E}_\infty(MUP, E) \twoheadrightarrow \mathbb{H}_\infty(MUP, E)$ for $E$ at height 2.

Better yet: proof only really uses smoothness of $\pi_0(E \otimes MUP)$ over $E_0$.

# Proof of theorem

The proof is obstruction-theoretic.

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## Comments

- Height $n \leq 2$ restriction will come in at step (3);
- Goal: describe some of what goes into (1) and (2).

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### Replacement for $\mathbb{H}_\infty$

- $E_*MUP$ is $E_*$-free, so homotopy maps $MUP \to E$ are the same as $E_*$-module maps $E_*MUP \to E_*$;
- Now replace "$\mathbb{H}_\infty$ map $MUP \to E$" with "map $E_*MUP \to E_*$ compatible with $E$-power operations".

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What does it mean to be compatible with $E$-power operations?

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What encodes operations?

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## Better question

What encodes operations?

## Answer

Operations are encoded by algebraic theories.

# Algebraic theories

## Definition

A (bounded Mal'cev algebraic) theory is a category $\mathcal{P}$ such that

1. $\mathcal{P}$ has all small coproducts;
2. $\mathcal{P}$ satisfies a size condition;
3. $\mathcal{P}$ satisfies the Mal'cev condition.

The category of *models* of $\mathcal{P}$ is the category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ of presheaves $X$ on $\mathcal{P}$ such that $X(\coprod_{i\in I} P_i) \simeq \prod_{i\in I} X(P_i)$.

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## Comments

- From (1), theories are encoding "product theories", or structures with operations with set-indexed arities and equational axioms;
- Theory can be $\infty$-category; models are $X\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}\mathrm{pd}_\infty$

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## Facts

- $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ freely adjoins geometric realizations to $\mathcal{P}$;
- $\mathcal{P}$ a 1-cat: $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ = Quillen's homotopy theory of $s\mathcal{S}\mathrm{et}$-models.

# Categories of algebraic approximations

## General recipe

- Input: homotopy theory $\mathcal{M}$ with free objects $\mathcal{P} \subset \mathcal{M}$;
- Output: theory $\mathrm{h}\mathcal{P}$ and homotopy functor $\pi\colon \mathcal{M} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$.

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## Examples

- $G$ a finite group, $A$ a $G$-equivariant $\mathbb{A}_\infty$ ring, $\mathcal{P} = \mathrm{L}\mathcal{M}\mathrm{od}_A^{\mathrm{free}} = \{\bigoplus \Sigma^\alpha A \otimes (G/H)_+\}$, get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} = \mathrm{L}\mathcal{M}\mathrm{od}_{\underline{\pi}_\star A}$;
- $R$ an $\mathbb{E}_\infty$ ring, $\mathcal{P} =$ free associative $R$-algebras, get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} =$ $R_*$-algebras (you can even add in $G$);
- $R \in \mathcal{C}\mathrm{Alg}_{H\mathbb{F}_p}$, $\mathcal{P} = \mathcal{C}\mathrm{Alg}_R^{\mathrm{free}}$, get $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} =$ commutative $R_*$-rings with Dyer-Lashof operations.

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## Example of choice of $\mathcal{P}$

- Take $\mathcal{M} = \mathcal{M}\mathrm{od}_{H\mathbb{Z}_p}^{\mathrm{Cpl}(p)}$, $\mathcal{P} = \{(\bigoplus \Sigma^\alpha H\mathbb{Z})_p^\wedge\}$, $\mathcal{P}' = \{\bigoplus \Sigma^\alpha H\mathbb{Z}/(p)\}$;
- $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} = \mathrm{Ext}$-$p$-completes in $\mathcal{M}\mathrm{od}_{H\mathbb{Z}_*}$, $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}'} = \mathcal{M}\mathrm{od}_{\Lambda_{\mathbb{F}_p}(\beta)}$.

# More homotopical structure

## Homotopy theories

- In all examples, $\mathcal{P} \subset \mathcal{M}$ is a theory before taking $h\mathcal{P}$;
- In all examples, $\mathcal{M} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is not an equivalence.
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## Homotopy theories

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## Definition: resolution theories

1. $\mathcal{P}$ such that for $P \in \mathcal{P}$, the tensor $S^1 \otimes P = \mathrm{colim}_{x \in S^1} P$ exists in $\mathcal{P}$.
2. $\mathrm{Model}_{\mathcal{P}}^{\Omega}$ = models $X$ such that $X(S^1 \otimes P) \simeq X(P)^{S^1}$.

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## History

- Examples first seen in Hopkins-Lurie's work on Brauer groups;
- Finitary pointed version used by Pstrągowski to reinterpret Blanc-Dwyer-Goerss obstruction theory for Π-algebras.

# Purpose of construction

## Metatheorem I

If $\mathcal{M}$ is a category in stable homotopy theory with a good subcategory $\mathcal{P} \subset \mathcal{M}$ of free objects, then $\mathcal{M} \simeq \mathrm{Model}_{\mathcal{P}}^{\Omega}$.

## Idea

Check directly for $\mathcal{M}$ stable, then extend to algebras over good monads.

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Check directly for $\mathcal{M}$ stable, then extend to algebras over good monads.

### Metatheorem II

Studying the context

$$\mathrm{Model}_{\mathcal{P}}^{\Omega} \leftrightsquigarrow \mathrm{Model}_{\mathcal{P}} \leftrightsquigarrow \mathrm{Model}_{\mathrm{h}\mathcal{P}}$$

makes it easy to produce obstruction theories, spectral sequences, etc.

$$\{\text{Derived algebra of } \mathrm{Model}_{\mathrm{h}\mathcal{P}}\} \Rightarrow \{\text{Homotopy theory of } \mathrm{Model}_{\mathcal{P}}^{\Omega}\}.$$

# Examples

## Left-derived functor spectral sequences

$\mathcal{P}$ "stable" if $\Sigma\colon \mathcal{P} \simeq \mathcal{P}$. Gives $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\mathcal{S}\mathrm{p}}$ and explicit localization $L\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\mathcal{S}\mathrm{p}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\mathcal{S}\mathrm{p},\Omega}$. Get "left-derived functor spectral sequences":

- Künneth spectral sequences, including multiplicativity;
- Basterra spectral sequences for TAQ of ring spectra;
- For $X \in \mathcal{S}\mathrm{p}_{\geq 0}$ and $M \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, $H_*(X; \pi_* M) \Rightarrow \pi_*(X \otimes M)$.

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## Mapping space obstruction theory

Given $\mathcal{P}$ resolution theory, $A, B \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ and $\phi\colon \pi_0 A \to \pi_0 B$, get

- Obstructions in $H^{m+1}_{\mathrm{h}\mathcal{P}/\pi_0 B}(\pi_0 A; \pi_0 B^{S^m})$ to realizing $\phi$;
- Given a lift $f$, fringed $H^{p-q}_{\mathrm{h}\mathcal{P}/\pi_0 B}(\pi_0 A; \pi_0 B^{S^p}) \Rightarrow \pi_q(\mathrm{Map}(A, B), f)$.

We can apply this one to $E$-theory.

# Getting back to orientations

## Where we're at

- Want $\mathbb{E}_\infty$ orientation $MUP \to E$, have $\mathbb{H}_\infty$ orientations;
- $\mathbb{H}_\infty$ orientation equivalent to $E^\wedge_* MUP \to E_*$ in $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$;
- Here $\mathcal{P} = \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$, and $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \simeq \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$;
- Get obstructions to lifting in $H^{m+1}_{\mathrm{h}\mathcal{P}/E_*}(E^\wedge_* MUP; \pi_* E^{S^m})$.

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## Rezk's monad

- Rezk studies a monad $\mathbb{T}$ on $\mathcal{M}\mathrm{od}_{E_*}$ encoding $E$-power operations;
- $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} \subset \mathcal{R}\mathrm{ing}_{\mathbb{T}}$ is reflective subcategory of $\mathfrak{m}$-complete objects;
- In particular can identify $H^*_{\mathrm{h}\mathcal{P}} = H^*_{\mathbb{T}}$.

So we can work with $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$ instead.

# Well-behaved theories

$\mathcal{R}\mathrm{ing}_{\mathbb{T}}$ is a "well-behaved" theory of operations.

## Fundamental facts

- Forgetful functor $\mathcal{R}\mathrm{ing}_{\mathbb{T}} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{E_*}$ is monadic and comonadic;
- Free $\mathbb{T}$-rings are (formally) smooth as $E_*$-rings.

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## Another example

- Let $R = \mathbb{E}_\infty$ ring with $p = 0$ in $\pi_0 R$, and $\mathcal{P} = \mathcal{C}\mathrm{Alg}_R^{\mathrm{free}}$;
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## General context

- Bimonadicity says $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$ is "algebra over the theory of $E_*$-rings";
- Examples can be unified using this notion (due to Freyd, Wraith).

# $\theta$-rings

## Definition: $\theta$-rings

Ring $R$ with $\theta\colon R \to R$ so $\psi(x) = x^p + p\theta(x)$ generically a ring map:

$$\theta(x+y) = \theta(x) + \theta(y) - p^{-1}((a+b)^p - a^p - b^p);$$
$$\theta(xy) = \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y);$$
$$\theta(1) = 0.$$

## Relevance

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## Facts

- $\theta\mathcal{R}\mathrm{ing} \to \mathrm{C}\mathcal{R}\mathrm{ing}$ is monadic and comonadic;
- $W$ = free $\theta$-ring = $\mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing},\mathcal{S}\mathrm{et})}(U,U)$ with $U$ = forgetful;
- Can identify $W \cong \mathbb{Z}[a, \theta a, \theta^{\circ 2} a, \ldots]$.

# $\theta$-rings: Quillen cohomology

## Abelianizations

Using the fact that $\theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}} \to \mathrm{C}\mathcal{R}\mathrm{ing}^{\mathrm{aug}}$ is bimonadic:

- Identify $\Delta := \mathrm{Hom}_{\mathrm{Fun}(\mathcal{R}\mathrm{ing}^{\mathrm{aug}}, \mathcal{S}\mathrm{et})}(Q, Q) = Q(W) = \mathbb{Z}[\theta]$;
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In general, $\mathcal{A}\mathrm{b}(\theta\mathcal{R}\mathrm{ing}/A) = \mathrm{L}\mathcal{M}\mathrm{od}_{A\otimes\Delta}$, $A \otimes \Delta = A\langle\theta\rangle/(\theta \cdot a = \psi(a) \cdot \theta)$.

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## Cohomology

Using fact that $W$ is smooth: for $A \in \theta\mathcal{R}\mathrm{ing}^{\mathrm{aug}}$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$,

$$H^*_{\theta^{\mathrm{aug}}}(A; M) = \mathrm{Ext}^*_\Delta(\mathbb{L}Q(A); M).$$

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## Comment

All of this is generic for "smooth algebras over theories".

# $\theta$-rings: Additive operations

## The additive bialgebra

- Identify $\Gamma := \mathrm{Hom}_{\mathrm{Fun}(\theta\mathcal{R}\mathrm{ing},\mathcal{A}\mathrm{b})}(U,U) = \mathrm{Prim}(W) = \mathbb{Z}[\psi]$;
- Identity $\psi(xy) = \psi(x)\psi(y)$ encoded by coproduct $\Delta(\psi) = \psi \otimes \psi$;
- Coproduct equivalent to: $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma \to \mathcal{M}\mathrm{od}_\mathbb{Z}$ symmetric monoidal.

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Recipe works for "algebras over theories of commutative rings".

Remember $\psi(x) = x^p + p\theta(x)$.

## The Wilkerson criterion

- If $R$ is torsion-free, then $\psi$ and $\theta$ determine each other;
- $\mathcal{R}\mathrm{ing}_\theta \to \mathrm{C}\mathcal{M}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma)$ is fully faithful on torsion-free objects, image characterized by the congruence $\psi(x) \equiv x^p \pmod p$;
- As $W$ is torsion-free, congruence determines structure of $\theta\mathcal{R}\mathrm{ing}$.

# $\mathbb{T}$-rings

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- T.f. objects of $\mathcal{R}\mathrm{ing}_{\mathbb{T}} =$ t.f. objects of $\mathcal{CM}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma)$ satisfying $Q(x) \equiv x^p \pmod{p}$ for certain $Q \in \Gamma$. Heuristic: $\mathbb{T}$-rings are like $\theta$-rings, but with more additive operations and relations.

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- $\Delta$ is Koszul, and $H^m(\Delta) = 0$ for $m > n$. Thus every $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\Delta}$ projective over $E_*$ admits length $n$ projective resolution.

# Application

## Theorem (B.)

Fix height $n \leq 2$, and $A, B \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ with $A_1 = 0 = B_1$ and $A_*$ "completed smooth" over $E_*$. Then

- Every $\mathbb{T}$-ring map $A_* \to B_*$ lifts to $A \to B$;
- $\pi_* \mathcal{C}\mathrm{Alg}_E(A, B)$ is algebraic, up to extensions at $n = 2$.

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## Proof sketch

1. Obstructions in $H^{k+1}_{\mathbb{T}/B_*}(A_*; \pi_* B^{S^k})$ for $k \geq 1$ to lifting;
2. By algebraic story + smoothness, this is $\mathrm{Ext}^{k+1}_{A_* \otimes \Delta}(\widehat{\Omega}_{A_*|E_*}; \pi_* \Omega^k B)$;
3. Can upgrade Rezk's theorem to show group vanishes for $k \geq n$;
4. $\mathrm{Ext}^2_{A_* \otimes \Delta}(\widehat{\Omega}_{A_*|E_*}; \pi_* \Omega B) = 0$ as source is even and target is odd.

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## Corollary with $A = E \mathbin{\widehat{\otimes}} MUP$ and $B = E$

Every $\mathbb{H}_\infty$ orientation $MUP \to E$ at $n \leq 2$ lifts to an $\mathbb{E}_\infty$ orientation.