ALGEBRAIC THEORIES OF POWER OPERATIONS (DRAFT)

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Abstract. We develop some aspects of algebraic theories that are useful for working with some of the algebraic structures that arise in stable homotopy theory, with an emphasis on their role in certain obstruction theories. We give a general treatment of Koszul complexes which may be of independent interest. We describe in detail the tools one obtains for computing with $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$ and over Lubin-Tate spectra. As an application, we demonstrate the existence of $\mathbb{E}_\infty$ periodic complex orientations at heights $h \leq 2$.

1. Introduction

Let R be an $\mathbb{E}_\infty$ ring spectrum. Then there is a theory of *R-power operations*: for any $\mathbb{E}_\infty$ algebra A over R, there are natural maps

$$R_b(S^a)^{\wedge n}_{\mathrm{h}\Sigma_n} \times \pi_a A \to \pi_b A$$

refining the n'th power map. As with all natural operations, these are immediately applicable to nonexistence theorems. For example, a map $\phi\colon \pi_* A \to \pi_* B$ that fails to be compatible with these operations must also fail to refine to an $\mathbb{E}_\infty$ map $A \to B$. The converse is false, making existence theorems more subtle. As a general heuristic, by understanding the global structure of these operations, one can start to quantify the failure of the converse by introducing a suitable obstruction theory.

A collection of methods for realizing this heuristic was studied in [Bal20], with obstruction groups given in terms of formally defined algebraic theories of operations. In some cases, these algebraic theories can be identified, and are sufficiently well-behaved the corresponding obstruction groups can be described. Two main examples of this are the theories of power operations for $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$, and for $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height h.

The purpose of this paper is to develop some of the algebra necessary for working with well-behaved theories of operations, and to apply this to the above examples. Very roughly, we consider a theory well-behaved when the relevant cohomology computations can be split into computations in more classical theories, such as of the André-Quillen cohomology of commutative rings, and ordinary homological algebra. The latter portion often turns out to permit a theory of Koszul complexes, and to access this, we give a general development of the Koszul story that allows for instability conditions, non-augmented algebras, and more.

The tools we produce for computing over Lubin-Tate spectra turn out to be very pleasant at heights $h \leq 2$, with frequently vanishing obstruction groups. As an application, we show that there exist $\mathbb{E}_\infty$ orientations $MUP \to E$ for E a Lubin-Tate spectrum of height $h \leq 2$, as well as for $E = KU$.

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Contents

1.1. **Summary.** This paper is split naturally into two halves. The first half consists of general theory, and takes places in the three sections following this one. These sections are carried out in a fairly general setting; in particular, they are not tied to power operations, or even to any kind of homotopy operations. Certainly the strength of any abstract framework lies in its examples and ease of use, and to that end we have given numerous examples throughout these sections illustrating all of the main definitions and results. The second half takes place in the final two sections, and consists of applications to the theories of power operations for $\mathbb{E}_\infty$ rings in characteristic p and over Lubin-Tate spectra, and in turn to applications of this algebra to the homotopy theory of these objects. These contexts are sufficiently rich that everything done in the first half sheds light on some aspect of one or both of these.

By algebraic theory, we refer to a form of Lawvere theory. We begin in Section 2 by giving the particular form that we will use, and recalling a number of classical definitions relevant to their use. In short, the form of algebraic theory that we will use is a category $\mathcal{P}$ with small coproducts satisfying some additional conditions which, in part, serve to make the infinitary aspects of $\mathcal{P}$ well-behaved; we will just call these *theories*. The category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ of set-valued models of $\mathcal{P}$ is then the category of set-valued presheaves on $\mathcal{P}$ that send coproducts in $\mathcal{P}$ to products of sets.

Among the definitions recalled in Section 2, the most important is that of an *algebra over a theory*, covered in Subsection 2.3. This paper is not motivated by arbitrary theories of power operations, but only those which are in some sense well-behaved. To give an example, consider the theory of power operations for $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$. Abstractly, this theory can be defined as follows. Let $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$ be the category of free $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$, and let $\mathcal{P} = \mathrm{h}\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$ be its homotopy category. Then $\mathcal{P}$ is a theory that deserves to be called the theory of mod p power operations, and we consider $\mathcal{P}$ to be a well-behaved theory due to a combination of two facts:

(1) There is a forgetful functor $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_{p*}}^{\heartsuit}$ to the category of graded commutative rings over $\mathbb{F}_p$ which admits a left and right adjoint, and is the forgetful functor of both a monadic and a comonadic adjunction;
(2) The free models of $\mathcal{P}$ have discrete and projective Quillen homology as objects of $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_{p*}}^{\heartsuit}$.

The first point tells us that we can understand $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ as being built in a simple way from $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_{p*}}^{\heartsuit}$. The second point tells us that this remains the case when we move beyond set-valued models and consider the homotopy theories of simplicial objects in these categories. In particular, the Quillen cohomology of models of $\mathcal{P}$ can be split into a classical portion, given by the ordinary André-Quillen cohomology of graded commutative rings over $\mathbb{F}_p$, and a purely linear portion, consisting of the homological algebra of graded vector spaces equipped with Dyer-Lashof operations subject to certain instability conditions. The use of commutative rings in particular is not necessary for this story, and the notion of an algebra over a theory serves to describe the general situation.

In Section 3, we specialize to additive theories; the main purpose of this section is to develop a sufficiently robust theory of *Koszul algebras*, and in particular of their associated Koszul resolutions and Koszul complexes. Koszul algebras were first introduced by Priddy in [Pri70], with the motivating example being given by the

Steenrod algebra. It is by now apparent that a more general notion of Koszul algebra is necessary for understanding the various examples that arise in homotopy theory. Even when considering just variations of the classical example of the Steenrod algebra, it is desirable to have a Koszul story that allows for instability conditions, algebras that are not augmented, algebras that do not contain their coefficient rings centrally, and algebras built on objects richer than just abelian groups (such as Mackey functors). We show that such a Koszul story indeed exists, and with the right definitions in place, is not any more difficult than the classical story.

Section 4 is independent from Section 3, and treats a notion of *plethory*, generalizing the biring triples of [TW70] and plethories of [BW05]. In short, a plethory is an algebra for the theory of commutative monoids over a symmetric monoidal additive theory. In some sense, for the purposes of understanding the Quillen cohomology of rings over a plethory, this definition is all there is to say: the general story is then just a specialization of the more general story already given in Subsection 2.9. Turning this around, our treatment of the Quillen cohomology of rings over plethory in Subsection 4.4 serves as a more concrete demonstration of how the general story can be applied in practice. In addition, we cover some further topics special to plethories which are necessary for understanding the examples which are of interest.

We turn in Section 5 to the study of power operations for $\mathbb{E}_\infty$ rings in characteristic p. Certain aspects of this story are classical, and so this material serves in part as an extended example of the general material developed in the preceding sections. We give a detailed recollection of the full $\mathbb{Z}$-graded theory of mod p power operations in Subsection 5.2. The linear part of this story, encoded by Dyer-Lashof operations subject to certain instability conditions, is a Koszul algebra in the general sense considered in Section 3; we describe the cohomology of this algebra in Subsection 5.5, thereby giving access to the associated Koszul complexes. Much of what one sees in this story is similar to what one sees in the study of unstable rings over the Steenrod algebra, and we describe the precise relationship between these contexts in Subsection 5.4.

So far, this material is still essentially algebraic. Though perhaps interesting in its own right, our real interest in this algebra comes from its relevance to the homotopy theory of $\mathbb{E}_\infty$ rings in characteristic p. We explain some of this connection in Subsection 5.6 and Subsection 5.7. In particular, we describe an obstruction theory for computing mapping spaces between these objects, as well as some spectral sequences for computing their topological André-Quillen homology and cohomology.

Section 6 is independent from Section 5, but is similar in form; this section gives applications to the study of $K(h)$-local $\mathbb{E}_\infty$ algebras over Lubin-Tate spectra. One subtlety that must be dealt with in this context is that the $K(h)$-local condition in homotopy is reflected in algebra by certain completeness conditions. This motivates the use of infinitary theories from the start, allowing for categories of completed objects to be treated as no different from any other category considered. With the right setup, which our use of algebraic theories makes easily accessible, many of the issues related to completions can be absorbed into the background. The result is that these issues are no longer issues, but merely additional aspects of the objects under consideration.

After describing some of the general algebra relevant to understanding power operations in this setting, and recalling some of their known structure, we turn to describing what this algebraic story can tell us about the homotopy theory of $K(h)$-local $\mathbb{E}_\infty$ rings over Lubin-Tate spectra. Our primary application is Theorem 6.5.1, which describes a general obstruction theory for computing mapping spaces between these objects. Abstractly, this obstruction theory is a special case of the general framework developed in [Bal20], but now we have developed the algebra necessary for understanding the obstruction groups.

A special case of this worth highlighting here is the following. Let A and B be $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height $h \leq 2$, and suppose that $A_1 = 0 = B_1$, and that A_0 is smooth (in a suitable completed, and not necessarily finitely generated, sense) over E_0; then every map $A_0 \to B_0$ compatible with E-power operations lifts to an $\mathbb{E}_\infty$ map $A \to B$. Moreover, under these assumptions, the homotopy groups of the space $\mathcal{C}\mathrm{Alg}_E(A, B)$ are built in a reasonably simple way out of algebraic invariants of A_* and B_*. We describe some applications of this in Subsection 6.5; for instance, it applies immediately to give the existence of various $\mathbb{E}_\infty$ orientations at heights $h \leq 2$, and we detail this application in Theorem 6.5.3. In Subsection 6.6, we give some spectral sequences for $K(h)$-local topological André-Quillen homology and cohomology, and explain how in certain cases of interest its computation reduces to a purely algebraic problem.

1.2. **Conventions.** Throughout the paper, we will freely use the theory of ∞-categories, which we refer to just as categories, as developed by Lurie in [Lur17]. However, the homotopy theories we consider rarely extend beyond those modeled nicely by simplicial objects in a category of set-valued models for an algebraic theory, as already considered by Quillen in [Qui67], and the reader who prefers to do so should have no trouble interpreting most of our constructions in this sense instead.

By default, all of our constructions should be interpreted in the derived sense. For example, if $\mathcal{C}$ is a small category, then $\mathrm{Psh}(\mathcal{C})$ refers to the category of presheaves of ∞-groupoids on $\mathcal{C}$. As another example, if R is a commutative ring, then $\mathcal{C}\mathcal{R}\mathrm{ing}_R$ is (the underlying ∞-category of) the category of simplicial R-rings, and we would instead use $\mathcal{C}\mathcal{R}\mathrm{ing}_R^\heartsuit$ to refer the category of discrete R-rings. As a small exception, the term R-ring will always refer to a discrete R-ring. Here, by R-ring we refer to an commutative R-algebra; all of our rings will be commutative, and we reserve the term algebra for a different concept. We will be explicit when we wish to refer instead to $\mathbb{E}_\infty$ rings.

Large portions of this paper are really about ordinary algebra, and so the default conventions of the preceding paragraph are only defaults. Certain examples and certain subsections take place entirely within 1-categories, and this will be noted explicitly when it occurs.

2. Theories

This section is primarily a collection of mostly classical definitions, together with various examples. In Subsections 2.1–2.4, we review some aspects of the theory of algebraic theories, which we just call theories, in the form we will use, in particular covering the notion of an algebra over a theory. In Subsection 2.5, we recall the notion of a distributive law; these are useful for identifying the structure of certain theories that get built out of other theories. In Subsections 2.6–2.8, we review the notions of left-derived functors and Quillen cohomology available in the context of theories. In Subsection 2.9, we describe some general aspects of how this story plays out in the context of models for an algebra over a theory.

2.1. **Mal'cev theories.** The categorical approach to universal algebra was pioneered by Lawvere in his thesis [Law04], leading to the notion of a *Lawvere theory*. These can be defined as categories $\mathcal{C}$ with object set $\mathbb{N}$ wherein n is the n-fold coproduct of 1 for all $n \in \mathbb{N}$. The category of set-valued models of $\mathcal{C}$ is then the category of functors $X\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ such that the canonical map $X(n) \to X(1)^{\times n}$ is an isomorphism for all $n \in \mathbb{N}$. We will make use of a variant of this, modifying it in three ways.

First, by only asking that $\mathcal{C}$ has finite coproducts, and not that a specified object generates $\mathcal{C}$ under coproducts, one obtains a multisorted variant of Lawvere theories. Note that in this definition no particular sorts are specified. This approach emphasizes the aspects of algebraic theories which are invariant under Morita equivalence.

Second, by relaxing the condition that $\mathcal{C}$ has only finite coproducts, one can obtain an infinitary form of Lawvere theory. The classic reference for these is Wraith [Wra70], although certain size issues are overlooked there. To deal with these, we restrict ourselves to bounded theories, i.e. those generated by κ-ary operations for some regular cardinal κ. This has the further benefit of making available to us all the tools from the theory of presentable categories.

Third, we can consider models not just in sets, but in simplicial sets, or in ∞-groupoids. The former were considered by Quillen [Qui67], and their equivalence with the latter is known due to work of Badzioch [Bad02], Bergner [Ber06], and Lurie [Lur17, Section 5.5.9]. In order for this to work as one would like in the infinitary setting, we must impose an additional condition on our theories, which essentially amounts to asking for all simplicial models to be fibrant.

We now proceed to the precise definitions. First, a *herd* is a set X together with a ternary operation $t\colon X \times X \times X \to X$ satisfying the identities

$$t(x,x,y) = y, \qquad t(x,y,y) = x.$$

Such operations t are also called *Mal'cev operations*, as they were first discovered by Mal'cev; we take the name herd from Lambek [Lam92], to which we refer the reader for more on their history.

2.1.1. **Example.** If X and Y are objects in some category, then $\mathrm{Iso}(X,Y)$ is a herd with Mal'cev operation $t(f,g,h) = fg^{-1}h$. In particular, every group is a herd. $\triangleleft$

Herds are modeled by an ordinary Lawvere theory, so we can freely speak of herd objects in arbitrary categories.

2.1.2. **Definition.** A *Mal'cev theory* is a category $\mathcal{P}$ such that

(1) $\mathcal{P}$ has all small coproducts;
(2) Every $P \in \mathcal{P}$ admits the structure of a coherd.

We say that $\mathcal{P}$ is *κ-bounded* for a regular cardinal κ if

(3) There exists a small full subcategory $\mathcal{P}_0 \subset \mathcal{P}$ closed under κ-small coproducts and satisfying the following κ-compactness condition: for every $P_0 \in \mathcal{P}_0$ and set of objects $\{P_i : i \in I\}$ of $\mathcal{P}$, the canonical map $\operatorname{colim}_{F\subset I, |F|<\kappa} \mathrm{Map}_{\mathcal{P}}(P_0, \coprod_{i\in F} P_i) \to \mathrm{Map}_{\mathcal{P}}(P_0, \coprod_{i\in I} P_i)$ is an equivalence.

We say that $\mathcal{P}$ is *bounded* if it is κ-bounded for some κ, and by *theory* we will refer to a bounded Mal'cev theory. By *discrete theory* we refer to a theory which is a 1-category. The category of *models* of a theory $\mathcal{P}$ is the category $\mathcal{M}\mathrm{odel}_{\mathcal{P}} = \mathrm{Psh}^{\Pi}(\mathcal{P})$ of presheaves X on $\mathcal{P}$ such that $X(\coprod_{i\in I} P_i) \simeq \prod_{i\in I} X(P_i)$ for any set $\{P_i : i \in I\}$

of objects of $\mathcal{P}$. We call the full subcategory $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} = \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et}) \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ the category of *discrete models* of $\mathcal{P}$. ◁

Throughout the paper, $\mathcal{P}$ will always refer to some theory. The Yoneda embedding $h\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ allows us to view $\mathcal{P}$ as a full subcategory of its category of models, and we will omit the h in doing so. In addition, we will often write $\mathrm{Map}_{\mathcal{P}}$ where we might more properly write $\mathrm{Map}_{\mathcal{M}\mathrm{odel}_{\mathcal{P}}}$. We will at times refer to a theory by its models when no confusion is likely to occur; for instance, if $\mathcal{P}$ is the category of free abelian groups, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is the category of abelian groups, and we could refer to $\mathcal{P}$ as just the theory of abelian groups.

2.1.3. **Remark.** We emphasize here that this is really a paper about discrete theories; our goal is to describe some tools with which one can access the derived algebra of certain discrete theories, and some applications of this. We have opted to make certain definitions and constructions in the context of more general theories, both as this is their natural home, and as one is at times forced into the setting of non-discrete theories when reasoning about the derived algebra of discrete theories. However, we will restrict ourselves to the discrete case whenever it is convenient to do so. ◁

If $\mathcal{P}$ is a general theory, then its homotopy category $\mathrm{h}\mathcal{P}$ is a discrete theory. The theories we are most interested in are those related to theories arising this way, such as the theory of mod p power operations described in Subsection 1.1 (treated in more detail in Section 5).

The structure of categories of models of a theory can be summarized as follows.

2.1.4. **Lemma** ([Bal20, Section 2])**.**

1. (1) $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is the free cocompletion of $\mathcal{P}$ under geometric realizations, which are preserved by the embedding $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \subset \mathrm{Psh}(\mathcal{P})$. In particular, if $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, then $\mathrm{Map}_{\mathcal{P}}(X, -)$ preserves geometric realizations if and only if X is a retract of some object of $\mathcal{P}$.
2. (2) Say $\mathcal{P}$ is κ-bounded, and fix $\mathcal{P}_0 \subset \mathcal{P}$ realizing this. Then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is equivalent to the category of presheaves on $\mathcal{P}_0$ which preserve κ-small coproducts. In particular, it is a κ-compactly generated presentable category.
3. (3) Suppose that $\mathcal{P}$ is discrete. Every simplicial object in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is a Kan complex, so the category of simplicial set-valued models of $\mathcal{P}$ has a canonical Quillen model structure [Qui67, Theorem II.4.4]. The underlying ∞-category of this is $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, with localization realized by the functor of geometric realization. □

We end this subsection by pointing out the following, which illustrates the role that theories play for us. For $P \in \mathcal{P}$, write $\mathrm{ev}_P\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{G}\mathrm{pd}_{\infty}$ for the functor given by evaluation at P.

2.1.5. **Proposition.** For $P, P' \in \mathcal{P}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}\mathrm{odel}_{\mathcal{P}},\, \mathcal{S}\mathrm{et})}(\pi_0 \mathrm{ev}_P, \pi_0 \mathrm{ev}_{P'}) \cong \pi_0 \mathrm{Map}_{\mathcal{P}}(P', P).$$

Proof. By the Yoneda lemma, we can identify

$$\mathrm{ev}_P(X) \cong \mathrm{Map}_{\mathcal{M}\mathrm{odel}_{\mathcal{P}}}(P, X),$$

and thus

$$\pi_0 \mathrm{ev}_P(X) \cong \pi_0 \mathrm{Map}_{\mathcal{M}\mathrm{odel}_{\mathcal{P}}}(P, X) \cong \mathrm{Map}_{\mathrm{h}\mathcal{M}\mathrm{odel}_{\mathcal{P}}}(P, X).$$

In other words, $\pi_0\mathrm{ev}_P$ is corepresented by P as a functor on the homotopy category of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$. We conclude with another application of the Yoneda lemma, yielding

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}\mathrm{odel}_{\mathcal{P}},\,\mathcal{S}\mathrm{et})}(\pi_0\mathrm{ev}_P,\pi_0\mathrm{ev}_{P'}) &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathrm{h}\mathcal{M}\mathrm{odel}_{\mathcal{P}},\,\mathcal{S}\mathrm{et})}(\pi_0\mathrm{ev}_P,\pi_0\mathrm{ev}_{P'})\\ &\cong \mathrm{Hom}_{\mathrm{h}\mathcal{M}\mathrm{odel}_{\mathcal{P}}}(P',P)\\ &\cong \mathrm{Hom}_{h\mathcal{P}}(P',P)\cong \pi_0\,\mathrm{Map}_{\mathcal{P}}(P',P). \qquad \square\end{aligned}$$

2.2. **Identifying categories of models.** Our main interests lie not in theories themselves, but in their models. We give here some ways of recognizing when a given category is in fact the category of models of a theory. These all ultimately follow from the characterization of Lemma 2.1.4.

2.2.1. **Lemma** ([Bal20, Proposition 3.3.3])**.** Let $\mathcal{P}$ be a discrete additive theory. Then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is an abelian category with enough projectives, and $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ its connective derived category. Conversely, if $\mathcal{A}$ is a cocomplete abelian category with enough projectives, and $\mathcal{P}\subset\mathcal{A}$ is a full subcategory consisting of projectives and closed under coproducts such that every object of $\mathcal{A}$ can be resolved by objects of $\mathcal{P}$, then $\mathcal{A}\simeq\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. □

2.2.2. **Example.**

(1) If R is an ordinary associative ring and $\mathcal{R}$ is the category of free left R-modules, then $\mathcal{R}$ is a theory and $\mathcal{M}\mathrm{odel}_{\mathcal{R}}\simeq \mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{cn}}$ is equivalent to the category of connective modules over the Eilenberg-MacLane spectrum HR.
(2) If G is a finite group and $\mathcal{B}_G$ is the additive completion of the Burnside category of finite G-sets, then $\mathcal{B}_G$ is a finitary theory and $\mathcal{M}\mathrm{odel}_{\mathcal{B}_G}^{\heartsuit}$ is equivalent to the category of G-Mackey functors.
(3) Let p be a prime and $\mathcal{P}$ be the category of p-completions of free abelian groups. Then $\mathcal{P}$ is an $\aleph_1$-bounded theory which is not associated to a finitary theory. We can identify $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ as the category of Ext-p-complete abelian groups and $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ as the category of connective $\mathbb{Z}$-modules which are p-complete in the sense studied, for instance, in [GM95] or [Lur18, Chapter 7]. ◁

2.2.3. **Remark.** Up to Morita equivalence, finitary additive theories are equivalent to *ringoids*, i.e. small $\mathcal{A}$b-enriched categories; if $\mathcal{C}$ is a finitary additive theory and $\mathcal{A}\subset\mathcal{C}$ is a subcategory generating $\mathcal{C}$ under finite sums and retracts, then $\mathcal{M}\mathrm{odel}_{\mathcal{C}}^{\heartsuit}$ is equivalent to the category of left $\mathcal{A}$-modules in the sense of [Mit72]. For example, if $\mathcal{C}$ is the theory of left modules over a ring R, then we may take $\mathcal{A}$ to be the full subcategory on the single object R. Then $\mathcal{A}$ is equivalent to R^{op} viewed as a one-object $\mathcal{A}$b-enriched category, and $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\heartsuit}\simeq\mathrm{Fun}^{\oplus}(R,\mathcal{A}\mathrm{b})$. ◁

Call a functor $U\colon\mathcal{D}\to\mathcal{C}$ *strongly monadic* if U preserves geometric realizations and is the forgetful functor of a monad adjunction. At least when $\mathcal{C}$ itself admits geometric realizations, it is equivalent to ask that $\mathcal{D}\simeq\mathcal{A}\mathrm{lg}_T$ for a monad T on $\mathcal{C}$ which preserves geometric realizations. The monads that we encounter will generally be of this form, as these are the monads which we can interpret in terms of theories. The following is a consequence of Lemma 2.1.4.

2.2.4. **Lemma.** If T is an accessible monad on $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ which preserves geometric realizations, and $T\mathcal{P}\subset\mathcal{A}\mathrm{lg}_T$ is the full subcategory spanned by the image of $\mathcal{P}$ under T, then $T\mathcal{P}$ is a theory and $\mathcal{M}\mathrm{odel}_{T\mathcal{P}}\simeq\mathcal{A}\mathrm{lg}_T$. Moreover, every accessible functor $U\colon \mathcal{D} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ which is preserves limits and geometric realizations arises this way. □

2.2.5. **Example.** Let R be a commutative ring, and $S\mathcal{R}$ the category of polynomial R-rings. This is the essential image of the theory of R-modules under the free functor $S\colon \mathcal{M}\mathrm{od}_R^\heartsuit \to \mathcal{CR}\mathrm{ing}_R^\heartsuit$, so we can identify $\mathcal{M}\mathrm{odel}_{S\mathcal{R}}^\heartsuit \simeq \mathcal{CR}\mathrm{ing}_R^\heartsuit$, and $\mathcal{M}\mathrm{odel}_{S\mathcal{R}} \simeq \mathcal{CR}\mathrm{ing}_R$ is the homotopy theory of simplicial R-rings. ◁

2.3. Bimodels and algebras.

Fix theories $\mathcal{P}$ and $\mathcal{P}'$.

2.3.1. **Lemma.** The following concepts are equivalent:

(1) Models of $\mathcal{P}$ in $\mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\mathrm{op}}$;
(2) Left adjoint, or colimit-preserving, functors $H\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$, or equivalently, coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$;
(3) Right adjoint, or limit-preserving accessible, or pointwise corepresentable, functors $H^\vee\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

Proof. These follow directly from either the universal property of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ given in Lemma 2.1.4 or the adjoint functor theorems for presentable categories [Lur17, Proposition 5.5.2.2, Corollary 5.5.2.9]. In addition, we can make the corepresentability condition of (3) explicit: if $H \dashv H^\vee$, then $H^\vee(M)(P) \simeq \mathrm{Map}_{\mathcal{P}}(H(P), M)$. □

We call the concept encoded in Lemma 2.3.1 that of a $\mathcal{P}$-$\mathcal{P}'$-*bimodel*; when $\mathcal{P} = \mathcal{P}'$, we will just call these $\mathcal{P}$-bimodels. We refer the reader to Wraith [Wra70] and Freyd [Fre66] for classical treatments of bimodels, as well as of algebras (defined below). We will consistently adhere to the convention that by bimodel we refer to the underlying left adjoint H, or on occasion the underlying coproduct-preserving functor H, and that $H^\vee$ is written for its right adjoint. For $P \in \mathcal{P}$ and $P' \in \mathcal{P}'$, we may write $H_{P,P'} = H(P)(P')$ and $H^\vee_{P,P'} = H^\vee(P)(P')$; note that these are covariant in the first variable and contravariant in the second.

2.3.2. **Example.** Let $\mathcal{P}$ be the theory of groups, R a commutative ring, and $\mathcal{P}'$ the theory of commutative R-rings. To a discrete commutative Hopf algebra H over R, we obtain the functor

$$\mathcal{CR}\mathrm{ing}_R(H, -)\colon \mathcal{CR}\mathrm{ing}_R \to \mathcal{G}\mathrm{rp}.$$

This is a $\mathcal{P}'$-$\mathcal{P}$-bimodel, and all discrete $\mathcal{P}'$-$\mathcal{P}$-bimodels arise this way. ◁

We will call a $\mathcal{P}'$-$\mathcal{P}$-bimodel H *projective* if $H(P)$ is projective for all $P \in \mathcal{P}$; equivalently, if H restricts to a functor $H\colon \mathcal{P} \to \mathcal{P}'$, at least up to idempotent completion of $\mathcal{P}'$.

If $\mathcal{P}''$ is another theory and H' is a $\mathcal{P}''$-$\mathcal{P}'$-bimodel, then we can compose to obtain the $\mathcal{P}''$-$\mathcal{P}$-bimodel $H' \circ H$. This has right adjoint $(H' \circ H)^\vee \simeq H^\vee \circ H'^\vee$.

2.3.3. **Remark.** We are mainly interested in structures built out of discrete bimodels. Even supposing that the theories in question are discrete, there are two possible ambiguities that arise:

(1) Coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^\heartsuit$ are not equivalent to coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ such that $H(P)$ is discrete for all $P \in \mathcal{P}$. Here, the issue is that discrete models need not be closed under coproducts in the category of all models.
(2) Even if $H\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ and $H'\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}''}$ deserve to be called discrete bimodels, the composite $H' \circ H$ need not.

The second ambiguity is not major, being no different than the ambiguity between a derived tensor product and a non-derived tensor product. The first ambiguity is more subtle, but will not be a major issue for us. When we are dealing with the purely discrete aspects of bimodels, we take as our discrete bimodels those which correspond to coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit}$. When we are dealing with homotopical aspects of bimodels, we take as our discrete bimodels those which correspond to coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ that land in $\mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit}$. In practice, in the latter case we will need to assume that H is projective, and both of these ambiguities disappear. ◁

2.3.4. **Example.** Let A and B be ordinary associative algebras with theories $\mathcal{A}$ and $\mathcal{B}$ of left modules. Then discrete $\mathcal{B}$-$\mathcal{A}$-bimodels are exactly B-A-bimodules. It is worth spelling out some aspects of this example explicitly to indicate the conventions that arise for working with bimodules. This example takes place in the 1-categorical setting, although similar observations hold in the derived setting (where general $\mathcal{B}$-$\mathcal{A}$-bimodels are equivalent to connective modules over the ring spectrum $B \otimes_{\mathbb{S}} A^{\mathrm{op}}$). To a discrete B-A-bimodule H, we have the functors

$$H\colon \mathrm{L}\mathcal{M}\mathrm{od}_A^{\heartsuit} \to \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}, \qquad H(M) = H \otimes_A M$$
$$H^\vee\colon \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit} \to \mathrm{L}\mathcal{M}\mathrm{od}_A^{\heartsuit}, \qquad H^\vee(M) = \mathrm{Hom}_B(H, M).$$

Here, B acts on $H \otimes_A M$ by

$$b \cdot (h \otimes m) = (bh) \otimes m,$$

and A acts on $\mathrm{Hom}_B(H, M)$ by

$$(a \cdot f)(h) = f(ha).$$

The bimodel H is projective precisely when the bimodule H is projective as a left B-module. The dual functor $H^\vee$ encodes more information than the ordinary dual bimodule $\mathrm{Hom}_B(H, B)$, and the latter can be recovered from the former by considering the restriction of $H^\vee$ to the category of finitely generated free B-modules. On the other hand, if H is projective, then we can equip $\mathrm{Hom}_B(H, B)$ with a natural topology as a right B-module such that $H^\vee(M) \simeq \mathrm{Hom}_B(H, B) \mathbin{\widehat{\otimes}}_B M$, where B acts on $\mathrm{Hom}_B(H, B)$ on the right by $(f \cdot b)(h) = f(bh)$.

If C is another ordinary associative algebra, $\mathcal{C}$ is its theory of left modules, and H' is a discrete $\mathcal{C}$-$\mathcal{B}$-bimodel, then under the correspondence between bimodules and bimodels we identify

$$H' \circ H \simeq H' \otimes_B H.$$

The isomorphism $(H' \circ H)^\vee \cong H^\vee \circ H'^\vee$ is given by the maps

$$\theta\colon\ \mathrm{Hom}_B(H, \mathrm{Hom}_C(H', M)) \to \mathrm{Hom}_C(H' \otimes_B H, M),$$
$$\theta(f)(h' \otimes h) = f(h)(h').$$

Taking $A = B = C$, this is essentially an enhancement of the duality pairing

$$\theta\colon\ \mathrm{Hom}_A(H, A) \otimes_A \mathrm{Hom}_A(H', A) \to \mathrm{Hom}_A(H' \otimes_A H, A),$$
$$\theta(f \otimes f')(h' \otimes h) = f'(h' f(h))$$

of bimodules. ◁

2.3.5. **Definition.** a $\mathcal{P}$-*algebra* consists of a $\mathcal{P}$-bimodel F together with the additional structure of a monad on F, or equivalently, of a comonad on $F^\vee$. An F-*model* is an algebra for the monad F, or equivalently, coalgebra for the comonad $F^\vee$. ◁

By Lemma 2.2.4, if F is a $\mathcal{P}$-algebra, then $\mathcal{A}\mathrm{lg}_F \simeq \mathcal{M}\mathrm{odel}_{F\mathcal{P}}$; we will generally abbreviate this to $\mathcal{M}\mathrm{odel}_F$. The forgetful functor $\mathcal{M}\mathrm{odel}_F \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is then *plethystic*: it is both monadic and comonadic. Conversely, every category plethystic over $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ arises from a $\mathcal{P}$-algebra. Heuristically, $\mathcal{P}$-algebras are those theories that can be obtained from $\mathcal{P}$ by adjoining suitable unary operations and relations.

2.3.6. **Example.** We have the following classes of examples of discrete algebras.

(1) Let R be an ordinary associative algebra and $\mathcal{R}$ the theory of left R-modules. Then a discrete $\mathcal{R}$-algebra is equivalent to a discrete R-bimodule A together with a map $A \otimes_R A \to A$ of R-bimodules making A into an associative algebra. We find that discrete $\mathcal{R}$-algebras are equivalent to ordinary associative algebras equipped with a map from R; we will just call these *ordinary R-algebras*. In particular, even when R is commutative, it need not be central in its algebras.

(2) Let R be a commutative ring, and $S\mathcal{R}$ the category of polynomial R-algebras as in Example 2.2.5. Then discrete $S\mathcal{R}$-algebras were studied by Tall-Wraith [TW70] under the name of biring triples, and more recently by Borger-Wieland [BW05] under the name of R-plethories. Our main examples of algebras over nonadditive theories are essentially of this form. We will study the relevant context in Section 4, allowing for bases more general than ordinary commutative rings.

(3) Let G be a finite group, $\mathcal{B}_G$ be as in Example 2.2.2, and $S\mathcal{B}_G$ the category of commutative green functors free on objects of $\mathcal{B}_G$, so that $\mathcal{M}\mathrm{odel}_{S\mathcal{B}_G}^{\heartsuit}$ is the category of commutative green functors. Let $\mathcal{T}\mathrm{amb}_G^{\heartsuit}$ be the category of discrete G-Tambara functors. Then $\mathcal{T}\mathrm{amb}_G^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{S\mathcal{B}_G}^{\heartsuit}$ preserves limits and colimits, and so realizes $\mathcal{T}\mathrm{amb}_G^{\heartsuit}$ as the category of models for an $S\mathcal{B}_G$-algebra. Thus G-tambara functors are $\mathcal{B}_G$-plethories in the sense that we will study in Section 4. See [BH19] for more on this context, and in particular for a study of the relevant right adjoints. ◁

2.3.7. **Example.** Plethystic functors also arise in more homotopical contexts.

(1) Let R be a commutative ring, $\mathcal{R}$ the theory of R-modules, and $\mathbb{P}\mathcal{R}$ the category of $\mathbb{E}_\infty$ algebras over R which are free on a discrete free R-module, i.e. of the form $\mathbb{P}R^{\oplus I}$ where $\mathbb{P}\colon \mathcal{M}\mathrm{od}_R \to \mathcal{C}\mathrm{Alg}_R$ is the free $\mathbb{E}_\infty$ algebra functor. We can then identify $\mathcal{M}\mathrm{odel}_{\mathbb{P}R} \simeq \mathcal{C}\mathrm{Alg}_R^{\mathrm{cn}}$ as the category of connective $\mathbb{E}_\infty$ algebras over R. The homotopy category $\mathrm{h}(\mathbb{P}\mathcal{R}) \simeq S\mathcal{R}$ is the category of polynomial R-rings, as in Example 2.2.5, and restriction along the truncation $\mathbb{P}\mathcal{R} \to S\mathcal{R}$ gives a forgetful functor $U\colon \mathcal{C}\mathcal{R}\mathrm{ing}_R \to \mathcal{C}\mathrm{Alg}_R^{\mathrm{cn}}$. The functor U automatically preserves limits and geometric realizations, and it preserves coproducts as these are given by $\otimes_R$ in either category. Thus U is plethystic, and realizes $S\mathcal{R}$ as a $\mathbb{P}\mathcal{R}$-algebra. We refer the reader to [Lur18, Chapter 25] for a more detailed discussion of the relation between $\mathcal{C}\mathcal{R}\mathrm{ing}_R$ and $\mathcal{C}\mathrm{Alg}_R^{\mathrm{cn}}$. We will briefly revisit this in Example 5.6.3.

(2) Let G be a finite group and $\mathcal{O}' \subset \mathcal{O}$ be G-coefficient systems in the sense of [BH15]. Then the forgetful functor $\mathcal{A}\mathrm{lg}_{\mathcal{O}} \to \mathcal{A}\mathrm{lg}_{\mathcal{O}'}$ is plethystic, where $\mathcal{A}\mathrm{lg}_{\mathcal{O}}$ is the category of algebras over the N_∞-operad associated to $\mathcal{O}$. ◁

2.4. **Monoidal products.** Suppose that $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ has been equipped with some form of monoidal product $\otimes$ which preserves colimits in each variable. If moreover the monoidal product preserves the full subcategory $\mathcal{P} \subset \mathrm{Model}_{\mathcal{P}}$, then it is determined by its restriction to $\mathcal{P}$, from which it can be recovered by Day convolution. In this case, one might call $\mathcal{P}$ a *monoidal theory*. The requirement that the monoidal product preserves colimits in each variable is equivalent to asking that its restriction to $\mathcal{P}$ preserves coproducts in each variable. If $X', X'' \in \mathrm{Model}_{\mathcal{P}}$, then $X' \otimes X''$ can be identified as the left Kan extension of the functor $(P', P'') \mapsto X'(P') \times X''(P'')$ along the product $\otimes\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$.

We note in particular the following: if $\mathcal{P}$ is a monoidal theory, then for any $P', P'' \in \mathcal{P}$, there is a natural pairing

$$\mathrm{ev}_{P'} \times \mathrm{ev}_{P''} \to \mathrm{ev}_{P' \otimes P''}$$

satisfying all the coherences one would expect coming from $\otimes$. This is an advantage of working with algebraic theories without specified sorts, as the presence of automorphisms of objects of $\mathcal{P}$ has not been hidden.

2.5. **Compositions.** This subsection takes place entirely in the 1-categorical setting. If k is an ordinary commutative ring, and A and B are ordinary k-algebras in which k is central, then the tensor product $A \otimes_k B$ naturally carries the structure of a k-algebra with product

$$m \otimes m \circ A \otimes \tau \otimes B\colon A \otimes_k B \otimes_k A \otimes_k B \cong A \otimes_k A \otimes_k B \otimes_k B \to A \otimes_k B.$$

This is not true for general k-algebras, or for k noncommutative: we have relied on centrality in order to use the switch map $A \otimes_k B \simeq B \otimes_k A$. Axiomatizing this leads to the notion of a *distributive law*, discovered by Beck [Bec69]. We summarize the relevant definitions here.

2.5.1. **Definition.** Let $\mathcal{C}$ be a 1-category and F and T be monads on $\mathcal{C}$.

(1) A *composition* of T with F is a monad structure on TF such that
   (a) Both $T \to TF \leftarrow F$ are maps of monads;
   (b) The composite

$$m_{TF} \circ T\eta_F \eta_T F\colon TF \to TFTF \to TF$$

   is the identity.

(2) A *distributive law* of F across T is a natural transformation $c\colon FT \to TF$ such that the diagrams

$$\begin{array}{ccc} & T & \\ {\scriptstyle \eta_F T}\swarrow & & \searrow{\scriptstyle T\eta_F} \\ FT & \xrightarrow{c} & TF \end{array} \qquad \begin{array}{ccc} & F & \\ {\scriptstyle F\eta_T}\swarrow & & \searrow{\scriptstyle \eta_T F} \\ FT & \xrightarrow{c} & TF \end{array}$$

$$\begin{array}{ccccc} FTT & \xrightarrow{cT} & TFT & \xrightarrow{Tc} & TTF \\ \downarrow{\scriptstyle Fm_T} & & & & \downarrow{\scriptstyle m_T F} \\ FT & & \xrightarrow{c} & & TF \end{array} \qquad \begin{array}{ccccc} FFT & \xrightarrow{Fc} & FTF & \xrightarrow{cF} & TFF \\ \downarrow{\scriptstyle m_F T} & & & & \downarrow{\scriptstyle Tm_F} \\ FT & & \xrightarrow{c} & & TF \end{array}$$

   commute.

(3) A *distributive square* is a diagram of categories

$$\begin{array}{ccc} \mathcal{D}' & \xrightarrow{V'} & \mathcal{D} \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathcal{C}' & \xrightarrow{V} & \mathcal{C} \end{array}$$

such that
  (a) The pairs $T' \dashv U'$ and $T \dashv U$ are adjoint, and the diagram commutes with T and T' omitted;
  (b) The mate $TV \to V'T'$ is an isomorphism.
(4) A *monadic distributive square* is a distributive square as above such that moreover
  (a) There are further left adjoints $F' \dashv V'$ and $F \dashv V$;
  (b) Each of these adjunctions are monadic adjunctions. ◁

We extend the definitions of distributive squares and monadic distributive squares to allow for the categories involved to not necessarily be 1-categories. Heuristically, a monadic distributive square is a square

$$\begin{array}{ccc} \mathcal{D}' & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathcal{D} \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathcal{C}' & \underset{F}{\overset{V}{\rightleftarrows}} & \mathcal{C} \end{array}$$

of monadic adjunctions such that "$T = T'$". In particular, it is dependent on the orientation that the square has been drawn.

2.5.2. **Lemma.** Let $\mathcal{C}$ be a 1-category and F and T be monads on $\mathcal{C}$. The following concepts are equivalent:
(1) Compositions of T with F;
(2) Distributive laws of F across T;
(3) Monadic distributive squares of the form

$$\begin{array}{ccc} \mathcal{D}' & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathcal{A}\mathrm{lg}_T \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathcal{A}\mathrm{lg}_F & \underset{F}{\overset{V}{\rightleftarrows}} & \mathcal{C} \end{array}$$

Proof. The equivalence of these notions is proved in [Bec69]; we just recall here the method of translation between the three. Given a composition of T with F, we obtain a distributive law of F across T by the composite

$$c = m_{TF} \circ \eta_T FT\eta_F : FT \to TFTF \to TF.$$

Conversely, given a distributive law $c: FT \to TF$, we can construct a composition of T with F via

$$\eta_{TF} = \eta_T\eta_F : I \to TF,$$
$$m_{TF} = m_T m_F \circ TcF : TFTF \to TTFF \to TF.$$

Given a composition of T with F, we obtain a diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{TF} & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathcal{A}\mathrm{lg}_T \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathcal{A}\mathrm{lg}_F & \underset{F}{\overset{V}{\rightleftarrows}} & \mathcal{C} \end{array}$$

of monadic functors. We claim this is distributive, i.e. $TV \cong V'T'$ where $T' \dashv U'$. Indeed, it is sufficient to verify this on free F-algebras and after forgetting to $\mathcal{C}$, where this is just the identification $UTVF \cong UV'T'F$. Conversely, given a monadic distributive square as in the statement of the lemma, we obtain a distributive law of F across T by the composite

$$FT = VFUT \to VU'F'T \cong UTVF = TF,$$

the arrow being obtained from the mate $FU \to U'F'$. □

2.5.3. **Example.** Let k be an ordinary associative algebra and A and B be ordinary k-algebras. Then

(1) Compositions of A with B are algebra structures on $A \otimes_k B$ such that
  (a) $(a' \otimes 1) \cdot (a'' \otimes 1) = a'a'' \otimes 1$ and $(1 \otimes b') \cdot (1 \otimes b'') = 1 \otimes b'b''$;
  (b) $(a \otimes 1) \cdot (1 \otimes b) = a \otimes b$.
(2) Distributive laws of B across A are maps $c\colon B \otimes_k A \to A \otimes_k B$ of k-bimodules such that
  (a) $c(1 \otimes a) = a \otimes 1$ and $c(b \otimes 1) = 1 \otimes b$;
  (b) If we write $c(b \otimes a) = \sum a_{(i)} \otimes b_{(i)}$ for a placeholder symbol i, then $\sum a'_{(1)} \otimes a''_{(2)} \otimes (b_{(1)})_{(2)} = \sum (a' \otimes a'')_{(3)} \otimes b_{(3)}$ and $\sum (a_{(1)})_{(2)} \otimes b'_{(2)} \otimes b''_{(1)} = \sum a_{(3)} \otimes (b'b'')_{(3)}$.
(3) If

$$\begin{array}{ccc} C & \longleftarrow & A \\ \uparrow & & \uparrow \\ B & \longleftarrow & k \end{array}$$

is a commutative diagram of algebra maps, then we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{L}\mathcal{M}\mathrm{od}_C^\heartsuit & \longrightarrow & \mathrm{L}\mathcal{M}\mathrm{od}_A^\heartsuit \\ \downarrow & & \downarrow \\ \mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit & \longrightarrow & \mathrm{L}\mathcal{M}\mathrm{od}_k^\heartsuit \end{array}$$

of monadic functors. The mate is given by maps $A \otimes_k M \to C \otimes_B M$ defined for a left B-module M, and is a natural isomorphism when it evaluates on B to an isomorphism $A \otimes_k B \cong C$.

Even when each of k, A, and B are commutative, these notions do not collapse. For example, if $\mathbb{H}$ is the ring of quaternions, then

$$\begin{array}{ccc} \mathbb{H} & \xleftarrow{f} & \mathbb{C} \\ {\scriptstyle g}\uparrow & & \uparrow \\ \mathbb{C} & \longleftarrow & \mathbb{R} \end{array}$$

satisfies the conditions of (3), where $f(i) = j$ and $g(i) = k$. The distributive law is the map $c\colon \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ given by $c(i \otimes i) = -i \otimes i$, and otherwise by the standard symmetry. ◁

For the most part, we will encounter distributive laws in the form of monadic distributive squares. The theory of distributive laws then provides a method for understanding the categories involved; this is an instance of a general philosophy we follow that it is often easier to construct the category of algebras over a monad than it is to construct the monad itself. The following are some typical examples of this sort of situation.

2.5.4. **Example.** Let k be an ordinary commutative ring, B an ordinary k-bialgebra, and A a monoid in the monoidal category $(\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}, \otimes_k)$, with resulting category $\mathrm{L}\mathcal{M}\mathrm{od}_A(\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit})$ of modules. Then

$$\begin{array}{ccc} \mathrm{L}\mathcal{M}\mathrm{od}_A(\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}) & \longrightarrow & \mathrm{L}\mathcal{M}\mathrm{od}_A^{\heartsuit} \\ \downarrow & & \downarrow \\ \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit} & \longrightarrow & \mathcal{M}\mathrm{od}_k^{\heartsuit} \end{array}$$

is a monadic distributive square, and thus $\mathrm{L}\mathcal{M}\mathrm{od}_A(\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{A\otimes_k B}^{\heartsuit}$ for some k-algebra structure on $A \otimes_k B$. Using Lemma 2.5.2, we find that this k-algebra structure is the "semi-tensor product" [MP65], given by

$$(a' \otimes b') \cdot (a'' \otimes b'') = \sum a'(b'_{(1)} \cdot a'') \otimes b'_{(2)} b'',$$

where we have written $\Delta(b) = \sum b_{(1)} \otimes b_{(2)}$. Equivalently, the distributive law is given by $c(b \otimes a) = \sum (b_{(1)} \cdot a) \otimes b_{(2)}$. ◁

2.5.5. **Example.** Let $\mathcal{P}$ be a discrete theory, F a discrete $\mathcal{P}$-algebra, and $B \in \mathcal{M}\mathrm{odel}_F^{\heartsuit}$. Then

$$\begin{array}{ccc} B/\mathcal{M}\mathrm{odel}_F^{\heartsuit} & \longrightarrow & B/\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \\ \downarrow & & \downarrow \\ \mathcal{M}\mathrm{odel}_F^{\heartsuit} & \longrightarrow & \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \end{array}$$

is a monadic distributive square. The distributive law is just the map

$$F(B \coprod -) \to F(B) \coprod F(-) \to B \coprod F(-)$$

given by the F-model structure of B and fact that F preserves coproducts. ◁

2.6. **Left-derived functors.** Fix two discrete theories $\mathcal{P}$ and $\mathcal{P}'$.

2.6.1. **Definition.** Let $\overline{F} \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ be an arbitrary functor and let f denote the composite

$$f \colon \mathcal{P}' \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}.$$

The *total left-derived functor* of $\overline{F}$ is the functor

$$F = f_! \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$$

obtained from f by left Kan extension. ◁

Total left-derived functors can be computed in the usual way, by taking projective resolutions. Their identification with a left Kan extension is a situation where the use of infinitary theories simplifies the story.

2.6.2. **Example.** Let $\overline{F} \colon \mathcal{M}\mathrm{od}_{\mathbb{Z}_p}^{\heartsuit} \to \mathcal{M}\mathrm{od}_{\mathbb{Z}_p}^{\heartsuit}$ denote the functor of p-adic completion. Then $\overline{F}$ is neither left nor right exact in general. Write $\mathcal{A}_p$ for the total left-derived functor of $\overline{F}$. Then

(1) $\mathcal{A}_p$ gives the correct notion of p-completion for the category $\mathcal{M}\mathrm{od}_{\mathbb{Z}_p}^{\mathrm{cn}}$ (see Example 2.2.2(3));
(2) $\pi_0\mathcal{A}_p$ is the functor of Ext-p-completion and $\pi_1\mathcal{A}_p$ is the functor of Hom-p-completion in the sense of [BK72a, Section VI.2.1].

As $\overline{F}$ restricts to the identity on the category of finitely generated $\mathbb{Z}_p$-modules, this is a purely infinitary construction. $\triangleleft$

2.7. **Unbounded derived categories.** If $\mathcal{P}$ is an additive theory, then we will write

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit} = \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}, \qquad \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} = \mathcal{M}\mathrm{odel}_{\mathcal{P}}, \qquad \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} = \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p}).$$

We then have fully faithful embeddings $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ is the stabilization of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}}$; this is just as in [Lur18, Proposition C.1.5.7, Remark C.1.5.9].

In particular, for $X, Y \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, we have a mapping spectrum $\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y)$ with

$$\Omega^{\infty-n}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y) \simeq \mathrm{Map}_{\mathcal{P}}(X, \Sigma^n Y).$$

We will write

$$\mathrm{Ext}_{\mathcal{P}}^n(X, Y) = \pi_{-n}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y).$$

When $\mathcal{P}$, X and Y are discrete, these are the usual Ext groups defined for the Abelian category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$.

2.8. **Quillen cohomology.** Fix a discrete theory $\mathcal{P}$. We can identify $\mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}) \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{A}\mathrm{b})$, and this category is seen to be strongly monadic over $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. Write the left adjoint as $\overline{D}$ and its left-derived functor as D, so that $\mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}) \simeq \mathcal{M}\mathrm{odel}_{D\mathcal{P}}^{\heartsuit}$. Here, $D\mathcal{P}$ is an additive theory, so the notation of Subsection 2.7 applies.

2.8.1. **Definition.** For $A \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ and $M \in \mathcal{M}\mathrm{odel}_{D\mathcal{P}}$, define

$$\mathcal{H}_{\mathcal{P}}(A; M) = \mathcal{E}\mathrm{xt}_{D\mathcal{P}}(DA, M), \qquad H_{\mathcal{P}}^n(A; M) = \pi_{-n}\mathcal{H}_{\mathcal{P}}(A; M) = \mathrm{Ext}_{D\mathcal{P}}^n(DA, M).$$

Equivalently,

$$\mathcal{H}_{\mathcal{P}}^n(A; M) = \Omega^{\infty-n}\mathcal{H}_{\mathcal{P}}(A; M) = \mathrm{Map}_{\mathcal{P}}(A, B^n M), \qquad H_{\mathcal{P}}^n(A; M) = \pi_0\mathcal{H}_{\mathcal{P}}^n(A; M).$$

These are the *Quillen cohomology groups of A with coefficients in M*. $\triangleleft$

Often the theory at hand is instead of the form $\mathcal{P}/B$ for some theory $\mathcal{P}$ and $B \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. Here, $\mathcal{M}\mathrm{odel}_{\mathcal{P}/B} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}/B$, and we will use these choices of notation interchangeably. Write $\overline{D}_B$ for the relevant functor of abelianization, and D_B for its total left-derived functor. We will call $B \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ *smooth* if $D_B B$ is discrete and projective. When $\mathcal{P}$ is the theory of R-rings for some commutative ring R, this is not really the standard notion of smoothness, as we impose no finiteness conditions. We have the following standard fact.

2.8.2. **Lemma.** Given $f\colon B \to C$, we can identify $D_C B \simeq f_! D_B B$, where $f_!$ is the derived functor of the left adjoint to the pullback functor $f^*\colon \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/C) \to \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/B)$. In particular, if B is smooth, then $D_C B$ is discrete and projective.

Proof. Observe that the diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/C) & \xrightarrow{f^*} & \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/B) \\ \downarrow & & \downarrow \\ \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/C & \xrightarrow{f^*} & \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/B \end{array}$$

commutes, and continues to commute upon passage to derived categories. The lemma follows upon taking left adjoints. $\square$

2.9. **Cohomology over an algebra.** Fix a discrete theory $\mathcal{P}$ and $\mathcal{P}$-algebra F. We would like to be able to compute the cohomology of F-models. We begin by noting the following.

2.9.1. **Lemma.** Suppose $U\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is strongly monadic. Then the induced map $V\colon \mathcal{M}\mathrm{odel}_{D\mathcal{P}}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{D\mathcal{P}'}^{\heartsuit}$ is strongly monadic, and is plethystic whenever U is.

Proof. It is easily seen that V is strongly monadic. As V is additive, to be plethystic it is sufficient for V to preserve filtered colimits, for which it is sufficient that U preserves filtered colimits, which holds if U is plethystic. $\square$

What makes F special is the existence of the limit-preserving comonad $F^\vee$. Heuristically, this is because $F^\vee$ preserves algebraic structure. In particular, for identifying abelian group objects, we have the following.

2.9.2. **Proposition.** Let $\mathcal{C}$ be a 1-category with finite products, and G a comonad on $\mathcal{C}$ which preserves these. Then

(1) $\mathrm{Co}\mathcal{A}\mathrm{lg}_G \to \mathcal{C}$ creates finite products;
(2) The resulting functor $\mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G) \to \mathcal{A}\mathrm{b}(\mathcal{C})$ is comonadic;
(3) The diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G) & \longrightarrow & \mathcal{A}\mathrm{b}(\mathcal{C}) \\ \downarrow{\scriptstyle U'} & & \downarrow{\scriptstyle U} \\ \mathrm{Co}\mathcal{A}\mathrm{lg}_G & \longrightarrow & \mathcal{C} \end{array}$$

of forgetful functors is Cartesian whenever U is fully faithful;
(4) The natural transformation $U' \circ G' \to G \circ U$ fitting in the diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G) & \xleftarrow{G'} & \mathcal{A}\mathrm{b}(\mathcal{C}) \\ \downarrow{\scriptstyle U'} & & \downarrow{\scriptstyle U} \\ \mathrm{Co}\mathcal{A}\mathrm{lg}_G & \xleftarrow{\quad G \quad} & \mathcal{C} \end{array}$$

is an isomorphism;
(5) If U admits a left adjoint D, then D lifts to a left adjoint $D'\colon \mathrm{Co}\mathcal{A}\mathrm{lg}_G \to \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G)$ making the diagram in (3) distributive.

Proof. Claim (1) is clear. For (2) and (4), as G preserves finite products, it lifts to a comonad G' on $\mathcal{A}\mathrm{b}(\mathcal{C})$. A G'-coalgebra consists of some $A \in \mathcal{A}\mathrm{b}(\mathcal{C})$ together with a coaction $A \to GA$ which is a map of abelian group objects, i.e. such that the diagram

$$\begin{array}{ccc} A \times A & \longrightarrow & GA \times GA \\ \downarrow & & \downarrow \\ A & \longrightarrow & GA \end{array}$$

commutes. Looking at it a different way, this is the same as asking for A to be an abelian group object in $\mathrm{Co}\mathcal{A}\mathrm{lg}_G$, so $\mathrm{Co}\mathcal{A}\mathrm{lg}_{G'} \simeq \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G)$. For (3), if $\mathcal{A}\mathrm{b}(\mathcal{C}) \to \mathcal{C}$ is fully faithful, then the above diagram automatically commutes for any choice of multiplication $A \times A \to A$ and coaction $A \to GA$, and the claim quickly follows. For (5), given the left adjoint D, we can identify a left adjoint

$D'\colon \mathrm{Co}\mathcal{A}\mathrm{lg}_G \to \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G)$ as sending a G-coalgebra $A \to GA$ to the G'-coalgebra with coaction given by the unique dashed arrow filling in

$$\begin{array}{ccc} A & \longrightarrow & GA \\ \downarrow & & \downarrow \\ DA & \dashrightarrow & GDA \end{array}$$

as a map of abelian group objects in $\mathcal{C}$. ☐

By Proposition 2.9.2, if $B \in \mathcal{M}\mathrm{odel}_F^\heartsuit$ and we are considering B as an model of $\mathcal{P}$ equipped with extra structure, then the notation $\overline{D}(B)$ is unambiguous, for the abelianization of B is the same when computed in $\mathcal{M}\mathrm{odel}_F^\heartsuit$ or $\mathcal{M}\mathrm{odel}_\mathcal{P}^\heartsuit$. However, the notation $D(B)$ is still ambiguous, and in particular we cannot safely write $DF\mathcal{P}$ for the theory of abelian group objects in $\mathcal{M}\mathrm{odel}_F^\heartsuit$. For now, write $D'F\mathcal{P}$ for this theory, and D' for the associated functor of derived abelianization of F-models. Call the algebra F *smooth* if $F(P)$ is smooth for all $P \in \mathcal{P}$.

2.9.3. **Proposition.** If F is smooth, then the diagram

$$\begin{array}{ccc} \mathrm{L}\mathcal{M}\mathrm{od}_{D'F}^{\mathrm{cn}} & \xrightarrow{V'} & \mathrm{L}\mathcal{M}\mathrm{od}_{D\mathcal{P}}^{\mathrm{cn}} \\ \downarrow & & \downarrow \\ \mathcal{M}\mathrm{odel}_F & \xrightarrow{V} & \mathcal{M}\mathrm{odel}_\mathcal{P} \end{array}$$

is distributive. In particular, $D'F\mathcal{P} = DF\mathcal{P}$.

Proof. As both DV and $V'D'$ preserve geometric realizations, it is sufficient to verify that the map $DV \to V'D'$ is an equivalence when restricted to $F\mathcal{P}$. Here, it follows from Proposition 2.9.2 and smoothness. ☐

We will postpone giving examples until Subsection 4.4. We end by noting the following, which illustrates the purpose of smooth algebras.

2.9.4. **Proposition.** Fix a smooth algebra F. For $B \in \mathcal{M}\mathrm{odel}_F$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{DF}$, there is a conditionally convergent spectral sequence

$$E_1^{p,q} = \mathrm{Ext}_{DF}^{q-p}(\pi_p D(B), M) \Rightarrow H_F^q(B; M), \qquad d_r^{p,q}\colon E_r^{p,q} \to E_r^{p-r,q+1}.$$

In particular, if B is smooth as an object of $\mathcal{M}\mathrm{odel}_\mathcal{P}^\heartsuit$, then

$$H_F^q(B; M) \simeq \mathrm{Ext}_{DF}^q(\overline{D}(B); M).$$

Proof. By definition, $\mathcal{H}_F(B; M) = \mathcal{E}\mathrm{xt}_{DF}(D(B), M)$. Smoothness of F ensures that the notation $D(B)$ is unambiguous. The spectral sequence is then associated to the filtration of $\mathcal{E}\mathrm{xt}_{DF}(D(B), M)$ by the Whitehead tower of $D(B)$. ☐

3. Additive Koszul algebras

This section is concerned with additive theories, and in particular in the notion of a Koszul algebra over an additive theory. We begin by covering some general topics relevant to homology and cohomology in the additive setting. In particular, in Subsection 3.2 we review the bar resolution in detail, in part to be explicit about our conventions. In Subsection 3.3, we give the relevant notion of the homology and cohomology of an augmented algebra. In Subsection 3.4, we generalize to our setting some standard facts about how homology and cohomology behave with respect to tensor products.

With the above in place, the remaining subsections are dedicated to the Koszul story. As the purpose is to obtain explicit complexes, these are carried out entirely in the 1-categorical setting. In Subsection 3.5, we give the definition of a Koszul algebra over an additive theory, and show how Koszulity formally implies the existence of Koszul resolutions and Koszul complexes. In Subsection 3.6, we begin to make this more explicit by introducing quadratic algebras and using them to describe the cohomology of a homogeneous Koszul algebra. This serves to describe Koszul complexes as graded objects, and in Subsection 3.7 we describe their differentials. In Subsection 3.8, we show how the standard PBW criterion for detecting Koszulity directly translates to our context.

3.1. **Coalgebras.** Fix additive theories $\mathcal{P}$ and $\mathcal{P}'$. To emphasize that we are working in the additive setting, we will generally refer to $\mathcal{P}'$-$\mathcal{P}$-bimodels as $\mathcal{P}'$-$\mathcal{P}$-*bimodules*; see Example 2.3.4. In addition, we can extend a $\mathcal{P}'$-$\mathcal{P}$-bimodule $H\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}'}^{\mathrm{cn}}$ to a colimit-preserving functor $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}'}$, and will do so with no change in notation. It happens on occasion that a bimodule H has the property that its right adjoint $H^\vee$ preserves colimits; in this case, $H^\vee$ is also a bimodule, with further right adjoint $H^{\vee\vee}$.

3.1.1. **Example.** Let A and B be ordinary associative algebras and H a discrete B-A-bimodule. As mentioned in Example 2.3.4, we can recover the ordinary dual bimodule $H_c^\vee = \mathrm{L}\mathcal{M}\mathrm{od}_B(H, B)$ of H by taking the left Kan extension of the restriction of $H^\vee$ to the category of finitely generated free B-modules. There is a comparison map

$$H_c^\vee \to H^\vee, \quad \theta\colon \mathrm{Hom}_B(H,B) \otimes_B M \to \mathrm{Hom}_B(H,M), \quad \theta(f\otimes m)(h) = f(h)m,$$

which is an isomorphism when H is finitely presented and projective as a left B-module. ◁

It is not necessary for $H^\vee$ to preserve colimits to talk about monad structures on $H^\vee$. These are equivalently comonad structures on H, and thus deserve to be called $\mathcal{P}$-*coalgebras*. In this case, $H^\vee$-modules are the analogues of H-*contramodules* in the sense of [EM65, Section III.5], but we will not use this name.

3.2. **Cobar complexes.** We review in this subsection bar resolutions and cobar complexes in some detail so as to make clear our conventions. Fix for the moment an arbitrary category $\mathcal{M}$ and monad T on $\mathcal{M}$. For $M \in \mathcal{A}\mathrm{lg}_T$, we may form the bar construction $B(T,T,M)$. This is the simplicial object augmented over M with

$$B_n(T,T,M) = T^{1+n}M, \qquad d_i = T^i m T^{n-i}\colon T^{1+1+n}M \to T^{1+n}M, \quad 0 \le i \le n{+}1.$$

Here, d_{n+1} is to be understood as given by the T-module structure on M. We then have the following standard fact.

3.2.1. **Lemma.** The simplicial object $B(T,T,M)$ is a resolution of M, in the sense that the augmentation extends to an equivalence

$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} B_n(T,T,M) \simeq M$$

in $\mathcal{A}\mathrm{lg}_T$. □

We now restrict ourselves to the case where $\mathcal{M} = \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ for a discrete additive theory $\mathcal{P}$ and $T = F$ is a discrete $\mathcal{P}$-algebra. In particular, everything that follows takes place in the 1-categorical setting. Certainly assuming that F is a colimit-preserving monad on a category of the form $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is much stronger than we actually use, as most of what follows is just an explicit comparison of formulas; this assumption is made for notational convenience.

In this additive setting, we can obtain from $B(F,F,M)$ the unreduced bar resolution $C^{\mathrm{un}}(F,F,M)$, which is a chain complex of F-modules of the form

$$C_n^{\mathrm{un}}(F,F,M) = F^{1+n}M, \qquad d = \sum_{0\leq i\leq n+1}(-1)^i d_i\colon F^{1+1+n}M \to F^{1+n}M$$

as well as the reduced bar resolution $C(F,F,M)$, which is the quotient chain complex of $C^{\mathrm{un}}(F,F,M)$ with

$$C_n(F,F,M) = FF^{+n}M, \qquad F^+ = \mathrm{Coker}(I\to F).$$

Given $M,M'\in \mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$, we can form

$$\begin{aligned} B_F(M,M') &= \mathrm{Hom}_{F\mathcal{P}}(B(F,F,M),M'),\\ B_F^n(M,M') &= \mathrm{Hom}_{F\mathcal{P}}(F^{1+n}M,M') \cong \mathrm{Hom}_{\mathcal{P}}(F^nM,M'). \end{aligned}$$

This is a cosimplicial abelian group modeling $\mathcal{E}\mathrm{xt}(M,M')$ provided that $B(F,F,M)$ consists of projective F-modules. We obtain from this the unreduced cobar complex $C_F^{\mathrm{un}}(M,M')$ and reduced cobar complex $C_F(M,M')\subset C_F^{\mathrm{un}}(M,M')$. Explicitly, the differential on $C_F^{\mathrm{un}}(M,M')$ is given by the maps

$$\begin{aligned} \delta\colon \mathrm{Hom}_{\mathcal{P}}(F^nM,M') &\to \mathrm{Hom}_{\mathcal{P}}(F^{1+n}M,M'),\\ \delta &= \delta_0 + \sum_{1\leq i\leq n}(-1)^i\delta_i + (-1)^{n+1}\delta_{n+1},\\ \delta_0(f) &= m\circ Ff\colon F^{1+n}M\to FM'\to M',\\ \delta_i(f) &= f\circ F^{i-1}mF^{n-i}\colon F^{1+n}M\to F^nM\to M',\\ \delta_{n+1}(f) &= f\circ F^n m\colon F^{1+n}M\to F^nM\to M'. \end{aligned}$$

We now come to the following.

3.2.2. **Lemma.** Fix $M,M',M''\in \mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$. Define

$$\wr\colon C_F^{\mathrm{un},n}(M,M')\otimes C_F^{\mathrm{un},n'}(M',M'')\to C_F^{\mathrm{un},n'+n}(M,M'')$$

as follows: given $f\colon F^nM\to M'$ and $f'\colon F^{n'}M'\to M''$, we set

$$(-1)^{nn'}f\wr f' = f'\circ F^{n'}f\colon F^{n'+n}M\to F^{n'}M'\to M''.$$

Then $\wr$ has the following properties:

1. If $f\in C_F^n(M,M')$ and $f'\in C_F^{n'}(M',M'')$, then $f\wr f'\in C_F^{n'+n}(M,M'')$.
2. We have $\delta(f\wr f') = \delta(f)\wr f' + (-1)^n f\wr\delta(f')$, and thus $\wr$ passes to pairings
$$\begin{aligned} C_F^{\mathrm{un}}(M,M')\otimes C_F^{\mathrm{un}}(M',M'') &\to C_F^{\mathrm{un}}(M,M''),\\ C_F(M,M')\otimes C_F(M',M'') &\to C_F(M,M'') \end{aligned}$$
of cochain complexes.
3. Suppose that $C(F,F,M)$ and $C(F,F,M')$ are projective resolutions of M and M'. Then the induced pairing
$$\wr\colon \mathrm{Ext}_F^n(M,M')\otimes \mathrm{Ext}_F^{n'}(M',M'')\to \mathrm{Ext}_F^{n'+n}(M,M'')$$

is the graded opposite of the standard Yoneda composition:

$$f \wr f' = (-1)^{nn'} f' \circ f.$$

Here, to be explicit, we can take the Yoneda composition as defined in [Mac67, Section III.5, Theorem III.6.4] as the standard.

Proof. Claim (1) is clear. For (2), fix $f\colon F^n M \to M'$ and $f'\colon F^{n'} M' \to M''$. The main point is that

$$\delta_{n+1}(f') \circ F^{n'+1} f = f' \circ F^{n'} \delta_0(f);$$

this allows us to compute

$$\begin{aligned}
\delta(f' \circ F^{n'} f) &= \sum_{i=0}^{n'+1+n} (-1)^i \delta_i(f' \circ F^{n'} f) \\
&= \sum_{i=0}^{n'} \delta_i(f' \circ F^{n'} f) + (-1)^{n'} \sum_{i=n'+1}^{n'+1+n} (-1)^i \delta_i(f' \circ F^{n'} f) \\
&= \sum_{i=0}^{n'} \delta_i(f') \circ F^{n'} f + (-1)^{n'} \sum_{i=1}^{1+n} (-1)^i f' \circ F^{n'} \delta_i(f) \\
&= \sum_{i=0}^{n'+1} \delta_i(f') \circ F^{n'} f + (-1)^{n'} \sum_{i=0}^{1+n} (-1)^i f' \circ F^{n'} \delta_i(f) \\
&= \delta(f') \circ F^{n'} f + (-1)^{n'} f' \circ F^{n'} \delta(f),
\end{aligned}$$

which yields

$$\begin{aligned}
\delta(f \wr f') &= (-1)^{nn'} \delta(f' \circ F^{n'} f) \\
&= (-1)^{nn'} \delta(f') \circ F^n f + (-1)^{nn'+n'} f' \circ F^{n'} \delta(f) \\
&= (-1)^{nn'+n(n'+1)} f \wr \delta(f') + (-1)^{nn'+n'+(n+1)n'} \delta(f) \wr f' \\
&= \delta(f) \wr f' + (-1)^n f \wr \delta(f')
\end{aligned}$$

as desired. For (3), we first introduce a bit of local notation. If $\mathcal{A}$ is an additive category, we can define the shift functors

$$\mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A}), \qquad C \mapsto C[p]$$

for $p \in \mathbb{Z}$ on objects by

$$C[p]_n = C_{n-p}, \qquad d_n^{C[p]} = (-1)^p d_{n-p}^C,$$

and on morphisms by

$$f[p]_n = f_{n-p}.$$

Now fix a map $f\colon F^n M \to M'$ in $\mathrm{LMod}_{\mathcal{P}}^{\heartsuit}$. We can lift this to a map

$$f^+\colon C^{\mathrm{un}}(F, F, M) \to C^{\mathrm{un}}(F, F, M')[n]$$

of graded objects by

$$f^+_{<n} = 0, \qquad f^+_{k+n} = (-1)^{nk} F^{1+k} f.$$

The map f^+ then satisfies

$$f^+_{k+n} \circ d - d \circ f^+_{1+k+n} = (-1)^{(n+1)k} F^{1+k} \delta(f);$$

thus f^+ is a chain map whenever f is a cocycle. If $f'\colon F^{n'}M' \to M''$ is another map in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, then $f' \circ f^+_{n'+n} = f \wr f'$. We conclude by observing that if $C(F,F,M)$ and $C(F,F,M')$ are projective resolutions of M and M', and f and f' are cocycles, then $f' \circ f^+_{n'+n}$ is a cocycle representing the standard Yoneda composition twisted by $(-1)^{nn'}$. □

When $\mathcal{P}$ is the theory of $\mathbb{Z}$-graded modules over an ordinary $\mathbb{Z}$-graded algebra, it is standard practice to insert additional signs in various places when developing the homological algebra of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, where these additional signs are dependent on internal degrees of elements involved. The degrees of elements of a graded module cannot be defined in a Morita-invariant way, so these signs cannot be incorporated at the present level of generality. In practice, one can simply modify the constructions of this section to be compatible with whatever conventions are most convenient for a given category. We give an example indicating the effect of this in a standard case.

3.2.3. **Example.** Let k be a commutative ring and A be an ordinary projective $\mathbb{Z}$-graded augmented k-algebra in which k is central. Let H be the cohomology algebra of A defined with conventions from [Pri70]. Write s^q for q-fold shift, so that $(s^qM)_n = M_{n-q}$ for a $\mathbb{Z}$-graded object M. Then H is a bigraded object with

$$H^{p,q} = \mathrm{Ext}^p(k, s^q k).$$

Write $\smile$ for the product on H and $\circ$ for the Yoneda composition on Ext. Then for $x \in H^{p,q}$ and $x' \in H^{p',q'}$ we have

$$x' \smile x = (-1)^{q(q'-p')} s^q x' \circ x$$

Closer to our conventions is the bigraded opposite algebra H^{op}. This is the algebra with $(H^{\mathrm{op}})^p_q = H^{p,-q}$ and product given for $x \in (H^{\mathrm{op}})^p_q$ and $x' \in (H^{\mathrm{op}})^{p'}_{q'}$ by

$$x \smile^{\mathrm{op}} x' = (-1)^{(q-p)(q'-p')} x' \smile x.$$

If we identify $(H^{\mathrm{op}})^p_q = \mathrm{Ext}^p(s^q k, k)$, then this product satisfies

$$x \smile^{\mathrm{op}} x' = (-1)^{pq'} s^{q'} x \wr x'.$$

◁

3.3. **Cohomology of augmented algebras.** Fix an additive theory $\mathcal{P}$ and $\mathcal{P}$-algebra F. Suppose that F is *augmented*; that is, that we have chosen a map $\epsilon\colon F \to I$ of algebras, where I is the identity functor. Restriction along the augmentation gives us a functor

$$\epsilon^*\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_F, \qquad \epsilon^*(M) = \overline{M}.$$

As ϵ^* preserves limits and colimits, it is part of an adjoint triple $\epsilon_! \dashv \epsilon^* \dashv \epsilon_*$, giving thus a (non-discrete) $\mathcal{P}$-coalgebra $\epsilon_!\epsilon^*$ with right adjoint monad $\epsilon_*\epsilon^*$. We can identify $\epsilon_!\epsilon^*$ as arising from a bar construction: for $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, we have

$$\epsilon_!\epsilon^* M \simeq \epsilon_! \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} B_n(F,F,\overline{M}) \simeq \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} B_n(I,F,\overline{M}),$$

where $B(I,F,-) = \epsilon_! B(F,F,-)$ by definition. Likewise, $\epsilon_*\epsilon^*$ can be identified as

$$(\epsilon_*\epsilon^* M)(P) = \mathrm{Map}_{\mathcal{P}}(P, \epsilon_*\epsilon^* M) = \mathrm{Map}_F(\overline{P}, \overline{M}),$$

which can be computed via a cobar construction.

We will make minimal use of these homotopical objects directly, but will make use of their algebraic shadows. Specialize then to the case where $\mathcal{P}$ is discrete and F

is projective. Under these assumptions, we can view $C(I, F, -)$ as a chain complex of $\mathcal{P}$-bimodules modeling $\epsilon_!\epsilon^*$. Define

$$H_n(F) = \pi_n \epsilon_! \epsilon^* \colon \mathcal{P} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \qquad H_n(F)_{P,P'} = H_n C(I, F, \overline{P})_{P'};$$
$$H^n(F) = \pi_{-n} \epsilon_* \epsilon^* \colon \mathcal{P} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \qquad H^n(F)_{P,P'} = \mathrm{Ext}^n_F(\overline{P}', \overline{P}).$$

These extend to endofunctors of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, and if each $H_n(F)_P$ is projective, then $H_n(F)$ is a bimodule with $H_n(F)^\vee = H^n(F)$. The products of Lemma 3.2.2 give us the pairings

$$\wr \colon H^{n'}(F)_{P',P''} \otimes H^n(F)_{P,P'} \to H^{n'+n}(F)_{P,P''},$$

and these make $H^*(F)$ into a graded monad. When each $H_n(F)$ is projective, the identification $H_n(F)^\vee \cong H^n(F)$ implies that $H_*(F)$ can be considered as a graded comonad, although we will not make use of this.

3.3.1. **Example.** Let k be an ordinary algebra and A an ordinary augmented projective k-algebra. Then treating A as an algebra for the theory of left k-modules, we have

$$H_*(A)_{k,k} = \mathrm{Tor}^A_*(k, k), \qquad H^*(A)_{k,k} = \mathrm{Ext}^*_A(k, k),$$

and $H^*(A)_{k,k}$ is itself an augmented k-algebra. On the other hand, suppose instead that A is an ordinary $\mathbb{Z}$-graded augmented projective k-algebra. Then still we have

$$H^*(A)_{e_p,e_q} = \mathrm{Ext}^*_A(e_q, e_p),$$

where e_a denotes a copy of k in degree a. In particular, we can extract from this a left k-module

$$H^*(A)_{e_0} = \mathrm{Ext}^*_A(e_*, e_0).$$

Heuristically, this is the ordinary cohomology algebra of A. However, note the following subtlety: to make $H^*(A)_{e_0}$ into an ordinary algebra requires the additional structure of the isomorphisms $\mathrm{Ext}^*_A(e_{a+b}, e_b) \cong \mathrm{Ext}^*_A(e_a, e_0)$. ◁

3.4. **Homology and compositions.** We will need some facts about how homology interacts with composing algebras in the sense of Subsection 2.5.

3.4.1. **Lemma.** Suppose given a square

$$\begin{array}{ccc} \mathcal{D}' & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathcal{D} \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathcal{C}' & \underset{F}{\overset{V}{\rightleftarrows}} & \mathcal{C} \end{array}$$

of adjunctions between 1-categories. Then there is a natural simplicial map

$$T'B(F, F, C') \to B(F', F', T'C')$$

defined for $C' \in \mathcal{C}'$, which is an isomorphism if the square is distributive.

Proof. In degree n, this simplicial map is given by the map

$$T'F(VF)^n VC' \to F'(V'F')^n V'T'C'$$

obtained by repeated application of the mate

$$T'FV \simeq F'TV \to F'V'T'.$$

That this is an isomorphism when the square is distributive is clear. □

3.4.2. **Lemma.** Let $\mathcal{C}$ be a 1-category, and T and F be monads on $\mathcal{C}$ together with a distributive law $c\colon FT \to TF$ allowing us to form the composition monad TF, and let

$$\begin{array}{ccc} \mathcal{D}' & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathcal{D} \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathcal{C}' & \underset{F}{\overset{V}{\rightleftarrows}} & \mathcal{C} \end{array}$$

be the associated monadic distributive square. Suppose that F is equipped with an augmentation ϵ such that $T\epsilon \circ c = \epsilon T$. Then ϵ lifts to an augmentation on F', and moreover

$$TB(I, F, \epsilon_F^* C) \cong B(I, F', \epsilon_{F'}^* TC)$$

for $C \in \mathcal{C}$.

Proof. The given assumption on the augmentation of F implies that $T\epsilon\colon TF \to T$ is a map of monads, and this gives rise to the augmentation on F'. The given isomorphism of bar constructions follows readily from Lemma 3.4.1 and the isomorphisms $T'\epsilon_F^* \simeq \epsilon_{F'}^* T$ and $\epsilon_{F'!}T' \simeq T\epsilon_{F!}$. □

3.4.3. **Proposition.** Let $\mathcal{P}$ be a discrete additive theory. Let T and F be discrete projective augmented $\mathcal{P}$-algebras. Suppose that we have chosen a distributive law $c\colon FT \to TF$ such that $\epsilon_T\epsilon_F \circ c = \epsilon_F\epsilon_T$. Then the composite monad $T \circ F$ is augmented, and if each $H_n(T)$ is projective, then $H_n(T \circ F) \cong \bigoplus_{i+j=n} H_i(F) \circ H_j(T)$.

Proof. It is easily verified that $\epsilon_T\epsilon_F$ makes TF into an augmented monad. Now write the monadic distributive square associated to the composite TF as

$$\begin{array}{ccc} \mathcal{D}' & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathcal{D} \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathcal{C}' & \underset{F}{\overset{V}{\rightleftarrows}} & \mathcal{C} \end{array} .$$

Then there is a natural map

$$\epsilon_{TF!}\epsilon_{TF}^* \simeq \epsilon_{F!}\epsilon_{T'!}\epsilon_{F'}^*\epsilon_T^* \to \epsilon_{F!}\epsilon_F^*\epsilon_{T!}\epsilon_T^*.$$

which we claim is an isomorphism; here, these functors are to be interpreted in the derived sense. It is sufficient to verify that this map induces

$$V\epsilon_{T'!}\epsilon_{F'}^* TP \simeq V\epsilon_F^*\epsilon_{T!}TP$$

for $P \in \mathcal{P}$. The right hand side is simply P, and we compute the left hand side to be

$$V\epsilon_{T'!}\epsilon_{F'*}TP \simeq V\epsilon_{T'!}T'\epsilon_F^* P \simeq V\epsilon_F^* P \simeq P.$$

When each $H_n(T)$ is projective, we can split $\epsilon_{T!}\epsilon_T^* P \simeq \bigoplus_{n\geq 0} \Sigma^n H_n(T)_P$ for $P \in \mathcal{P}$. Thus in this case we have

$$\begin{aligned} H_n(T \circ F)_P &= \pi_n \epsilon_{TF!}\epsilon_{TF}^* P \cong \pi_n \epsilon_{F!}\epsilon_F^*\epsilon_{T!}\epsilon_T^* P \\ &\cong \pi_n \bigoplus_{k\geq 0} \Sigma^k \epsilon_{F!}\epsilon_F^* H_k(T)_P \cong \bigoplus_{i+j=n} (H_i(F) \circ H_j(T))_P \end{aligned}$$

as claimed. □

3.5. **Koszul resolutions.** Our goal for the rest of this section is to generalize Priddy's theory of Koszul algebras and Koszul resolutions [Pri70] to the setting of algebras over additive theories. For us, the purpose of this theory is to give concrete tools for certain homological computations. As such, for the rest of this section, everything will take place in the 1-categorical setting. Fix an additive theory $\mathcal{P}$ and $\mathcal{P}$-algebra F.

3.5.1. **Definition.**

(1) F is a *filtered algebra* if we have chosen subfunctors $F_{\leq n} \subset F$ such that
  (a) $I = F_{\leq 0} \subset F$ is the unit;
  (b) The product on F restricts to maps $F_{\leq n} \circ F_{\leq m} \to F_{\leq n+m}$.

(2) F is a *graded algebra* if we have chosen a decomposition $F = \oplus_{n \geq 0} F[n]$ of bimodules such that
  (a) $I = F[0] \subset F[n]$ is the unit;
  (b) The product on F restricts to maps $F[n] \circ F[m] \to F[n+m]$.

(3) The *associated graded algebra* of a filtered algebra F is the algebra $\operatorname{gr} F$ given by
$$\operatorname{gr} F = \bigoplus_{m \geq 0} F[m], \qquad F[m] = \operatorname{Coker}(F_{\leq m-1} \to F_{\leq m}).$$

(4) F is a *projective filtered algebra* if both F and $\operatorname{gr} F$ are projective. ◁

Suppose now that F is a projective filtered algebra. For $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, there is a filtration $C^{\mathrm{un}}(F,F,M) = \operatorname{colim}_{m \to \infty} C^{\mathrm{un}}(F,F,M)[\leq m]$ obtained by declaring
$$C^{\mathrm{un}}_n(F,F,M)[\leq m] = \operatorname{Im}\left(\bigoplus_{m_1 + \cdots + m_n = m} F F_{\leq m_1} \cdots F_{\leq m_n} \to C^{\mathrm{un}}_n(F,F,M) \right).$$
This induces a filtration on $C(F,F,M)$. In particular, we obtain the associated graded complex
$$\operatorname{gr} C = \bigoplus_{m \geq 0} C(F,F,M)[m],$$
where by definition $C(F,F,M)[m]$ fits into a short exact sequence
$$0 \to C(F,F,M)[\leq m-1] \to C(F,F,M)[\leq m] \to C(F,F,M)[m] \to 0.$$

3.5.2. **Lemma.** Fix notation as above. Then

(1) $\operatorname{gr} C(F,F,M) = FC(I, \operatorname{gr} F, \overline{M})$;
(2) Explicitly,
$$C_n(F,F,M)[m] = \bigoplus_{\substack{m_1 + \cdots + m_n = m \\ m_1, \ldots, m_n \geq 1}} F F[m_1] \cdots F[m_n] M.$$
(3) In particular, $C_n(F,F,M)[\leq m] = 0$ for $n > m$, and $C_n(F,F,M)[\leq n] = C_n(F,F,M)[n] = F[1]^{\circ n}(M)$.

Proof. Immediate from the definitions. □

If F is augmented, then the above filtration on $C(F,F,M)$ induces a filtration on $C(I,F,M)$. When F is graded, this filtration is split on $C(I,F,\overline{P})$, yielding gradings $H_*(F)_P = H_* \bigoplus_{n \geq 0} C(I,F,\overline{P})[n]$ and $H^*(F) = \prod_{n \geq 0} H^*(F)[n]$.

3.5.3. **Definition.** Let F be a $\mathcal{P}$-algebra. We say that F is a *homogeneous Koszul* $\mathcal{P}$*-algebra* if

(1) F is projective;
(2) F has been equipped with a grading;
(3) $H_n(F)[m] = 0$ for $n \neq m$.

We say that F is a *Koszul* $\mathcal{P}$*-algebra* if

(1) F has been equipped with a projective filtration;
(2) $\operatorname{gr} F$ is a homogeneous Koszul $\mathcal{P}$-algebra. ◁

Say F is still a projective filtered algebra and M is a $\mathcal{P}$-projective F-module. The filtration $C(F,F,M) \simeq \operatorname{colim}_{m\to\infty} C(F,F,M)[\leq m]$ gives rise to a spectral sequence

$$E^1_{p,q} = FH_q(\operatorname{gr} F)[p](M) \Rightarrow H_q C(F,F,M), \qquad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

3.5.4. **Lemma.** Suppose either of the following is satisfied:

(1) Filtered colimits in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ are exact;
(2) The connectivity of $C(F,F,M)[m]$ goes to ∞ as m goes to ∞.

Then the above spectral sequence converges.

Proof. The first case is classical, so consider the second. Abbreviate $C = C(F,F,M)$. It is sufficient to show that for each $n \geq 0$, we can identify $H_n C \cong \operatorname{colim}_{m\to\infty} H_n C[\leq m]$, and that this colimit stabilizes at a finite step. As F and M are projective, $C \cong \operatorname{colim}_{m\to\infty} C[\leq m]$ is a homotopy colimit, and so $\tau_{\leq n} C \simeq \operatorname{colim}_{m\to\infty} \tau_{\leq n} C[\leq m]$ in the category of n-truncated objects for each n. This colimit stabilizes at a finite step by induction, proving the claim. □

Define the chain complex $K(F,F,M)$ by $K_p(F,F,M) = E^1_{p,p}$, with differential obtained from the d^1 differential of the above spectral sequence. Then we can identify $K(F,F,M)$ as a chain complex of the form

$$FH_0(\operatorname{gr} F)(M) \leftarrow FH_1(\operatorname{gr} F)(M) \leftarrow FH_2(\operatorname{gr} F)(M) \leftarrow \cdots,$$

or more memorably,

$$K(F,F,M) = FH_*(\operatorname{gr} F)(M),$$

and this sits as a subcomplex of the bar resolution.

3.5.5. **Theorem.** Let F be a Koszul $\mathcal{P}$-algebra, and let M be an F-module which is projective over $\mathcal{P}$. Then we can split $C(F,F,M) \simeq K(F,F,M) \oplus C'$, where C' is a contractible chain complex. In particular,

$$M \leftarrow K(F,F,M)$$

is a projective resolution of M.

Proof. As F is Koszul, the spectral sequence $FH_q(\operatorname{gr} F)[p](M) \Rightarrow H_q C(F,F,M)$ converges by Lemma 3.5.4, and collapses into a projective resolution $K(F,F,M) \to M$. The cokernel of the inclusion $K(F,F,M) \to C(F,F,M)$ is an acyclic complex of projectives, allowing for the indicated splitting. □

Fix a filtered $\mathcal{P}$-algebra F and F-modules M and M'. We then have the *Koszul complex*

$$K_F(M,M') = \mathrm{L}\mathcal{M}\mathrm{od}_F(K(F,F,M),M'),$$

sometimes called a co-Koszul complex. This is a quotient of the cobar complex $C_F(M, M')$, and models $\mathcal{E}\mathrm{xt}_F(M, M')$ when F is Koszul and M is projective over $\mathcal{P}$.

3.5.6. **Example.**

(1) The motivating example of a Koszul algebra is the Steenrod algebra [Pri70]. For simplicity, take $\mathcal{A}$ to be the mod 2 Steenrod algebra. Then $\mathcal{A}$ is Koszul with respect to the length filtration on $\mathcal{A}$. The resulting complex $K_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ can be enriched to a complex of graded vector spaces, and is known as the *lambda algebra*.

(2) More generally, one has Steenrod algebras for other forms of mod p cohomology, such as in motivic and equivariant stable homotopy, and examples suggest that one can expect these to be Koszul as well. These examples require a fairly general notion of Koszul algebra: they are not generally augmented, their coefficients rings do not generally live in their center, and they need not be ordinary graded algebras at all, such as in the equivariant setting where one has additional Mackey functor structure. The motivic case is studied in detail in [BCQ20]. ◁

We end by noting the following stability properties of Koszulity.

3.5.7. **Lemma.** Fix a monadic distributive square

$$\begin{array}{ccc} \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{F'} & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathcal{P}'} \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{F} & \underset{F}{\overset{V}{\rightleftarrows}} & \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathcal{P}} \end{array},$$

where $\mathcal{P}$ and $\mathcal{P}'$ are additive theories and F and F' are projective algebras over them. In particular, we have a distributive law $c\colon FT \to TF$, and $\mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{F'} \simeq \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{T\circ F}$ where $T \circ F$ is composition of T and F as a monad on $\mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathcal{P}}$.

(1) Suppose that F is a projective filtered algebra, with filtration compatible with the distributive law. If F is Koszul over $\mathcal{P}$, then F' is Koszul over $\mathcal{P}'$.

(2) Suppose that F and T are projective filtered algebras, with filtrations compatible with the distributive law. Filter $T \circ F$ by $(T \circ F)_{\leq n} = \mathrm{Im}(\bigoplus_{i+j=n} T_{\leq i} \circ F_{\leq j} \to T \circ F)$, so that $\mathrm{gr}(T \circ F) \cong \mathrm{gr}\, T \circ \mathrm{gr}\, F$. Then $T \circ F$ is Koszul.

Proof. Given Theorem 3.5.5, these follow from Lemma 3.4.2 and Proposition 3.4.3. □

3.6. **Quadratic algebras.** As in the previous subsection, everything here takes place in the 1-categorical setting. Our goal in this subsection is to describe the structure and cohomology of homogeneous Koszul algebras. Fix an additive theory $\mathcal{P}$, and let H be a $\mathcal{P}$-bimodule. We can then form the free algebra on H as a type of tensor algebra:

$$TH = \bigoplus_{n\geq 0} T_n H, \qquad T_n H = H^{\circ n},$$

with standard multiplication. The right adjoint to this is

$$\widehat{T}H^{\vee} = (TH)^{\vee} = \prod_{n\geq 0} (H^{\circ n})^{\vee} = \prod_{n\geq 0} H^{\vee\circ n}.$$

This also carries an obvious multiplication, but we want the slightly less obvious multiplication, obtained by twisting the identifications

$$\widehat{T}_i(H^\vee)\circ\widehat{T}_j(H^\vee)=H^{\vee\circ i}\circ H^{\vee\circ j}\cong H^{\vee\circ(i+j)}\cong\widehat{T}_{i+j}(H^\vee)$$

by $(-1)^{ij}$; we write this multiplication as $\wr$. In fact, these two choices are isomorphic by $x\mapsto(-1)^{\frac{|x|(|x|-1)}{2}}x$, so the distinction is essentially invisible in the quadratic algebras we will consider; the purpose of this choice is to ensure compatibility with the cobar complex.

Given a subfunctor $R\subset H\circ H$, we may form the quadratic algebra

$$T(H,R)=T(H)/R=\bigoplus_{n\geq 0}T_n(H,R),$$

$$T_n(H,R)=H^{\circ n}/\left(\sum_{i+j=n}H^{\circ i-1}\circ R\circ H^{\circ j-1}\right).$$

Likewise, given some $R'\subset H^\vee$, we may form the monad

$$\widehat{T}(H^\vee,R')=\prod_{n\geq 0}\widehat{T}_n(H^\vee,R'),$$

$$\widehat{T}_n(H^\vee,R')=H^{\vee\circ n}/\left(\sum_{i+j=n}H^{\vee\circ i-1}\circ R'\circ H^{\vee\circ j-1}\right),$$

although this is no longer guaranteed to preserve limits.

3.6.1. **Lemma.** The following are equivalent:

(1) $T(H,R)$ is projective;
(2) H is projective, and for all $P\in\mathcal{P}$, the map $R_P\to(H\circ H)_P$ admits a splitting.

Proof. If $T(H,R)$ is projective, so too is H, being a summand of the former. As

$$0\to R\to H\circ H\to T_2(H,R)\to 0$$

is exact, and $T_2(H,R)$ is projective, it is levelwise split. Conversely, if H is projective and $R\subset H\circ H$ is levelwise split, then each $H^{\circ i-1}\circ R\circ H^{\circ j-1}\subset H^{\circ n}$ is levelwise split, and thus $T_n(H,R)$ is projective. ☐

Call a pair (H,R) satisfying the conditions of Lemma 3.6.1 a *quadratic datum*. Fix now a quadratic datum (H,R). By projectivity, the short exact sequence

$$0\to R\to H\circ H\to T_2(H,R)\to 0$$

dualizes to a short exact sequence

$$0\leftarrow R^\vee\leftarrow H^\vee\circ H^\vee\leftarrow R^\perp\leftarrow 0;$$

the pair $(H^\vee,R^\perp)$ could be called the *dual quadratic datum* to (H,R), though with this name dual quadratic data are not themselves quadratic data.

3.6.2. **Lemma.** Fix a quadratic datum (H,R). Then the monad $\widehat{T}(H^\vee,R^\perp)$ preserves limits, and is thus the right adjoint of a coalgebra.

Proof. As in the proof of Lemma 3.6.1, the hypotheses imply that $\widehat{T}(H^\vee,R^\perp)$ is levelwise a summand of $\widehat{T}(H^\vee)$, and this proves the lemma. ☐

3.6.3. **Remark.** We can identify the coalgebra left adjoint to $\widehat{T}(H^\vee, R^\perp)$ explicitly as the coquadratic coalgebra

$$\bigoplus_{n\geq 0}\left(\bigcap_{i+j=n} H^{\circ i-1}\circ R\circ H^{\circ j-1}\right)\subset T(H),$$

but we will not have a chance to make use of this. ◁

3.6.4. **Theorem.**

(1) Let (H, R) be a quadratic datum. Then we can identify $H^1(T(H,R))[1]\cong H^\vee$, and the inclusion $H^\vee\subset H^*(T(H,R))$ extends multiplicatively to an isomorphism $\widehat{T}(H^\vee, R^\perp)\cong\prod_{n\geq 0} H^n(T(H,R))[n]$.

(2) Let $F=\bigoplus_{n\geq 0} F[n]$ be a homogeneous Koszul algebra, and let $R=\operatorname{Ker}(F[1]\circ F[1]\to F[2])$. Then $F\cong T(F[1],R)$ and $H^*(F)\cong\widehat{T}(F[1]^\vee, R^\perp)$.

Proof. Consider first (1), and abbreviate $C=C(I,T(H,R),-)$. By Lemma 3.5.2, we can identify

$$H_nC[n]=\operatorname{Ker}\left(H^{\circ n}\to\bigoplus_{i+j=n} H^{\circ i-1}\circ T_2(M,R)\circ H^{\circ j-1}\right).$$

This is left adjoint to $\widehat{T}_n(H^\vee, R^\perp)$, proving (1) additively, and multiplicative compatibility follow readily from the definition of the product on $\widehat{T}(H^\vee, R^\perp)$ and construction given in Lemma 3.2.2. Consider next (2). We must show only $F\simeq T(F[1],R)$, for the remaining claims follow from Koszulity and (1). By construction, the inclusion $F[1]\to F$ extends to a map $T(F[1],R)\to F$ of algebras, and we must only verify that this is an isomorphism. By Koszulity, the sequences

$$\bigoplus_{\substack{i+j=n\\ i,j>0}} F[i]\circ F[j]\to F[n]\to 0$$

$$\bigoplus_{\substack{i+j+k=m\\ i,j,k>0}} F[i]\circ F[j]\circ F[k]\to\bigoplus_{\substack{r+s=m\\ r,s>0}} F[r]\circ F[s]\to F[m]$$

are exact for $n>1$ and $m>2$. The first implies that each $F[1]^{\circ n}\to F[n]$ is surjective, and thus so too is $T(F[1],R)\to F$. The second implies that any relation seen in the multiplication $F[r]\circ F[s]\to F[m]$ with $r+s=m$ and either $r>1$ or $s>1$ is already generated in relations among $F[i]$ with $i<r$ or $i<s$. Thus $T_n(F[1],R)\to F[n]$ is an injection, so an isomorphism. We learn that $T(F[1],R)\to F$ is a direct sum of isomorphisms, and so is an isomorphism. □

3.6.5. **Example.** Let $R=W[[a]]$ where $W=W(\kappa)$ is the ring of 2-typical Witt vectors on a perfect field κ of characteristic 2. Define an R-bimodule $\Gamma[1]$ as follows. As a left R-module, $\Gamma[1]\simeq R\{Q_0,Q_1,Q_2\}$. The right R-module structure is determined by

$$\begin{aligned}
Q_i\lambda &= \lambda^\sigma Q_i,\quad \lambda\in W;\\
Q_0 a &= a^2Q_0-2aQ_1+6Q_2;\\
Q_1 a &= 3Q_0+aQ_2;\\
Q_2 a &= -aQ_0+3Q_1.
\end{aligned}$$

Here, $(-)^\sigma$ is the Frobenius automorphism of W. Define $R \subset \Gamma[1] \circ \Gamma[1]$ to be spanned by the relations

$$Q_1 Q_0 = 2Q_2Q_1 - 2Q_0Q_2,$$
$$Q_2Q_0 = Q_0Q_1 + aQ_0Q_2 - 2Q_1Q_2.$$

Now define $\Gamma = T(\Gamma[1], R)$. This is the algebra of additive power operations for a certain Morava E-theory at height $h = 2$ and $p = 2$ computed by Rezk [Rez08], and is Koszul. So by Theorem 3.6.4 and Koszulity, we have $H^*(\Gamma) = \widehat{T}(\Gamma[1]^\vee, R^\perp)$; we can identify this explicitly as follows. As $\Gamma[1]$ is finitely generated and free as a left R-module, $\Gamma[1]^\vee$ is a bimodule, and is finitely generated and free as a right R-module on a basis dual to that of $\Gamma[1]$; write this as $\Gamma[1]^\vee = \{Q^0, Q^1, Q^2\}R$. The left R-module structure is then given by

$$\lambda Q^i = Q^i \lambda^\sigma, \quad \lambda \in W;$$
$$aQ^0 = Q^0a^2 + 3Q^1 - Q^2a;$$
$$aQ^1 = -2Q^0a + 3Q^2;$$
$$aQ^2 = 6Q^0 + Q^1a.$$

The space $R^\perp \subset \Gamma[1]^\vee \circ \Gamma[1]^\vee$ is spanned by

$$Q^0Q^0, \qquad Q^1Q^1, \qquad Q^2Q^2, \qquad Q^1Q^0 + Q^0Q^2,$$
$$Q^1Q^2 + 2Q^0Q^1, \qquad Q^2Q^1 - 2Q^0Q^2,$$
$$Q^2Q^0 - 2Q^0Q^1 + Q^0Q^2a.$$

We find that

$$H^*(\Gamma) \cong \{1, Q^0, Q^1, Q^2, Q^0Q^1, Q^0Q^2\}R,$$

with multiplicative structure determined by the preceding. ◁

3.7. **Koszul complexes.** As in the previous subsection, everything here takes place in the 1-categorical setting. Our goal in this section is to describe the structure of the Koszul complexes computing Ext over a Koszul algebra. We begin with the homogeneous case.

Fix an additive theory $\mathcal{P}$. Fix a quadratic datum (H, R), and write $F = T(H, R)$. Fix $M, N \in \mathrm{LMod}_F^\heartsuit$ with M projective over $\mathcal{P}$. Recall from Subsection 3.5 the Koszul complex $K_F(M, N)$, which is a quotient of the cobar complex $C_F(M, N)$ satisfying $K_F^n(M, N) = H^n(F)[n](N)(M)$. This is described by Theorem 3.6.4, and it remains only to determine the differential. In some cases, this can be determined by analyzing the surjective map $C_F(M, N) \to K_F(M, N)$ directly, but we can also proceed as follows. Observe first that the composition pairings of Lemma 3.2.2 pass to pairings

$$\wr \colon K_F(M, N) \otimes K_F(N, L) \to K_F(M, L).$$

Observe next that as F is generated by $F[1] = H$, the F-module structure on M is determined by a map $H(M) \to M$. This is nothing more than an element of $H^\vee(M)(M) = K_F^1(M, M)$; write Q^M for this element twisted by -1, and likewise define $Q^N \in H^\vee(N)(N) = K_F^1(N, N)$.

3.7.1. **Theorem.** The differential on $K_F(M, N)$ is given by

$$\delta \colon K_F^n(M, N) \to K_F^{n+1}(M, N), \qquad \delta(f) = Q^M \wr f - (-1)^n f \wr Q^N$$

Proof. Recall $C^n_F(M,N) = \mathrm{Hom}_{\mathcal{P}}(F^{+\circ n}M, N)$, where $F \cong I \oplus F^+$, and recall the differential on $C^n_F(M,N)$ from Subsection 3.2, which is of the form

$$\delta = \delta_0 + \sum_{1\le i\le n} (-1)^i \delta_i + (-1)^{n+1}\delta_{n+1} \colon C^n_F(M,N) \to C^{n+1}_F(M,N).$$

By construction, the inner sum $\sum_{1\le i\le n}(-1)^i\delta_i$ is killed by the quotient mapping $C_F(M,N) \to K_F(M,N)$. On the other hand,

$$\delta_0(f) = m \circ Tf = -(-1)^n f \wr Q^N;$$
$$\delta_{n+1}(f) = f \circ T^n m = -(-1)^n Q^M \wr f.$$

Combining these proves the theorem. □

Now fix a possibly non-homogeneous Koszul algebra F, together with $M, N \in \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_F$. As before, we have the Koszul complex $K_F(M,N) = H^*(\mathrm{gr}\,F)(N)(M)$, still a quotient of $C_F(M,N)$, and would like to identify its differential.

3.7.2. **Lemma.** The inclusion $F_{\le 1} \to F$ induces an epimorphism $T(F_{\le 1}) \to F$.

Proof. We induct to show $T_{\le n}F_{\le 1} \to F_{\le n}$ is an epimorphism. The lemma then follows as epimorphisms are closed under unions. When $n \le 1$, there is nothing to show. In the inductive step, by the ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\le n-1}F_{\le 1} & \longrightarrow & T_{\le n}F_{\le 1} & \longrightarrow & F^{\circ n}_{\le 1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_{\le n-1} & \longrightarrow & F_{\le n} & \longrightarrow & F[n] & \longrightarrow & 0 \end{array}$$

of short exact sequences, it is sufficient for $F^{\circ n}_{\le 1} \to F[n]$ to be a surjection. This factors as the composite $F^{\circ n}_{\le 1} \to F[1]^{\circ n} \to F[n]$, where the first map is the quotient mapping, and the second is a surjection by Koszulity as in Theorem 3.6.4. □

Now write $qR = \mathrm{Ker}(F_{\le 1} \circ F_{\le 1} \to F_{\le 2})$. Then $(F_{\le 1}, qR)$ is a quadratic datum, and we have a surjection $T(F_{\le 1}, qR) \to F$. We may view M and N as $T(F_{\le 1}, qR)$-modules by restriction along this map.

3.7.3. **Theorem.** With notation as above, $T(F_{\le 1}, qR)$ is a homogeneous Koszul algebra, and the surjection $T(F_{\le 1}, qR) \to F$ gives rise to a short exact sequence

$$0 \to H^*(\mathrm{gr}\,F) \to H^*(T(F_{\le 1}, qR)) \to H^{*-1}(\mathrm{gr}\,F) \to 0,$$

which is split when F is augmented. In particular, $K_F(M,N) \subset K_{T(F_{\le 1},qR)}(M,N)$ is an identifiable subcomplex with differential on the target described by Theorem 3.6.4.

Proof. Define

$$\Gamma_{\le k}[m] = \mathrm{Im}\left(\bigoplus_{\substack{\epsilon_1+\cdots+\epsilon_m = k \\ \epsilon_1,\ldots,\epsilon_m \in \{0,1\}}} F_{\le \epsilon_1} \circ \cdots \circ F_{\le \epsilon_m} \to T_m(F_{\le 1}, qR) \right).$$

Then $T(F_{\le 1}, qR) = \mathrm{colim}_{k\to\infty} \Gamma_{\le k}$ is a multiplicative filtration, though not necessarily satisfying $\Gamma_{\le 0} = I$. We can identify $\mathrm{gr}\,\Gamma \simeq T(I) \circ \mathrm{gr}\,F$ as a "polynomial algebra" on $\mathrm{gr}\,F$; in particular, by Lemma 3.5.7, $\mathrm{gr}\,\Gamma$ is a homogeneous Koszul algebra with $H^n(\mathrm{gr}\,\Gamma) = H^n(\mathrm{gr}\,F) \oplus H^{n-1}(\mathrm{gr}\,F)$. The spectral sequence associated to this filtration then collapses into a two-step filtration of $H^*(T(F_{\leq 1}, qR))$ both proving Koszulity and providing the indicated short exact sequences. □

3.7.4. **Example** ([Bru88]). Let $\mathcal{A}$ be the mod 2 Steenrod algebra, so that $\mathcal{A}$ equipped with its length filtration is Koszul. As $\mathcal{A}$ is an ordinary $\mathbb{Z}$-graded algebra, we can form the ordinary cohomology algebra $H^*(\operatorname{gr}\mathcal{A})$; this is the mod 2 lambda algebra. Explicitly, $\mathcal{A}$ is generated by $\mathcal{A}_{\leq 1} = \mathbb{F}_2\{\mathrm{Sq}^n : n \geq 0\}$ subject to the quadratic relations

$$\mathrm{Sq}^{2s-r-1}\mathrm{Sq}^s = \sum_i \binom{r-i-1}{i} \mathrm{Sq}^{2s-1-i}\mathrm{Sq}^{s-r+i}$$

for $r \geq 0$ together with the additional nonquadratic relation $\mathrm{Sq}^0 = 1$. Thus $\operatorname{gr}\mathcal{A}$ and $T(\mathcal{A}_{\leq 1}, qR)$ are generated by Sq^n for $n \geq 1$ and $n \geq 0$ respectively, subject to the same quadratic relations. We learn that the ordinary cohomology algebras $H^*(\operatorname{gr}\mathcal{A})$ and $H^*(T(\mathcal{A}_{\leq 1}, qR))$ are generated by elements λ_n dual to Sq^{n+1} subject to the dual relations

$$\lambda_a \lambda_{2a+b+1} = \sum_j \binom{b-j-1}{j} \lambda_{a+b-j}\lambda_{2a+1+j}$$

for $b \geq 0$; the distinction between them is that λ_{-1} lives in the latter and not the former. These relations imply the identity $\lambda_{-1}^2 = 0$, which is also predicted by Theorem 3.7.3. The action of $\mathcal{A}$ on $\mathbb{F}_2$ lifts to the action of $T(\mathcal{A}_{\leq 1}, qR)$ on $\mathbb{F}_2$ with Sq^0 acting by the identity. We can identify $K_{T(\mathcal{A}_{\leq 1}, qR)}(\mathbb{F}_2, \mathbb{F}_2) = H^*(T(\mathcal{A}_{\leq 1}, qR))$, where we have enriched the former over graded vector spaces, and Theorem 3.7.1 tells us that the Koszul differential is obtained as the commutator with λ_{-1}. It follows that the lambda algebra is closed under the commutator with λ_{-1}, and we recover this description of its differential. ◁

3.8. **The PBW criterion.** As in the previous subsection, everything here takes place in the 1-categorical setting. Fix a quadratic algebra $F = T(H, R)$.

3.8.1. **Definition.** An *additive decomposition* of F consists of a decomposition

$$H \simeq \bigoplus_{i \in B} H_i$$

of bimodules, together with a subset S of the set B^* of words on B such that $\bigoplus_{w \in S} F_w \simeq F$, where if $w = (s_1, \ldots, s_n)$ then $F_w = H_{s_1} \circ \cdots \circ H_{s_n}$. This is a *PBW decomposition* if moreover

(1) A word $w = (s_1, \ldots, s_n)$ lives S if and only if each pair (s_i, s_{i+1}) lives in S;
(2) B is equipped with an order, and so B^* with the lexicographic order, such that for all $w', w'' \in S$, either $w'w'' \in S$ or the composite

$$F_{w'} \circ F_{w'} \to F \circ F \to F \to \bigoplus_{w \leq w'w''} F_w$$

is null. ◁

Suppose now that F is equipped with a PBW decomposition, with notation as in the definition. Abbreviate $C = C(I, F, \overline{P})$ for varying P. For $w \in B^*$, define

$$C_k[\leq w] = \bigoplus_{\substack{w_1, \ldots, w_k \in S \\ w_1 \cdots w_k \leq w}} F_{w_1} \circ \cdots \circ F_{w_k} \subset C_k,$$

and similarly define $C_k[< w]$ and $C_k[w]$. The PBW criterion implies that $C_k[\leq w]$ and $C_k[< w]$ are quotient complexes of C, and by construction we have short exact sequences

$$0 \to C[w] \to C[\leq w] \to C[< w] \to 0.$$

3.8.2. **Proposition.** Suppose that F is equipped with a PBW decomposition, and fix notation for this as above. Suppose that the lexicographic ordering on B^* is well-founded when restricted to subsets of words of a fixed length, and that $H_*C \to H_* \lim_w C[\leq w]$ is an injection. Then F is Koszul.

Proof. The proof is exactly as in [Pri70, Theorem 5.3]; we recall the construction in dual form. Under the given hypothesis, it is sufficient to fix a word w of length m and verify that $C[w]$ is acyclic outside degree m. To that end, one constructs $s\colon C[w]_k \to C[w]_{k+1}$ such that $sd + ds$ is the identity on $C[w]_k$ for $k < m$ as follows. Write $w = (r_1, \ldots, r_m)$, and denote decompositions $w = w_1 \cdots w_k$ by $(r_1, \ldots, r_{n_1}; \ldots; r_{n_{k-1}+1}, \ldots, r_m)$. Then s is defined on a summand indexed by a decomposition $w = w_1 \cdots w_k$ as follows. If this decomposition is of the form $(r_1; \ldots; r_{j-1}; r_j, \ldots, r_{j+l}; \ldots)$ with $l \geq 1$ and $r_i r_{i+1} \notin S$ for $i < j$, then s is given by twisting the identification with the summand indexed by $(r_1; \ldots; r_j; r_{j+1}, \ldots, r_{j+l}; \ldots)$ with a sign of $(-1)^j$. On all other summands, $s = 0$. □

The finiteness conditions of Proposition 3.8.2 are satisfied in settings where one may reasonably call F a locally finite algebra. On the other hand, there are settings where one may reasonably say that F has a PBW basis which fails to respect the $\mathcal{P}$-bimodule structure of F and therefore does not give rise to a PBW decomposition. In such cases, it may nonetheless be possible to deduce Koszulity by filtering the failure away, as was done in the proof of Theorem 3.7.3.

3.8.3. **Example.** Let $\mathcal{A}$ denote the mod 2 Steenrod algebra. Following [Pri70, Section 7], $\operatorname{gr}\mathcal{A}$ has a PBW basis of admissibles, and this provides a proof of its Koszulity. Now let $\mathcal{U}$ be the monad on the category on graded $\mathbb{F}_2$-vector spaces whose algebras are the unstable $\mathcal{A}$-modules. Then $\mathcal{U}$ is a quotient algebra of $\mathcal{A}$, for our general definition of an algebra, and the admissible basis of $\operatorname{gr}\mathcal{A}$ projects to a PBW decomposition of $\operatorname{gr}\mathcal{U}$. Thus $\mathcal{U}$ is itself a Koszul algebra. This in fact recovers the unstable lambda algebra:

$$K_{\mathcal{U}}(e_n, e_*) \cong \Lambda(n)$$

as chain complexes, up to choices of grading, where e_a denotes a copy of $\mathbb{F}_2$ in degree a. We will cover a variant of this example in greater detail in Subsection 5.5. ◁

4. Plethories

This section is concerned with a generalization of the biring triples of Tall-Wraith [TW70], or plethories of [BW05]. We give the definition in Subsection 4.1. In Subsection 4.2, we introduce the notion of a cobialgebroid, and in Subsection 4.3 show how the additive operations on rings over a plethory naturally form a cobialgebroid, at least under appropriate flatness assumptions. In Subsection 4.4, we describe what abelianization looks like for rings over a plethory; this serves as an example of the general theory of Subsection 2.9.

Everything in this section takes place in the 1-categorical setting. In particular, we will write $\otimes$ for non-derived tensor products, write D for non-derived abelianization, and so forth. We do expect that an interesting theory of higher algebraic plethories, modeled on $\mathbb{E}_\infty$ objects and possibly taking into account stability conditions, exists. Examples of these would include those seen in Example 2.3.7. For instance, given such a theory, the higher analogue of the discussion of Subsection 4.4 would interpret the $\mathbb{A}_\infty$ ring spectrum DB of [Sch01, Section 7.9] (see also [Lur18, Section 25.3.3]) as the cotangent space of the higher plethory associated to $\mathrm{C}\mathcal{R}\mathrm{ing}_B \to \mathcal{C}\mathrm{Alg}_B^{\mathrm{cn}}$ for an ordinary commutative ring B; there are also clear analogues between the constructions of [GL20] and parts of what we consider in Subsection 4.3 and Subsection 4.5.

4.1. **Exponential monads.** Let $\mathcal{P}$ be a symmetric monoidal additive theory. Write $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ for the category of commutative monoids in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, so that $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit} \simeq \mathcal{M}\mathrm{odel}_{S\mathcal{P}}^{\heartsuit}$ where $SP = \bigoplus_{n\geq 0} P^{\otimes n}/\Sigma_n$. We will extend our general conventions to refer to the objects of $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ as $\mathcal{P}$-rings. In particular, $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}$ is not the category of $\mathbb{E}_\infty$ monoids in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, although the natural forgetful functor is plethystic under suitable flatness assumptions.

4.1.1. **Lemma.** The following concepts are equivalent:

(1) Colimit-preserving monads T on $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$;
(2) Monads $\mathbb{T}$ on $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ which preserve sifted colimits and are equipped with the structure of a strong monoidal functor $\mathbb{T}\colon (\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \oplus) \to (\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \otimes)$ in such a way that for all $X, Y \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, the dashed arrow in
$$\begin{array}{ccc} \mathbb{T}(\mathbb{T}(X)\otimes\mathbb{T}(Y)) & \xrightarrow{\simeq} & \mathbb{T}\mathbb{T}(X\oplus Y) \\ \downarrow & & \downarrow \\ \mathbb{T}(X)\otimes\mathbb{T}(Y) & \xrightarrow{\simeq} & \mathbb{T}(X\oplus Y) \end{array}$$
endows $\mathbb{T}(X)\otimes\mathbb{T}(Y)$ with the structure of a $\mathbb{T}$-algebra.

Proof. This is implicit in [Rez09]. Given the monad T, one can verify that $\mathbb{T} = T\circ S$ has the indicated structure. Conversely, given $\mathbb{T}$, any $B \in \mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit}$ is naturally an object of $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ via the map
$$B\otimes B \to \mathbb{T}(B)\otimes\mathbb{T}(B) \simeq \mathbb{T}(B\oplus B) \to \mathbb{T}(B) \to B,$$
giving a monadic functor $\mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ which preserves sifted colimits. One can then verify that for $A, B \in \mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit}$, we have $A \coprod B = A\otimes B$, with $\mathbb{T}$-algebra structure analogous to the diagram in (2), and thus $\mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ preserves all colimits. □

Monads as in Lemma 4.1.1 are called *exponential monads*.

4.1.2. **Definition.** The equivalent data of Lemma 4.1.1 is a $\mathcal{P}$-*plethory*. ◁

We will generally treat a $\mathcal{P}$-plethory as its underlying exponential monad. Given a $\mathcal{P}$-plethory Λ, we will make use of the notation $\Lambda_{P,P'} = \Lambda(P)(P')$. We extend our conventions to refer to objects of $\mathcal{M}\mathrm{odel}_{\Lambda}^{\heartsuit}$ as Λ-rings.

Among the main pieces of structure of a $\mathcal{P}$-plethory Λ are maps

$$\begin{aligned}\Delta^+ &\colon \Lambda_P \to \Lambda_{P\oplus P} \simeq \Lambda_P \otimes \Lambda_P;\\ \epsilon^+ &\colon \Lambda_P \to \Lambda_0 \simeq \mathbb{1};\\ \Delta^\times &\colon \Lambda_{P\otimes P'} \to \Lambda_P \otimes \Lambda_{P'};\\ \epsilon^\times &\colon \Lambda_{\mathbb{1}} \to \mathbb{1}.\end{aligned}$$

Here, Δ^+ and ϵ^+ come from the diagonal $P \to P \oplus P$ and unique map $P \to 0$. The map $\Delta^\times$ is equivalent to a natural transformation $\mathrm{ev}_P \times \mathrm{ev}_{P'} \to \mathrm{ev}_{P\otimes P'}$, and classifies the multiplication present on Λ-rings, and likewise $\epsilon^\times$ classifies the multiplicative identity. Strictly speaking, these maps are present given just the underlying $S\mathcal{P}$-bimodel of Λ, only one can no longer interpret them as corresponding to natural operations.

4.2. **Cobialgebroids.** Fix a symmetric monoidal additive theory $\mathcal{P}$.

4.2.1. **Definition.** A (discrete) $\mathcal{P}$-*cobialgebroid* is a $\mathcal{P}$-algebra Γ with a lift of Γ to a monoid in the category of oplax symmetric monoidal endofunctors of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, or equivalently, a lift of $\Gamma^\vee$ to a comonoid in the category of lax symmetric monoidal endofunctors of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. ◁

Denote the category of $\mathcal{P}$-cobialgebroids by $\mathrm{co}\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}_{\mathcal{P}}^{\heartsuit}$.

4.2.2. **Lemma.** Let $\mathcal{C}$ be a symmetric monoidal 1-category, and $U\colon \mathcal{D} \to \mathcal{C}$ be a plethystic functor with associated monad F and comonad G. Then the following concepts are equivalent:

(1) The structure of a symmetric monoidal category on $\mathcal{D}$ and strong symmetric monoidal functor on U;
(2) A lift of F to a monoid in the category of oplax symmetric monoidal endofunctors of $\mathcal{C}$;
(3) A lift of G to a comonoid in the category of lax symmetric monoidal endofunctors of $\mathcal{C}$.

Proof. The equivalence of (2) and (3) is readily seen, so we consider their relation with (1). Given the data of (2), we can make $\mathcal{D} = \mathcal{A}\mathrm{lg}_F$ into a symmetric monoidal category, where for F-algebras A and B, their tensor product is $A\otimes B$ with F-algebra structure $F(A\otimes B) \to F(A)\otimes F(B) \to A\otimes B$. This is seen to refine to the data of (1). Consider then given the data of (1). As U is strong monoidal, for $A, B \in \mathcal{C}$ we have a map $F(A\otimes B) \to F(A)\otimes F(B)$ adjoint to $A\otimes B \to UF(A)\otimes UF(B) \simeq U(F(A)\otimes F(B))$. This is seen to refine to the data of (2). □

4.2.3. **Remark.** Let Γ be a $\mathcal{P}$-algebra. Then one can be more explicit about the structure necessary to upgrade Γ to a $\mathcal{P}$-cobialgebroid: to lift Γ to an oplax symmetric monoidal functor, we require maps

$$\Gamma(M\otimes N) \to \Gamma(M)\otimes\Gamma(N), \qquad \Gamma(\mathbb{1}) \to \mathbb{1}$$

natural in M and N and subject to the evident counity, coassociativity, and cocommutativity conditions, and for this to make Γ into a $\mathcal{P}$-cobialgebroid we further require that the product $\Gamma\circ\Gamma \to \Gamma$ respects this structure. ◁

4.2.4. **Remark.** Let Γ be a $\mathcal{P}$-cobialgebroid. Let A be a monoid in the monoidal category $\mathcal{M}\mathrm{odel}_\Gamma^\heartsuit$. In particular, A overlies a monoid in $\mathcal{M}\mathrm{odel}_\mathcal{P}^\heartsuit$, giving $A \otimes -$ the structure of a monad. Then there is a distributive law of Γ across $A \otimes -$ given by

$$\Gamma(A \otimes M) \to \Gamma(A) \otimes \Gamma(M) \to A \otimes \Gamma(M),$$

and this rise to a composite monad $A \otimes \Gamma$. Algebras for this monad are exactly modules over the monoid A in $\mathcal{M}\mathrm{odel}_\Gamma^\heartsuit$. This generalizes Example 2.5.4. ◁

We will write $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit$ for the category of commutative monoids in $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit$. Observe the forgetful functor $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit \to \mathrm{C}\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit$ is plethystic.

4.2.5. **Example.** Let R be an ordinary commutative ring, $\mathcal{R}$ be the theory of R-modules with its usual symmetric monoidal structure, and F be an $\mathcal{R}$-cobialgebroid. Unwinding the definitions, we see this amounts to the following. First, as F is a bimodule, we can write $F(M) = \Gamma \otimes M$ for an ordinary R-bimodule Γ. Abbreviate $\otimes = \otimes_R$, and use subscripts l and r to denote tensoring with respect to the left or right R-module structure on Γ. Then we have left R-module maps

$$\begin{aligned} m\colon \Gamma\,{}_r{\otimes}_l\,\Gamma &\to \Gamma; \\ \epsilon^\times\colon \Gamma &\to R; \\ \Delta^\times\colon \Gamma &\to \Gamma\,{}_l{\otimes}_l\,\Gamma. \end{aligned}$$

The map m makes Γ into an R algebra, and $\epsilon^\times$ with $\Delta^\times$ satisfy evident counity, coassociativity, and cocommutativity conditions. The map $\Delta^\times$ is required in addition to be a map of right R-modules with respect to the two right R-module structures on the target given by the action of R on the left and right factor. This corresponds to the fact that $F(M \otimes N) \to F(M) \otimes F(N)$ is natural in both variables, and is what is necessary to extend $\Delta^\times$ to the natural transformation

$$\Gamma\,{}_r{\otimes}\, M \otimes N \to (\Gamma\,{}_r{\otimes}\, M)\,{}_l{\otimes}_l(\Gamma\,{}_r{\otimes}\, N), \qquad \gamma \otimes m \otimes n \mapsto \sum \gamma^\times_{(1)} \otimes m \otimes \gamma^\times_{(2)} \otimes n.$$

Compatibility of m with $\epsilon^\times$ amounts to asking for

$$\epsilon^\times(1) = 1, \qquad \epsilon^\times(\gamma\gamma') = \epsilon^\times(\gamma \cdot \epsilon^\times(\gamma')),$$

and compatibility of m with $\Delta^\times$ amounts to asking for

$$\sum \gamma^\times_{(1)}\gamma'^\times_{(1)} \otimes \gamma^\times_{(2)}\gamma'^\times_{(2)} = \sum (\gamma\gamma')^\times_{(1)} \otimes (\gamma\gamma')^\times_{(2)}$$

in $\Gamma\,{}_l{\otimes}_l\,\Gamma$. In the above, we have written $\Delta^\times(\gamma) = \sum \gamma^\times_{(1)} \otimes \gamma^\times_{(2)}$. We find that Γ is a *twisted R-bialgebra*, for instance as discussed in [BW05, Section 9]. A Γ-ring is a commutative R-ring A equipped with an action of Γ such that $\gamma \cdot (aa') = \sum(\gamma^\times_{(1)} \cdot a)(\gamma^\times_{(2)} \cdot a')$ for all $a, a' \in A$ and $\gamma \in \Gamma$. ◁

4.2.6. **Example.** Let Γ be the algebra of Example 3.6.5. Then Γ is an R-cobialgebroid, with augmentation

$$\epsilon^\times(Q_0) = 1, \qquad \epsilon^\times(Q_1) = 0, \qquad \epsilon^\times(Q_2) = 0,$$

and coproduct

$$\begin{aligned} \Delta^\times(Q_0) &= Q_0 \otimes Q_0 + 2Q_1 \otimes Q_2 + 2Q_2 \otimes Q_1; \\ \Delta^\times(Q_1) &= Q_0 \otimes Q_1 + Q_1 \otimes Q_0 + aQ_1 \otimes Q_2 + aQ_2 \otimes Q_1 + 2Q_2 \otimes Q_2; \\ \Delta^\times(Q_2) &= Q_0 \otimes Q_2 + Q_2 \otimes Q_0 + Q_1 \otimes Q_1 + aQ_2 \otimes Q_2. \end{aligned}$$

In this way, if M and N are Γ-modules, then so too is $M \otimes_R N$. ◁

4.3. **Additive operations.** Fix a symmetric monoidal additive theory $\mathcal{P}$. In dealing with $\mathcal{P}$-plethories, one wants to avoid dealing with nonlinear structure whenever possible. The use of $\mathcal{P}$-cobialgebroids is one thing that enables this.

There is a purely formal means by which one can extract from any $\mathcal{P}$-plethory a $\mathcal{P}$-cobialgebroid. Going in the other direction, if Γ is a $\mathcal{P}$-cobialgebroid, then as $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit \to \mathrm{C}\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit$ is plethystic, we obtain a $\mathcal{P}$-cobialgebroid $S\Gamma$. The functor $S\colon \mathrm{co}\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}_\mathcal{P}^\heartsuit \to \mathcal{P}\mathrm{leth}_\mathcal{P}^\heartsuit$ so obtained can be described more explicitly as follows. If Γ is a $\mathcal{P}$-cobialgebroid, then as $\Gamma^\vee$ is lax symmetric monoidal, it passes to a limit-preserving comonad $S\Gamma^\vee$ on $\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit$, right adjoint to the colimit-preserving monad $S\Gamma$ on $\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit$. The construction S so described preserves colimits, so has a right adjoint sending a plethory Λ to a $\mathcal{P}$-cobialgebroid $\Gamma(\Lambda)$. We do not know if $\Gamma(\Lambda)$ admits a nice description in general, but it will under certain hypotheses.

Now fix a $\mathcal{P}$-plethory Λ. As in Proposition 2.1.5, we may identify

$$\Lambda_{P_1\oplus\cdots\oplus P_n,P} \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit,\mathcal{S}\mathrm{et})}(\mathrm{ev}_{P_1}\times\cdots\times\mathrm{ev}_{P_n},\mathrm{ev}_P).$$

Let $\Gamma_{(P_1,\ldots,P_n),P} \subset \Lambda_{P_1\oplus\cdots\oplus P_n,P}$ be the subset consisting of those operations which are n-multilinear. By allowing $P_1,\ldots,P_n,P$ to vary, this can be seen as a functor $\mathcal{P}^{\times n} \to \mathrm{L}\mathcal{M}\mathrm{od}_\mathcal{P}^\heartsuit$.

4.3.1. **Lemma.** The $\mathcal{P}$-module $\Gamma_{(P_1,\ldots,P_n)}$ arises as the total fiber of the n-cube obtained by tensoring together the maps

$$\Delta^+ - \eta\otimes P_i - P_i\otimes\eta\colon \Lambda_{P_i} \to \Lambda_{P_i\oplus P_i} \simeq \Lambda_{P_i}\otimes\Lambda_{P_i}.$$

Proof. For purely notational convenience, we consider the case $n=1$; the general case is identical in nature. Here we are claiming that there is an equalizer diagram

$$\Gamma_P \longrightarrow \Lambda_P \underset{\eta\otimes P+P\otimes\eta}{\overset{\Delta^+}{\rightrightarrows}} \Lambda_{P\oplus P}\ .$$

Evaluating on $P'\in\mathcal{P}$, this is asking for an equalizer diagram

$$\begin{array}{c}\mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}\mathrm{odel}_\Lambda^\heartsuit,\mathcal{A}\mathrm{b})}(\mathrm{ev}_P,\mathrm{ev}_{P'})\\ \downarrow\\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}\mathrm{odel}_\Lambda^\heartsuit,\mathcal{S}\mathrm{et})}(\mathrm{ev}_P,\mathrm{ev}_{P'})\\ {\scriptstyle\eta\otimes P+P\otimes\eta}\downarrow\ \downarrow{\scriptstyle\Delta^+}\\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}\mathrm{odel}_\Lambda^\heartsuit,\mathcal{S}\mathrm{et})}(\mathrm{ev}_P\times\mathrm{ev}_P,\mathrm{ev}_{P'})\end{array}\ .$$

If we fix a natural operation $\sigma\colon \mathrm{ev}_P \to \mathrm{ev}_{P'}$, then

$$\Delta^+(\sigma)(x,y) = \sigma(x+y), \qquad (\eta\otimes P + P\otimes\eta)(x,y) = \sigma(x)+\sigma(y),$$

and these agree precisely when σ is additive. □

We have in particular maps $\Gamma_{P_1}\otimes\cdots\otimes\Gamma_{P_n} \to \Gamma_{(P_1,\ldots,P_n)}$. Call Λ *good* if this is an isomorphism for all $P_1,\ldots,P_n\in\mathcal{P}$, and moreover $\Gamma = \Gamma_{(-),(=)}$ is a $\mathcal{P}$-bimodule.

4.3.2. **Theorem.** Suppose that Λ is good. Then the $\mathcal{P}$-plethory structure of Λ restricts to a $\mathcal{P}$-cobalgebroid structure on Γ, and $\Gamma(\Lambda) \cong \Gamma$.

Proof. We begin by building the $\mathcal{P}$-cobialgebroid structure on Γ. By hypothesis, Γ is a $\mathcal{P}$-bimodule. As additive operations are closed under composition, we have for $P, P', P'' \in \mathcal{P}$ a bilinear composition map $\Gamma_{P',P''} \times \Gamma_{P,P'} \to \Gamma_{P,P''}$, and these together with the identity maps make Γ into a $\mathcal{P}$-algebra. The counit of Γ is the composite

$$\epsilon^\times : \Gamma_{\mathbb{1}} \subset \Lambda_{\mathbb{1}} \to \mathbb{1}.$$

For the coproduct on Γ, consider the diagram

$$\begin{array}{ccc} \Gamma_{P\otimes P'} & \longrightarrow & \Lambda_{P\otimes P'} \\ \downarrow{\scriptstyle \Delta^\times} & & \downarrow{\scriptstyle \Delta^\times} \\ \Gamma_P \otimes \Gamma_{P'} & \longrightarrow & \Lambda_P \otimes \Lambda_{P'} \end{array} .$$

The right vertical map classifies multiplication for Λ-rings, so the clockwise composite lands in $\Gamma_{(P,P')} \subset \Lambda_{P\oplus P'} \simeq \Lambda_P \otimes \Lambda'_P$. This is isomorphic to $\Gamma_P \otimes \Gamma_{P'}$ by assumption, so we can fill in the dashed map. The axioms of counity, coassociativity, cocommutativity, and compatibility with the composition product follow from the corresponding facts about operations on Λ-rings. It remains to verify that $\Gamma \cong \Gamma(\Lambda)$. Observe first that the functor S on $\mathcal{P}$-cobialgebroids satisfies, as the notation suggests, $(S\Gamma)(P) = S(\Gamma(P))$. Indeed, we can identify

$$\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}((S\Gamma)(P), R) \simeq (S\Gamma^\vee)(R)(P) \simeq \Gamma^\vee(R)(P) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(\Gamma(P), R),$$

so that $(S\Gamma)(P)$ has the necessary universal property. It follows from the definition of Γ that the composite $\Gamma(\Lambda) \to S\Gamma(\Lambda) \to \Lambda$ factors uniquely through $\Gamma \subset \Lambda$, so $S\Gamma(\Lambda) \to \Lambda$ factors uniquely through $S\Gamma \to \Lambda$. So $S\Gamma \to \Lambda$ has the necessary universality property to be the counit of the adjunction, giving $S\Gamma \simeq S\Gamma(\Lambda)$, from which it follows that $\Gamma \simeq \Gamma(\Lambda)$. □

From here on, we will always assume that our plethories are good when we speak of their associated cobialgebroids, as this will be the case for the examples we are interested in. In fact, all of our examples will be *very good*: the inclusion $\Gamma \to \Lambda$ will be additively split. This ensures that the goodness property of Λ is preserved under various operations, such as base change.

4.3.3. **Remark.** Let Λ be a $\mathcal{P}$-plethory and $B \in \mathcal{R}\mathrm{ing}_\Lambda^\heartsuit$. In particular, $B \in \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit$, so there is a category $\mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit$ of left B-modules. The forgetful functor $B/\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit \to B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit$ is plethystic, so realizing $B/\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit$ as the category of rings for a B-plethory $B \otimes \Lambda$. The monadic structure is as given by Example 2.5.5. There is a diagram

$$\begin{array}{ccc} \Gamma(B\otimes\Lambda) & \longleftarrow & B \\ \uparrow & & \uparrow \\ \Gamma(\Lambda) & \longleftarrow & I \end{array}$$

of $\mathcal{P}$-algebras, where B stands for the $\mathcal{P}$-algebra $B \otimes -$, which extends to a map

$$B \otimes \Gamma(\Lambda) \to \Gamma(B \otimes \Lambda)$$

of algebras, which is an isomorphism in the nice cases we will consider. Here, $B \otimes \Gamma(\Lambda)$ has algebra structure as indicated in Remark 4.2.4. See Example 4.4.4 for an explicit example of this. ◁

4.4. **Cotangent spaces.** Fix a symmetric monoidal additive theory $\mathcal{P}$ and a $\mathcal{P}$-plethory Λ. We are interested in the cohomology of Λ-rings. In particular, we need to identify the relevant categories of abelian group objects. This is as indicated by the general theory of Subsection 2.9, but it is worth making this more explicit. We remind the reader that here, as throughout this section, all constructions are to be interpreted in the 1-categorical sense.

We first review the essentially classical case where $\Lambda = S$. Given $B \in \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$, we may form the square-zero extension $B \ltimes M \in \mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$. As an object of $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit} = \mathrm{C}\mathcal{M}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit})$, this is given by $B \ltimes M = B \oplus M$ with multiplication

$$\begin{aligned}(B \oplus M) \otimes (B \oplus M) &\simeq B \otimes B \oplus B \otimes M \oplus M \otimes B \oplus M \otimes M \\ &\simeq B \otimes B \oplus B \otimes M \oplus B \otimes M \oplus M \otimes M \to B \oplus M\end{aligned}$$

arising from the multiplication on B, the B-module structure of M, and killing $M \otimes M$. This has obvious structure as an object of $B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$, and structure as an abelian group object therein of

$$(B \ltimes M) \times_B (B \ltimes M) \cong B \oplus M \oplus M \to B \oplus M, \qquad ((b, m'), (b, m'')) \mapsto (b, m' + m'').$$

4.4.1. **Lemma.** The above construction describes an equivalence between the following categories:

(1) The category $\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$;
(2) The full subcategory of $B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$ spanned by the square-zero extensions of B;
(3) The category $\mathcal{A}\mathrm{b}(\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B) \simeq \mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$.

Moreover,

(4) Abelianization $D_B \colon B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit} \to \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$ is given by $D_B(A) = B \otimes_A \Omega_{A|\mathcal{P}}$, where $\Omega_{A|\mathcal{P}} = J/J^2$, where $J = \mathrm{Ker}(A \otimes A \to A)$;
(5) Abelianization $Q_B \colon B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B \to \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$ is given by $Q_B(A) = I/I^2$ where $I = \mathrm{Ker}(A \to B) = \mathrm{Coker}(B \to A)$.

Proof. These statements are standard in the case where $\mathcal{P}$ is the theory of commutative rings, and the same proofs carry over. That $M \mapsto B \ltimes M$ yields an equivalence from $\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$ to the category of square-zero extensions of B is clear, with inverse sending a square-zero extension to its augmentation ideal. We can identify $\mathcal{A}\mathrm{b}(\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B) \simeq \mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$ as abelian groups are in particular pointed. We can identify $\mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$ with the category of square-zero extensions of B as follows. Fix $A \in \mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$. Additively we may split $A = B \oplus M$, and we must show that in fact $A \cong B \ltimes M$ multiplicatively. As A is a B-ring, M is a B-module, and the multiplication on A is necessarily determined by the multiplication on B, the B-module structure of M, and some map $M \otimes M \to M$ which we must show is zero. Fixing $P, P' \in \mathcal{P}$, we must show that the map $m \colon M(P) \otimes M(P') \to M(P \otimes P')$ is zero. Write $\mu \colon A \times_B A \to A$ for the abelian group object structure of A; we will only use the fact that μ is a unital pairing. As μ is unital, the maps $\mu \colon A(P) \times_{B(P)} A(P) \to A(P)$ satisfy $\mu(x, 0) = x = \mu(0, x)$ for $x \in M(P)$. Thus for $x \in M(P)$ and $x' \in M(P')$, we may identify $m(x \otimes x') = m(\mu(x, 0) \otimes \mu(0, x')) = \mu(m(x, 0) \otimes m(0, x')) = \mu(0, 0) = 0$.

So we have shown the categories of (1)–(3) to be equivalent, and it remains only to verify the claims of (4) and (5). Write D_B and Q_B for the abelianization functors in question, and D'_B and Q'_B for their proposed descriptions. Then $D_B(A) = Q_B(B\otimes A)$ for $A \in \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$, and we claim that also $D'_B(A) = Q'_B(B \otimes A)$. By definition, $Q_B(B \otimes A) = I/I^2$ where $I = \mathrm{Ker}(B \otimes A \to B) = \mathrm{Ker}(B \otimes_A (A \otimes A) \to B \otimes_A A)$. As $A \otimes A \to A$ admits an A-linear splitting, we call pull B out to get $I = B \otimes_A J$ where $J = \mathrm{Ker}(A \otimes A \to A)$. Thus $D'_B(A) = B \otimes_A (J/J^2) = I/I^2 = Q'_B(B \otimes A)$ as claimed. So it is sufficient to verify just that $Q_B = Q'_B$. Fix $A \in B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$, and write $A = B \oplus I$ additively, so that $Q'_B(A) = I/I^2$. Let $B \ltimes M$ be some square-zero extension of B. Then maps $A \to B \ltimes M$ in $B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$ are equivalent to maps $I \to M$ of nonunital B-rings. As M has trivial multiplication, this factors uniquely through the quotient nonunital ring I/I^2, which has square-zero multiplication. We find that the quotient ring $B \ltimes I/I^2$ of A is the square-zero extension of B associated to $Q_B(A)$, and thus $Q_B(A) = I/I^2 = Q'_B(A)$. □

We now turn to considering a general $\mathcal{P}$-plethory Λ. Observe that $\mathbb{1}$ is naturally a Λ-ring by the unique map $\mathbb{1} \to \Lambda^\vee(\mathbb{1})$ of $\mathcal{P}$-rings. Equivalently, $\mathbb{1}$ is a Λ-ring by the fact that $\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ preserves the empty colimit. Thus there is a good category of $\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}$ of augmented Λ-rings. For all $P \in \mathcal{P}$, the map $P \to 0$ of $\mathcal{P}$-modules gives a map $\Lambda_P \to \Lambda_0 = \mathbb{1}$ of Λ-rings, so we can regard each Λ_P as an object of $\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}$. Define now

$$\Delta(\Lambda)\colon \mathcal{P} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \qquad \Delta(\Lambda)_{P,P'} = Q(\Lambda_P)_{P'}.$$

In later sections, where Q is to be interpreted as giving derived indecomposables, we continue to take this discrete object as the definition of $\Delta(\Lambda)$. We call $\Delta(\Lambda)$ the *cotangent algebra* of Λ. The functor $\Delta(\Lambda)$ preserves coproducts, so can be regarded as a $\mathcal{P}$-bimodule.

4.4.2. **Theorem.**

(1) We can identify $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit})$ as the full subcategory of $\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}$ spanned by those Λ-rings whose underlying $\mathcal{P}$-ring is a square-zero extension of $\mathbb{1}$. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}) & \longrightarrow & \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit} \\ \downarrow & & \downarrow \\ \mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit} & \longrightarrow & \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\mathrm{aug},\heartsuit} \end{array}$$

is distributive.

(2) The underlying $\mathcal{P}$-bimodule of the $\mathcal{P}$-algebra associated to the plethystic functor $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}) \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ is given by $\Delta(\Lambda)$.

(3) Fix $B \in \mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit}$, so that $B/\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit} \simeq \mathcal{R}\mathrm{ing}_{B\otimes\Lambda}^{\heartsuit}$ as in Example 2.5.5, so that $B/\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit}/B \simeq \mathcal{R}\mathrm{ing}_{B\otimes\Lambda}^{\mathrm{aug},\heartsuit}$. Then as a $\mathcal{P}$-algebra, $\Delta_B(B \otimes \Lambda) \cong B \otimes \Delta(\Lambda)$ is a composition of the monad $B \otimes -$ with $\Delta(\Lambda)$. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit}/B) & \longrightarrow & \mathcal{A}\mathrm{b}(\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B) \\ \downarrow & & \downarrow \\ \mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit}/B & \longrightarrow & \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B \end{array}$$

is distributive.

Proof. Given Lemma 4.4.1, these are just specializations of the general theory of Subsection 2.9. ☐

4.4.3. **Remark.** Observe that the composite

$$\Gamma(\Lambda) \to \Lambda \to \Delta(\Lambda)$$

is a map of $\mathcal{P}$-algebras. Moreover, we observe that the constructions of $\Gamma(\Lambda)$ and $\Delta(\Lambda)$ are formally dual: if we split $\Lambda_P = \widetilde{\Lambda}_P \oplus \mathbb{1}$, then

$$\Delta(\Lambda)_P = \mathrm{Coker}(\widetilde{\Lambda}_{P\oplus P} \to \widetilde{\Lambda}_P)$$
$$\Gamma(\Lambda)_P = \mathrm{Ker}(\widetilde{\Lambda}_P \to \widetilde{\Lambda}_{P\oplus P}),$$

the maps obtained from the codiagonal $P \oplus P \to P$ and diagonal $P \to P \oplus P$ respectively. In other words, $\Delta(\Lambda)$ is the linearization of the functor Λ, and dually we might call $\Gamma(\Lambda)$ the colinearization of Λ. ◁

4.4.4. **Example.** Let Λ be the $\mathbb{Z}$-plethory of θ-rings, also known as δ-rings [Joy85], as well as by other names. In brief, θ-rings are commutative rings B equipped with a nonlinear operation $\theta\colon B \to B$ satisfying all the identities necessary to make

$$\psi(b) = b^p + p\theta(b)$$

generically a ring map. The underlying commutative ring of Λ can be identified as

$$\Lambda = \mathbb{Z}[\theta_n : n \geq 0],$$

where $\theta_n = \theta^{\circ n}$. Here, we are making use of the correspondence between elements of the ring Λ and natural operations on θ-rings, so for example the operation ψ is given by the element

$$\psi = \theta_0^p + p\theta_1.$$

The operation ψ freely generates the additive operations, and we can identify $\Gamma(\Lambda) = \mathbb{Z}[\psi]$ as a $\mathbb{Z}$-algebra. As ψ is a ring homomorphism, the cobialgebroid structure is

$$\epsilon^\times(\psi) = 1, \qquad \Delta^\times(\psi) = \psi \otimes \psi.$$

Evidently $\Delta(\Lambda) = \mathbb{Z}[\theta]$ as a $\mathbb{Z}$-algebra, and the map $\Gamma(\Lambda) \to \Delta(\Lambda)$ is given by

$$\mathbb{Z}[\psi] \to \mathbb{Z}[\theta], \qquad \psi \mapsto p\theta.$$

Now say B is a θ-ring, so that $B \otimes \Lambda$ is a B-plethory. The general recipe of Example 2.5.5 for computing the plethory structure on $B \otimes \Lambda$ translates into the following. Note $B \otimes \Lambda = B[\theta_n : n \geq 0]$, and it is sufficient to just determine the composition $\theta_1 \circ b$ for $b \in B$. As an element of $B \otimes \Lambda$, the element $\theta_1 \circ b$ represents the natural operation on θ-rings A under B given by

$$(\theta \circ b)(a) = \theta(b \cdot a) = \theta(b)a^p + b^p\theta(a) + p\theta(b)\theta(a).$$

Thus we have

$$\theta \circ b = \theta(b)\theta_0^p + b^p\theta_1 + p\theta(b)\theta_1.$$

This is nothing but a specialization of the general formula

$$(b \otimes \sigma) \circ (b' \otimes \sigma') = \sum b\sigma^\times_{(1)}(b') \otimes \sigma^\times_{(2)} \circ \sigma'.$$

We can identify $\Gamma(B \otimes \Lambda) = B \otimes \Gamma(\Lambda) = B[\psi]$ as $\mathbb{Z}$-bimodules. For clarity, write $\psi_1 = \psi$ as an element of $B \otimes \Gamma(\Lambda)$. The algebra structure is determined by the general distributive law of Remark 4.2.4, which is in turn described by Example 2.5.4, and we find

$$\Gamma(B\otimes\Lambda) = B\langle\psi_1\rangle/(\psi_1\cdot b = \psi(b)\cdot\psi_1)$$

as a B-algebra. We likewise identify $\Delta(B\otimes\Lambda) = B\otimes\Delta(\Lambda) = B[\theta]$ as $\mathbb{Z}$-bimodules, but as $\Delta(\Lambda)$ does not carry a coproduct, the algebra structure cannot be determined in the same way. There are two good ways to proceed. The method that works in general is to observe that the composition law on $B\otimes\Lambda$ implies that $\theta\circ b\equiv\psi(b)\theta$ mod indecomposables, and thus

$$B\otimes\Delta(\Lambda) = B\langle\theta\rangle/(\theta\cdot b = \psi(b)\cdot\theta).$$

The second method is to observe that although $\Gamma(\Lambda)\to\Delta(\Lambda)$ is not surjective, the existence of an algebra map

$$B\langle\psi_1\rangle/(\psi_1\cdot b = \psi(b)\cdot\psi_1)\to B\otimes\mathbb{Z}[\theta],\qquad \psi_1\mapsto p\theta$$

is nonetheless sufficient to determine the algebra structure on $B\otimes\mathbb{Z}[\theta]$, at least when B is p-torsion free. ◁

We might call Λ *smooth* when its associated $S\mathcal{P}$-algebra is smooth in the sense of Subsection 2.9. In these cases, following Propositions 2.9.3 and 2.9.4, we can split computations of the cohomology of Λ-rings into computations of the cohomology of $\mathcal{P}$-rings plus Ext computations over the additive algebra $\Delta(\Lambda)$. We will need a relative version of this. Given a map $\Lambda'\to\Lambda$ of $\mathcal{P}$-plethories, we can view Λ as an algebra for the theory of Λ'-rings, and so call Λ smooth over Λ' when it is smooth in this sense.

4.4.5. **Example.** Let R be an ordinary commutative ring, and $\mathcal{P}$ the theory of $\mathbb{Z}$-graded R-modules, regarded as a symmetric monoidal theory with symmetrizer employing the Koszul sign rule. Then the resulting category $\mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$ is the category of ordinary $\mathbb{Z}$-graded R-algebras B such that

$$bb' = (-1)^{|b||b'|}b'b$$

for all $b,b'\in B$. In particular, if we write $R\{e_n\}$ for a copy of R in degree n, then

$$SR\{e_n\} = R[e_n]/((1-(-1)^n)e_n^2).$$

In particular, when 2 is neither zero nor a unit in R, the monad S need not preserve projective objects, and the homotopy theory of simplicial R_*-rings may not behave as one would like. One can fix this by working instead with *alternating R-algebras*, i.e. those objects $B\in\mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$ such that moreover $b^2=0$ whenever $|b|$ is odd. Denote this category by $\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$. Then the inclusion $\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}\to\mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$ is plethystic, and we can even identify $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}/B)\simeq\mathcal{M}\mathrm{od}_{B_*}^{\heartsuit}\simeq\mathcal{A}\mathrm{b}(\mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}/B)$, so everything we have done for R_*-plethories carries over to the relative setting over $\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$. ◁

4.5. **Suspension maps.** We record here a definition of an additional piece of structure present in plethories that encode homotopy operations. Fix $\mathcal{P}$ as before. Let E be an object of $\mathcal{P}$ which is invertible under the tensor product. Then we obtain an automorphism of the category of endofunctors of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ by

$$H\mapsto H^E,\qquad H^E(M) = E\otimes H(E^{-1}\otimes M).$$

This is compatible with compositions of endofunctors, and thus passes to automorphisms of categories of monads and comonads. In addition, it preserves bimodules, and we can identify $(H^E)^\vee = (H^\vee)^{E^{-1}}$.

4.5.1. **Definition.**

(1) If F is a $\mathcal{P}$-algebra, we say that F is equipped with *E-suspensions* if we have chosen a map $\sigma\colon F^E \to F$ of algebras.

(2) If Γ is a $\mathcal{P}$-bialgebroid, we say that Γ is equipped with *E-suspensions* if we have equipped the underlying algebra of Γ with E-suspensions in such a way that for all $M \in \mathrm{LMod}_{\mathcal{P}}^{\heartsuit}$, the diagram

$$\begin{array}{ccccc} \Gamma^E(M) & \longrightarrow & & \longrightarrow & \Gamma(M) \\ \downarrow{=} & & & & \uparrow{\epsilon^\times \otimes \Gamma(M)} \\ E\otimes\Gamma(E^{-1}\otimes M) & \xrightarrow{E\otimes\Delta^\times} & E\otimes\Gamma(E^{-1})\otimes\Gamma(M) & \xrightarrow{\sigma\otimes\Gamma(M)} & \Gamma(\mathbb{1})\otimes\Gamma(M) \end{array}$$

commutes.

(3) If Λ is a $\mathcal{P}$-plethory, we say that Λ is equipped with *E-suspensions* if we have chosen a map $\sigma\colon \Delta(\Lambda)^E \to \Gamma(\Lambda)$ of algebras such that the composite $\Gamma(\Lambda)^E \to \Delta(\Lambda)^E \to \Gamma(\Lambda)$ equips $\Gamma(\Lambda)$ with E-suspensions. ◁

This definition is not intended to cover all cases where one may wish to speak of suspension maps, but only those we will need further on. In all of our explicit examples, $\mathrm{LMod}_{\mathcal{P}}$ will be a category of $\mathbb{Z}$-graded objects. Here we will always take E to be a copy of the monoidal unit in degree 1, and will just say "equipped with suspensions", as we trust no confusion should arise.

4.5.2. **Example.** Let k be an ordinary commutative ring, and consider the theory of $\mathbb{Z}$-graded left k-modules. Let E denote a copy of k in degree 1. If B is an ordinary $\mathbb{Z}$-graded k-algebra, then underlying monad of B is equipped with E-suspensions given by the identifications

$$E\otimes B\otimes E^{-1}\otimes M = B\otimes M.$$

As remarked in Example 3.3.1, when B is augmented, this is the sort of structure needed to define $H^*(B)$ as an ordinary $\mathbb{Z}$-graded algebra. ◁

Given a suitable object T equipped with a suspension map $\sigma\colon T^E \to T$, we can define the *costabilization* of T to be $\lim_{n\to\infty} T^{E^n}$. This is a monad when σ is a map of monads.

4.5.3. **Example.** Let $\mathcal{U}$ be the monad on the category of $\mathbb{Z}$-graded $\mathbb{F}_2$-modules whose algebras are the unstable modules over the mod 2 Steenrod algebra $\mathcal{A}$. Then $\mathcal{U}$ is naturally equipped with suspensions $\sigma\colon \mathcal{U}^E \to \mathcal{U}$ given by $\sigma(\mathrm{Sq}^I) = \mathrm{Sq}^I$, with the understanding that this element may be zero in the target even when nonzero in the source. The costabilization of $\mathcal{U}$ is exactly the Steenrod algebra $\mathcal{A}$. ◁

5. Power operations and $\mathbb{E}_\infty$ rings in characteristic p

We now come to more involved applications of the machinery we have introduced. In Subsection 5.1, we show how one can, by purely formal considerations, construct a plethory DL encoding the structure of mod p power operations. This serves as a typical example of how a theory of homotopy operations can be fit into the story we have given. In Subsection 5.2, we summarize the classic structure of these operations, showing how the plethystic language gives a clean way of packaging them together. In particular, the Quillen cohomology of DL-rings fits into the general story of Subsection 4.4; we indicate some features special to DL in Subsection 5.3. In Subsection 5.4, we describe the relation between DL-rings and unstable rings over the Steenrod algebra. The cotangent algebra of DL turns out to be Koszul over the theory of $\mathbb{Z}$-graded $\mathbb{F}_p$-modules, and we describe its cohomology in Subsection 5.5. With all this in place, in Subsection 5.6 and Subsection 5.7 we connect the homology and cohomology of DL-rings to the homotopy theory of $\mathbb{E}_\infty$-rings in characteristic p, describing some obstruction theories and spectral sequences for mapping spaces and for topological André-Quillen homology and cohomology.

Throughout this section, we will write e_a for a generic $\mathbb{Z}$-graded module generated by an element in degree a.

5.1. **Plethories of power operations.** To illustrate the relevant ideas, we show how one can identify that there is a plethory of mod p power operations, even before the hard work of computing their structure has been carried out. The approach taken here generalizes readily to other contexts, although additional hypotheses must be imposed to replace the simplifications obtained from the fact that $\mathbb{F}_p$ is a field.

Let $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ denote the category of $\mathbb{F}_p$-modules, i.e. $H\mathbb{F}_p$-module spectra. This is a symmetric monoidal category under $\otimes = \otimes_{\mathbb{F}_p}$. Let $\mathbb{P}$ be the free $\mathbb{E}_\infty$ algebra monad on $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$,

$$\mathbb{P}V = \bigoplus_{n\geq 0} \mathbb{P}_n V, \qquad \mathbb{P}_n V = V^{\otimes n}_{h\Sigma_n},$$

and $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ the resulting category of $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$. We will also write $\widetilde{\mathbb{P}} = \bigoplus_{n\geq 1}\mathbb{P}_n$ for the free nonunital $\mathbb{E}_\infty$ algebra monad, and generally identify its category of algebras with the category $\mathcal{C}\mathrm{Alg}^{\mathrm{aug}}_{\mathbb{F}_p} = \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}/\mathbb{F}_p$. Let $\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p} \subset \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ denote the essential image of $\mathbb{P}$. Both $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ and $\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p}$ are theories, and so too are their homotopy categories $\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ and $\mathrm{h}\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p}$. The former is easily identified:

$$\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p} \simeq \mathcal{M}\mathrm{od}^{\heartsuit}_{\mathbb{F}_{p*}}, \qquad \mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p}} \simeq \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}},$$

where we write $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$ for the category of $\mathbb{Z}$-graded $\mathbb{F}_p$-modules, which we will at times simply call $\mathbb{F}_{p*}$-modules. Moreover, the symmetric monoidal structure on $\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ obtained from that on $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ is exactly the standard symmetric monoidal structure on $\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathbb{F}_{p*}}$ with symmetrizer obeying the Koszul sign rule. The theory $\mathrm{h}\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p}$ is more complicated, but abstractly we can say that it is exactly the theory of operations acting on the homotopy groups of $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$. For example, following Proposition 2.1.5, we can identify

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p},\mathcal{S}\mathrm{et})}(\pi_p,\pi_q) \simeq \pi_q\mathbb{P}\Sigma^p\mathbb{F}_p \simeq \mathrm{Hom}_{h\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p}}(\mathbb{P}\Sigma^q\mathbb{F}_p,\mathbb{P}\Sigma^p\mathbb{F}_p).$$

By construction, $\mathbb{P}$ induces a map $\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p} \to \mathrm{h}\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p}$ of theories, and restriction along this makes the category of $\mathrm{h}\mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p}$-models strongly monadic over $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$, so we can view its objects as $\mathbb{F}_{p*}$-modules equipped with some extra structure. Write DL for the resulting monad on $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$. Then we have the following.

5.1.1. **Proposition.** The natural isomorphisms $\mathbb{P}(U\oplus V)\simeq \mathbb{P}U\otimes\mathbb{P}V$ give natural isomorphisms $\pi_*\mathbb{P}(U\oplus V)\simeq \pi_*\mathbb{P}U\otimes\pi_*\mathbb{P}V$ which equip DL with the structure of an exponential monad on $\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathbb{F}_{p*}}$, and thus DL is a $\mathbb{F}_{p*}$-plethory. Moreover, the natural maps $\Sigma\widetilde{\mathbb{P}}V \to \widetilde{\mathbb{P}}\Sigma V$ equip DL with suspensions. □

5.1.2. **Remark.** Essentially by definition, the category $\mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathrm{DL}}$ is equivalent to the category of $\mathbb{H}_\infty$ algebras over $\mathbb{F}_p$. ◁

This is as far as purely formal considerations can take us; to get further, one needs real knowledge of the structure of mod p power operations.

5.2. **Dyer-Lashof operations.** We summarize in this subsection the structure of mod p power operations, showing how it can be nicely packaged into the plethystic framework. We find it most convenient to proceed by introducing some of the relevant algebra first. We begin by recalling an algebra $\mathcal{B}$ of power operations, known as the big Steenrod algebra, or as the Kudo-Araki-May algebra, or by other names. We follow the convention that the binomial coefficient $\binom{n}{m}$ vanishes unless $0 \leq m \leq n$.

5.2.1. **Definition** ($p = 2$)**.** $\mathcal{B}$ is the ordinary $\mathbb{Z}$-graded associative $\mathbb{F}_2$-algebra generated by symbols Q^s of degree s for all $s \in \mathbb{Z}$, and subject to the relations

$$Q^{2s+r+1}Q^s = \sum_{0 \leq i < \frac{r}{2}} \binom{r-i-1}{i} Q^{2s+i+1}Q^{r+s-i}$$

for $r \geq 0$. Here, the bounds of summation are not necessary, but indicate when the binomial coefficients may be nonzero. Given a sequence $I = (r_1, \ldots, r_k)$, write $Q^I = Q^{r_1} \cdots Q^{r_k}$, and call I and Q^I *admissible* if $r_i \leq 2r_{i+1}$ for each i. Define the *excess* of I by $e(I) = r_1 - r_2 - \cdots - r_k$. Given an integer u, call a $\mathcal{B}$-module M *u-unstable* if $Q^r m = 0$ for any $m \in M$ with $r < |m| + u$; when $u = 0$ we omit it from the name and notation. ◁

5.2.2. **Definition** ($p > 2$)**.** $\mathcal{B}$ is the ordinary $\mathbb{Z}$-graded associative $\mathbb{F}_p$-algebra generated by symbols Q^s_ϵ of degree $2s(p-1) - \epsilon$ for $\epsilon \in \{0, 1\}$ and $s \in \mathbb{Z}$ (we abbreviate $Q^s = Q^s_0$, and one often writes instead $Q^s_\epsilon = \beta^\epsilon P^s$), and subject to the relations

$$Q^{ps+r+1}Q^s = \sum_{0 \leq i < \frac{p-1}{p}s} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i+1}Q^{r+s-i},$$

$$Q^{ps+r}Q^s_1 = \sum_{0 \leq i \leq \frac{p-1}{p}r} (-1)^{i} \binom{(p-1)(r-i)}{i} Q^{ps+i}_1 Q^{r+s-i}$$
$$+ \sum_{0 \leq i < \frac{p-1}{p}r} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i}Q^{r+s-i}_1,$$

$$Q^{ps+r+1}_1 Q^s = \sum_{0 \leq i < \frac{p-1}{p}r} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i+1}_1 Q^{r+s-i},$$

$$Q^{ps+r}_1 Q^s_1 = \sum_{0 \leq i < \frac{p-1}{p}r} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i}_1 Q^{r+s-i}_1$$

for $r \geq 0$. Here, the bounds of summation are not necessary, but indicate when the binomial coefficients may be nonzero. Given a sequence $I = (\epsilon_1, r_1, \ldots, \epsilon_k, r_k)$ in $\{0, 1\} \times \mathbb{Z}$, write $Q^I = Q^{r_1}_{\epsilon_1} \cdots Q^{r_k}_{\epsilon_k}$, and call I and Q^I *admissible* if $r_i \leq pr_{i+1} - \epsilon_{i+1}$ for each i. Define the *length* of I to be k and the *excess* of I to be $e(I) = 2r_1 - \epsilon_1 - (2r_2(p-1) - \epsilon_2) - \cdots - (2r_k(p-1) - \epsilon_k)$. Given an integer u, call a $\mathcal{B}$-module M *u-unstable* if $Q^r_\epsilon m = 0$ for any $m \in M$ with $2r - \epsilon < |m|$; when $m = 0$, we omit it from the name and notation. ◁

For any integer u, write F^u for the free u-unstable $\mathcal{B}$-module functor. Then F^u is a quotient algebra of $\mathcal{B}$ in our general sense, as well as of $F^{u'}$ for $u' < u$. Write $E = e_1$, and for $M \in \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$, write $sM = E \otimes M$. If M is an F^u-module, then sM is an F^{u-1}-module, and this provides an isomorphism $(F^u)^E \cong F^{u-1}$; together with the quotient maps $F^{u-1} \to F^u$, this equips each algebra F^u with suspensions, and for the most part allows us to consider just $F = F^0$.

5.2.3. **Lemma.**

(1) $\mathcal{B}$ has a basis consisting of Q^I for all admissible sequences I;
(2) $F(e_n)$ has a basis consisting of $Q^I e_n$ with I admissible of excess at least n.

Proof. This is [Man01, Proposition 11.2, 12.2]; see also [Lur07, Lectures 6-7] for a detailed algebraic proof when $p = 2$ which, at least if one assumes the analogous fact for unstable modules over the Steenrod algebra [Sch94, Proposition 1.6.2], readily generalizes to $p > 2$. □

5.2.4. **Remark.** The abelian category $\mathrm{L}\mathcal{M}\mathrm{od}_F^\heartsuit$ has, of course, enough projectives, given by the free F-modules. In addition, the right adjoint $F^\vee$ supplies it with enough injectives. By definition, we have

$$F^\vee(e_a)_b = \mathcal{M}\mathrm{od}_{\mathbb{F}_p}(F(e_b), e_a),$$

so the modules $F^\vee(e_a)$ can be seen as $\mathbb{Z}$-graded analogues of the Brown-Gitler modules seen in the study of unstable modules over the Steenrod algebra. ◁

5.2.5. **Definition.** A DL-*ring* is a graded commutative $\mathbb{F}_{p*}$-ring equipped with an F-module structure such that

(1) $Q^0(1) = 1$, and otherwise $Q^r_\epsilon(1) = 0$.
(2) We have
  (a) For $p = 2$, $Q^r x = x^2$ when $r = |x|$;
  (b) For $p > 2$, $Q^r x = x^p$ when $2r = |x|$.
(3) We have
  (a) For $p \geq 2$, $Q^r(xy) = \sum_{i+j=r} Q^i(x)Q^j(y)$;
  (b) For $p > 2$, $Q^r_1(xy) = \sum_{i+j=r}(Q^i_1(x)Q^j(y) + (-1)^{|x|}Q^i(x)Q^j_1(y))$. ◁

Here we have reassigned the name DL from Subsection 5.1; this will justified shortly. From the definition, we see that DL-rings are the models of a finite product theory living over $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}^\heartsuit$; we write the associated monad as DL. Let DL_n denote the free $\mathbb{F}_{p*}$-ring on symbols $Q^I e_n$ where I is an admissible sequence satisfying $e(I) > n$, graded so that $|Q^I e_n| = |Q^I| + n$. Then there is an evident map $\mathrm{DL}_n \to \mathrm{DL}(e_n)$ of $\mathbb{F}_{p*}$-rings described by the action of $\mathcal{B}$ on the canonical element of $\mathrm{DL}(e_n)_n$.

5.2.6. **Theorem** ([BMMS86, III.1.1, IX.2.1])**.** The structure of mod p power operations can be summarized as follows.

(1) The homotopy groups of any object of $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ naturally form a DL-ring, and the resulting maps $\mathrm{DL}_n \to \mathrm{DL}(e_n) \to \pi_*\mathbb{P}\Sigma^n\mathbb{F}_p$ are isomorphisms;
(2) In particular, DL agrees with the $\mathbb{F}_{p*}$-plethory of Subsection 5.1, and is a smooth $\mathbb{F}_{p*}$-plethory with suspensions;
(3) We can identify $\Gamma(\mathrm{DL}) \cong F$ and $\Delta(\mathrm{DL}) \cong F^1$, and the suspension map $\sigma\colon \Delta(\mathrm{DL})^E \to \Gamma(\mathrm{DL})$ is given by the isomorphism $(F^1)^E \cong F$. □

5.2.7. **Example.** The costabilization $\lim_{n\to\infty}\Gamma(\mathrm{DL})^{E^n}$ of DL can be identified as the completion of $\mathcal{B}$ with respect to excess. This object arises naturally when considering the endomorphism spectrum of the forgetful functor $U\colon \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}\to \mathcal{S}\mathrm{p}$; see [Lur07, Lecture 24] and [GL20, Section 10]. ◁

Throughout the rest of this section, we will abbreviate $\Delta = \Delta(\mathrm{DL}) = F^1$.

5.3. **Cohomology of** DL**-rings.** The general aspects of the cohomology of DL-rings are as given in Subsection 4.4. Explicitly, let $R\in\mathcal{R}\mathrm{ing}_{\mathrm{DL}}^{\heartsuit}$, $S\in\mathcal{R}\mathrm{ing}_{R\otimes\mathrm{DL}}^{\heartsuit}$, $A\in\mathcal{R}\mathrm{ing}_{R\otimes\mathrm{DL}/S}^{\heartsuit}$, and $M\in\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{R\otimes\mathrm{DL}/S}^{\heartsuit})\simeq \mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\heartsuit}$. Here, $S\otimes\Delta$ has multiplication given by

$$(s\otimes Q^r)\cdot(s'\otimes Q^{I'}) = \sum_{i+j=r} s\,Q^i(s')\otimes Q^jQ^{I'},$$

and if $p>2$, then

$$(s\otimes Q_1^r)\cdot(s'\otimes Q^{I'}) = \sum_{i+j=r}\left(s\,Q_1^i(s')\otimes Q^jQ^{I'} + (-1)^{|s'|}s\,Q^i(s')\otimes Q_1^jQ^{I'}\right).$$

We can then identify

$$\mathcal{H}_{R\otimes\mathrm{DL}/S}(A;M)\simeq \mathcal{E}\mathrm{xt}_{S\otimes\Delta}(S\otimes_A\Omega_{A|R},M),$$

where the terms in this expression are to be interpreted in the derived sense. In particular, if A is smooth over R, then $H^*_{R\otimes\mathrm{DL}/S}(A;M)$ is given by ordinary Ext groups over the algebra $S\otimes\Delta$.

There is a complementary method by which these Quillen cohomology computations can, in certain cases, be reduced to linear computations. Observe that the forgetful functor $\mathcal{R}\mathrm{ing}_{\mathrm{DL}}^{\heartsuit}\to\mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$ admits a left adjoint $\overline{S}_F$, described by

$$\overline{S}_FM = \begin{cases} SM/(Q^rx = x^2 \text{ for } r=|x|), & \text{for } p=2;\\ SM/(Q^rx=x^p \text{ for } 2r=|x|), & \text{for } p>2.\end{cases}$$

In fact this is already derived, i.e. agrees with its total derived functor S_F on discrete objects, as can be seen from the description

$$\overline{S}_FM = SM\,{}_{\psi}\!\otimes_{S\Psi M}\mathbb{F}_p;$$

$$(\Psi M)_n = \begin{cases} M_{n/p}, & \text{when } 2,p|n;\\ 0, & \text{otherwise;}\end{cases}\qquad \psi(m)=\begin{cases} m^2-Q^{|m|}m, & \text{for } p=2;\\ m^p-Q^{|m|/2}m, & \text{for } p>2.\end{cases}$$

For $M\in\mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$ and $B\in\mathcal{R}\mathrm{ing}_{\mathrm{DL}}$ we identify $\mathrm{Map}_{\mathrm{DL}}(S_FM,B)\simeq\mathrm{Map}_F(M,B)$, providing an approach to computing the cohomology of DL-rings in the image of S_F. This readily extends to other bases than $\mathbb{F}_p$ and to augmented settings. In particular, if $M\in\mathrm{L}\mathcal{M}\mathrm{od}_F$ and $N\in\mathrm{L}\mathcal{M}\mathrm{od}_\Delta$, then

$$\mathcal{H}_{\mathrm{DL}/\mathbb{F}_p}(S_FM;N)\simeq\mathcal{E}\mathrm{xt}_F(M,N).$$

5.3.1. **Example.** The module e_n always carries the F-module structure where each Q_ϵ^r acts by zero; this is the action arising from the augmentation on F. If $n\geq 0$, then e_{-n} carries another F-module structure with Q^0 acting by the identity. With this structure, we can identify $S_F(e_{-n})$ as the cohomology of the n-sphere, where if n is even then we must take the "homotopy theorist's even sphere" $J_{p-1}S^n$ [Gra93]. ◁

5.4. **Unstable $\mathcal{A}$-modules.** Let $\mathcal{A}$ denote the mod p Steenrod algebra. As observed by Mandell [Man01, Theorem 1.4], there is a quotient map

$$\mathcal{B} \to \mathcal{B}/(Q^0 = 1) \cong \mathcal{A}, \qquad \begin{cases} Q^s \mapsto \mathrm{Sq}^{-s}, & \text{when } p = 2; \\ Q^s_\epsilon \mapsto \beta^\epsilon P^{-s}, & \text{when } p > 2. \end{cases}$$

5.4.1. **Definition.** An *unstable $\mathcal{A}$-module*, resp., *unstable $\mathcal{A}$-ring*, is an F-module, resp., DL-ring, whose underlying $\mathcal{B}$-module structure factors through the quotient map $\mathcal{B} \to \mathcal{A}$. ◁

Unstable $\mathcal{A}$-modules and unstable $\mathcal{A}$-rings (more commonly called unstable $\mathcal{A}$-algebras) have been the study of much rich study; see [Sch94] for a textbook account, and [Lur07] for an account that treats the relation with $\mathcal{B}$. Essentially everything we have done, and will do, with F-modules and DL-rings has an analogue for unstable $\mathcal{A}$-modules and unstable $\mathcal{A}$-rings.

Write $\mathcal{U}$ for the $\mathbb{F}_{p*}$-algebra such that $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{U}}^{\heartsuit}$ is the category of unstable $\mathcal{A}$-modules (itself often written $\mathcal{U}$). Write $\mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit}$ for the category of unstable $\mathcal{A}$-rings. By their definition, unstable $\mathcal{A}$-rings embed fully faithfully into DL-rings. This is no longer the case at the level of simplicial rings, i.e. the functor $\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$ is no longer fully faithful. The situation here is exactly the same as that appearing in [Man01], and can be dealt with the same way.

5.4.2. **Lemma.** Let $T\colon \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}} \to \mathcal{R}\mathrm{ing}_{\mathcal{U}}$ denote the free unstable $\mathcal{U}$-ring functor. Then for all $n \in \mathbb{Z}$ there is a (homotopy) pushout square

$$\begin{array}{ccc} \mathrm{DL}(e_n) & \xrightarrow{\phi} & \mathrm{DL}(e_n) \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & T(e_n) \end{array}$$

in $\mathcal{R}\mathrm{ing}_{\mathrm{DL}}$, where ϕ classifies the element $e_n - Q^0 e_n$.

Proof. See [Man01, Section 12]. □

If R is a discrete commutative $\mathbb{F}_p$-ring, then R can be viewed as an $\mathbb{E}_\infty$ algebra over $\mathbb{F}_p$. The resulting DL-ring structure on $R = \pi_* R$ is forced by the axioms to satisfy $Q^0 x = x^p$ and otherwise $Q^r_\epsilon x = 0$.

Call a field κ of characteristic p *Artin-Schreier closed* if the map $\lambda \mapsto \lambda - \lambda^p$ is surjective on κ. In particular, this holds if κ is algebraically closed. We can now give the following.

5.4.3. **Proposition.** Let κ be an Artin-Schreier closed field. Then the composite

$$\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}} \to \mathcal{R}\mathrm{ing}_{\kappa\otimes\mathrm{DL}}$$

is fully faithful.

Proof. We first verify this on discrete objects. Fix $R, S \in \mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit}$. Then we have $\mathrm{Hom}_{\mathcal{U}}(R,S) = \mathrm{Hom}_{\mathrm{DL}}(R,S)$, and we must show this to be isomorphic to $\mathrm{Hom}_{\kappa\otimes\mathrm{DL}}(\kappa\otimes R, \kappa\otimes S)$. This is isomorphic to $\mathrm{Hom}_{\mathrm{DL}}(R, \kappa\otimes S)$, so we must show that every map $f\colon R \to \kappa\otimes S$ of DL-rings factors through $\mathbb{F}_p \otimes S \subset \kappa \otimes S$. As Q^0 acts by the identity on R, every such map lands in the fixed points of Q^0 on $\kappa\otimes S$. As Q^0 acts on $\kappa \otimes S$ by $Q^0(\lambda\otimes s) = \lambda^p \otimes s$, the set of fixed points of Q^0 is exactly $\mathbb{F}_p \otimes S$, proving the claim.

Now fix $R, S \in \mathcal{R}\mathrm{ing}_{\mathcal{U}}$ not necessarily discrete, and consider the map

$$\mathrm{Map}_{\mathcal{U}}(R,S) \to \mathrm{Map}_{\mathrm{DL}}(R, \kappa \otimes S)$$

which we are claiming is an isomorphism. As $\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$ is stable under colimits, we may by resolving R reduce to the case where $R = T(e_n)$ is a free unstable $\mathcal{A}$-ring, and so reduce to verify that the map

$$S_n = \mathrm{Map}_{\mathcal{U}}(T(e_n), S) \to \mathrm{Map}_{\mathrm{DL}}(T(e_n), \kappa \otimes S)$$

is an isomorphism. By Lemma 5.4.2 there is a fiber sequence

$$\mathrm{Map}_{\mathrm{DL}}(T(e_n), \kappa \otimes S) \to \kappa \otimes S_n \to \kappa \otimes S_n,$$

where the second map ϕ is given on homotopy groups by $\phi(\lambda \otimes s) = (\lambda - \lambda^p) \otimes s$. The result follows from the long exact sequence in homotopy groups. □

5.5. **The big lambda algebra.** We turn now to the construction of Koszul complexes computing Ext_F. This discussion applies equally well to Ext_Δ, or to Ext_{F^u} for $u \in \mathbb{Z}$. Moreover, it extends to $\mathrm{Ext}_{B \otimes F}$ for $B \in \mathcal{R}\mathrm{ing}_F^\heartsuit$ using Lemma 3.5.7.

These Koszul complexes have a number of predecessors, particularly in the connective setting with work of Miller [Mil78], and in work on the unstable Adams spectral sequence [BK72b]. We have found that working in the full $\mathbb{Z}$-graded setting serves to clarify some of the algebra.

We begin by observing that F is a quadratic algebra. If we write its length grading as $F = \bigoplus_{n \geq 0} F[n]$, then the generating bimodule is

$$F[1](e_a) = \begin{cases} \mathbb{F}_2\{Q^r e_a : r \geq a\}, & \text{when } p = 2; \\ \mathbb{F}_p\{Q^r_\epsilon e_a : 2r - \epsilon \geq a\}, & \text{when } p > 2. \end{cases}$$

The relations are just the image of the relations for $\mathcal{B}$ under $\mathcal{B}[1] \otimes \mathcal{B}[1] \to F[1] \circ F[1]$. Moreover, F is locally finite and its admissible basis gives a PBW decomposition, so by Proposition 3.8.2, it is a Koszul algebra. Thus there is indeed a theory of Koszul resolutions for computing Ext_F, which we obtain as soon as we understand the cohomology of F. We will describe the cohomology of F in two ways. The first method is perhaps more familiar, and proceeds by comparing the cohomology of F with the cohomology of the ordinary algebra $\mathcal{B}$. This is a plausible approach due to the following.

5.5.1. **Lemma.** The surjection $\mathcal{B} \to F$ yields an injection $H^*(F) \subset H^*(\mathcal{B})$.

Proof. It is sufficient to show dually that $H_*(\mathcal{B}) \to H_*(F)$ is a surjection. As F is Koszul, we need consider only the map on diagonal cohomology. This is given in degree m by

$$\bigcap_{i+j=m} B[1]^{\otimes i-1} \otimes R \otimes B[1]^{\otimes j-1} \to \bigcap_{i+j=m} F[1]^{\circ i-1} \circ R' \circ F[1]^{\circ j-1},$$

where $R \subset B[1] \otimes B[1]$ is the bimodule of Adem relations and $R' \subset F[1] \circ F[1]$ its image, so this is clear. □

And it is an appealing approach due to the following.

5.5.2. **Lemma.** Let D be the diagonal cohomology algebra of $\mathcal{B}$, defined with conventions as in [Pri70]. Let $\hat{Q}^r_\epsilon \in D[1]$ be dual to Q^r_ϵ. Then there is an injection $\mathcal{B} \to D$ of algebras, with dense image, given by

$$\begin{cases} Q^r \mapsto \hat{Q}^{-r-1}, & \text{when } p = 2; \\ Q^r_\epsilon \mapsto \hat{Q}^{-r}_{1-\epsilon}, & \text{when } p > 2. \end{cases}$$

Proof. Though not quite stated in this form, [Pri70, Theorem 2.5] gives generators and relations for the diagonal cohomology of an arbitrary ordinary quadratic algebra over a field, with the caveat that if the algebra in question is not locally finite, then these generators may only be topological generators, and the relations obtained may involve infinite sums. In our case, it follows by direct computation that the relations obtained between the topological generators $\hat{Q}^r_\epsilon \in D$ are finite, and upon the indicated change of indices, are exactly the relations defining $\mathcal{B}$. □

From here it is not difficult to proceed to fully describe the cohomology of F. We first lay out some conventions. In the present setting, it is most natural to compute the cohomology of F with conventions that are standard when dealing with $\mathbb{Z}$-graded modules, only with pairings still in reverse order to the Yoneda composition. So for $x \in \operatorname{Ext}^n(e_a, e_b)$ and $y \in \operatorname{Ext}^m(e_b, e_c)$, write

$$xy = (-1)^{n(b-c+m)} y \circ x,$$

where $\circ$ is the Yoneda composition of extensions; in this way our pairings are compatible with the graded opposite of [Pri70] as discussed in Example 3.2.3. Now if we let $\lambda_r \in \operatorname{Ext}^1_{\mathcal{B}}(e_a, e_{a-r-1})$ be the image of Q^r under Lemma 5.5.2 when $p = 2$, and $\lambda^\epsilon_r \in \operatorname{Ext}^1_{\mathcal{B}}(e_a, e_{a-2r(p-1)+\epsilon-1})$ be the image of Q^r_ϵ under Lemma 5.5.2 when $p > 2$, then the multiplicative relations between the λ's are exactly the relations in the graded opposite $\mathcal{B}^{\mathrm{op}}$.

In particular, the subspace of $\operatorname{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ generated under finite sums and products by the λ's is isomorphic, up to shifts in degree, to $\mathcal{B}^{\mathrm{op}}[n]$, and so has basis given by elements λ_I, where if $p = 2$, then $I = (r_1, \ldots, r_n)$ with $2r_i \geq r_{i+1}$ for each i, and if $p > 2$, then $I = (r_1, \epsilon_1, \ldots, r_n, \epsilon_n)$ with $pr_i - \epsilon_i \geq r_{i+1}$ for each i; call these *coadmissible sequences*.

Observe that as F is locally finite, the inclusion $\operatorname{Ext}^n_F(e_a, e_{a-*}) \subset \operatorname{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ lands in the subspace isomorphic to $\mathcal{B}^{\mathrm{op}}[n]$ generated by the elements λ_I. We have now all but proved the following.

5.5.3. **Theorem.** With the above notation, $\operatorname{Ext}^n_F(e_a, e_{a-*}) \subset \operatorname{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ has as basis those λ_I where I is a coadmissible sequence satisfying

(1) If $p = 2$, then $I = (r_1, \ldots, r_n)$ with $r_1 < -a$;
(2) If $p > 2$, then $I = (r_1, \epsilon_1, \ldots, r_n, \epsilon_n)$ with $2r_1 - \epsilon_1 < -a$.

Proof. As F has both generators and admissible basis induced by those of $\mathcal{B}$, we can view F as a subfunctor of $\mathcal{B}$, though not as a subalgebra. With Lemma 5.5.1, we find that $\operatorname{Ext}^n_F(e_a, e_{a-*}) \subset \operatorname{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ has image spanned by those coadmissible λ_I which lift to elements of $\operatorname{Hom}_{\mathbb{F}_p}(\mathcal{B}[1]^{\otimes n} e_a, e_{a-*})$ dual to simple tensors in $F[1]^{\circ n} e_a$.

Consider first $p = 2$, and choose a coadmissible sequence $I = (r_1, \ldots, r_n)$. Then we are asking for $\lambda_{r_1} \otimes \cdots \otimes \lambda_{r_n} \in \operatorname{Hom}_{\mathbb{F}_p}(\mathcal{B}[1]^{\otimes n} e_a, e_{a-*})$ to be dual to an element of $F[1]^{\circ n} e_a$. By definition, this element is dual to $Q^{-r_n-1} \otimes \cdots \otimes Q^{-r_1-1}$, and we require this to satisfy the instability condition

$$-r_{i+1} - 1 \geq (-r_i - 1) + \cdots + (-r_1 - 1) + a$$

for each i. Write $I' = (s_1, \dots, s_n) = (-r_n - 1, \dots, -r_1 - 1)$, so that this instability condition is

$$s_i \geq s_{i+1} + \cdots + s_n + a$$

for each i. Coadmissibility of I is equivalent to the complete unadmissibility condition on I' of $s_i > 2s_{i+1}$ for each i; thus if $s_i \geq s_{i+1} + \cdots + s_n + a$ for some i, then

$$s_{i-1} \geq 2s_i = s_i + s_i \geq s_i + s_{i-1} + \cdots + s_n + a.$$

So in fact the instability condition is equivalent to just $s_n \geq a$, which itself is equivalent to $r_1 < -a$ as claimed.

Consider next $p > 2$. Choose a coadmissible sequence $I = (r_1, \epsilon_1, \dots, r_n, \epsilon_n)$, so that we are asking for $Q^{-r_n}_{1-\epsilon_n} \otimes \cdots \otimes Q^{-r_1}_{1-\epsilon_1}$ to be an element of $F[1]^{\circ n} e_a$. Writing $I' = (\delta_1, s_1, \dots, \delta_n, \epsilon_n) = (1 - \epsilon_n, -r_n, \dots, 1 - \epsilon_1, -r_1)$, the relevant instability condition is

$$2s_i - \delta_i \geq (2s_{i+1}(p-1) - \delta_{i+1}) + \cdots + (2s_k(p-1) - \delta_k) + a$$

for each i. Coadmissibility of I is equivalent to the complete unadmissibility condition on I' of $s_i > ps_{i+1} - \delta_{i+1}$ for each i; thus if the above is satisfied for some i then

$$2s_{i-1} - \delta_{i-1} > 2(ps_i - \delta_i) - \delta_{i-1} \geq 2s_i(p-1) - \delta_i + \cdots + 2s_n(p-1) - \delta_n + a - \delta_{i-1},$$

which in turn implies the instability condition at $i - 1$ as $\delta_{i-1} \in \{0, 1\}$. So we conclude that the instability condition is equivalent to just $2s_n - \delta_n \geq a$, which in turn is equivalent to $2r_1 - \epsilon_1 < -a$. □

For the second approach to computing the cohomology of F, we appeal directly to Theorem 3.6.4.

5.5.4. **Theorem.** For $a \in \mathbb{Z}$, the graded vector space $H^*(F)(e_a)$ has basis given by those λ_I where I is a coadmissible sequence satisfying

(1) If $p = 2$, then $I = (r_1, \dots, r_n)$ with $2r_1 + r_2 + \cdots + r_n + n < -a$;
(2) If $p > 2$, then $I = (r_1, \epsilon_1, \dots, r_n, \epsilon_n)$ with $2(pr_1 - \epsilon_1) + 2r_2(p-1) - \epsilon_2 + \cdots + 2r_n(p-1) - \epsilon_n + n < -a$.

Proof. By Theorem 3.6.4, we can identify $H^*(F) = \widehat{T}(F[1]^\vee, R^\perp)$ where $R \subset F[1] \circ F[1]$ is the image of the Adem relations. Here, to maintain consistency with conventions set out above, we should replace $R^\perp$ with $(1 \otimes t)(R^\perp)$ where $t(x) = (-1)^{|x|} x$. Note

$$F[1]^\vee(e_a)_b = \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}(F[1](e_b), e_a).$$

As before, when $p = 2$, write λ_r for the element of $F[1]^\vee(e_a)$ dual to Q^{-r-1}, with the understanding that this does not exist for all a, and when $p > 2$, write λ_r^ϵ for the element of $F[1]^\vee(e_a)$ dual to $Q^{-r}_{1-\epsilon}$ with the same understanding. Either by appealing to Lemma 5.5.2, or else by carrying out the same sorts of computations, we find that $R^\perp$ is exactly the space of relations opposite to the Adem relations, only restricted to those elements which live in $F[1]^\vee$. Thus $H^*(F)(e_a)$ has basis given by those λ_I where I is coadmissible and λ_I lifts to $F[1]^{\vee \circ n}(e_a)$.

Consider first $p = 2$. Here $F[1]^\vee(e_a)_b$ contains an element $\hat{Q}^{a-b}$ dual to Q^{a-b} whenever $a - b \geq b$, i.e. $2b \leq a$. So $F[1]^\vee(e_a)$ is the space of $\hat{Q}^r$ with $2r \geq a$, and in general $F[1]^{\vee \circ n}(e_a)$ is the space of $\hat{Q}^{r_1} \otimes \cdots \otimes \hat{Q}^{r_n}$ with $2r_i + r_{i+1} + \cdots + r_n \geq a$ for each i. This is the space of $\lambda_{r_1} \otimes \cdots \otimes \lambda_{r_n}$ with $2r_i + r_{i+1} + \cdots + n - i + 1 < -a$

for each i. Coadmissibility of I reduces this condition to the case $i = 1$, which is the condition given.

Consider next $p > 2$. Here $F[1]^\vee(e_a)$ contains an element $\hat{Q}^r_\epsilon$ dual to Q^r_ϵ when $2(pr - \epsilon) \geq a$, so in general $F[1]^{\vee\circ n}(e_a)$ consists of those elements $\hat{Q}^{r_1}_{\epsilon_1} \otimes \cdots \otimes \hat{Q}^{r_n}_{\epsilon_n}$ with $2(pr_i - \epsilon_i) + 2r_{i+1}(p-1) - \epsilon_{i+1} + \cdots + 2r_n(p-1) - \epsilon_n \geq a$ for each i. These are the elements $\lambda^{\epsilon_1}_{r_1} \otimes \cdots \otimes \lambda^{\epsilon_n}_{r_n}$ with $2(pr_i - \epsilon_i) + 2r_{i+1}(p-1) - \epsilon_{i+1} + \cdots + 2r_n(p-1) - \epsilon_n + n - i + 1 < -a$ for each i, and again coadmissibility allows one to reduce to the condition with $i = 1$, where we find the given condition. $\square$

Of course, the preceding theorems are the same theorem, and it is not difficult to directly translate one into the other.

5.5.5. **Remark.** In the above, we have avoided the subtle point that though $\mathcal{B}$ has a PBW basis, we cannot apply Proposition 3.8.2 to deduce that it is Koszul, as the necessary finiteness conditions are not obviously satisfied. Nonetheless $\mathcal{B}$ is Koszul; this was considered in [BC07], and we can give an alternate proof as follows. Let H_u be the subalgebra, in our general sense, of B^{op}, defined so that $H_u(e_a) = H^*(F_u)(e_a) = H^*(F)(e_{a+u})$ is as computed in Theorem 5.5.4, up to the necessary shifts in degree. Then H_u is a locally finite quadratic algebra with PBW basis, so is Koszul. That B^{op}, and thus B, is Koszul follows from the further observation that $B^{\mathrm{op}} \cong \operatorname{colim}_{u\to-\infty} H_u$. $\triangleleft$

We end by noting the following, previously mentioned in Example 3.8.3. The proof is identical to the preceding story, so we omit it.

5.5.6. **Proposition.** The unstable Steenrod algebra $\mathcal{U}$ is Koszul with respect to the length filtration. Moreover, $\operatorname{gr}\mathcal{U} \simeq F/(Q^r : r \geq 0)$, and under this we can identify $\operatorname{Ext}^n_{\operatorname{gr}\mathcal{U}}(e_a, e_b)$ with the subspace of $\operatorname{Ext}^n_F(e_a, e_b)$ spanned by those λ_I with I consisting of nonnegative integers. $\square$

5.6. **Mapping spaces.** In the rest of this section, we explain the relevance of the preceding machinery to the homotopy theory of $\mathbb{E}_\infty$ algebras in characteristic p. We begin with the following.

5.6.1. **Theorem.** Let $R \in \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$, and choose $S \in \mathcal{C}\mathrm{Alg}_R$ such that $R_* \to S_*$ is surjective (such as $S = 0$ or $S = R$). Choose $A, B \in \mathcal{C}\mathrm{Alg}_{R/S}$, and fix a map $\phi\colon A_* \to B_*$ in $\mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}/S_*}$. Let $\mathcal{C}\mathrm{Alg}^\phi_{R/S}(A,B)$ be the space of lifts of ϕ to a map in $\mathcal{C}\mathrm{Alg}_{R/S}$. Then there is a decomposition

$$\mathcal{C}\mathrm{Alg}^\phi_{R/S}(A,B) \simeq \lim_{n\to\infty} \mathcal{C}\mathrm{Alg}^{\phi,\leq n}_{R/S}(A,B),$$

with layers fitting into fiber sequences

$$\mathcal{C}\mathrm{Alg}^{\phi,\leq n}_{R/S}(A,B) \to \mathcal{C}\mathrm{Alg}^{\phi,\leq n-1}_{R/S}(A,B) \to \mathcal{H}^{n+1}_{R_*\otimes\mathrm{DL}/B_*}(\pi_*A;\pi_*\Omega^n F)$$

for $n \geq 1$, where $F = \mathrm{Fib}(B \to S)$. In particular,

(1) There are successively defined obstructions in $H^{n+1}_{R_*\otimes\mathrm{DL}/B_*}(\pi_*A;\pi_*\Omega^n F)$ for $n \geq 1$ to exhibiting a point of $\mathcal{C}\mathrm{Alg}^\phi_{R/S}(A,B)$;

(2) Once a point f of $\mathcal{C}\mathrm{Alg}^\phi_{R/S}(A,B)$ is chosen, there is a fringed spectral sequence of signature

$$E_1^{p,q} = H^{p-q}_{R_*\otimes\mathrm{DL}/B_*}(\pi_*A;\pi_*\Omega^p F) \Rightarrow \pi_q(\mathcal{C}\mathrm{Alg}_{R/S}(A,B), f),$$

with differential $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-1}$.

Proof. This is a special case of [Bal20, Theorem 5.3.1] as applied to $\mathcal{P} = \mathcal{C}\mathrm{Alg}^{\mathrm{free}}_{\mathbb{F}_p}$. ☐

5.6.2. **Example.** Let $\mathcal{F}\mathrm{in}_p$ denote the category of p-finite spaces, i.e. those spaces X such that X is truncated, $\pi_0 X$ is finite, and $\pi_n(X,x)$ is a finite p-group for all $x \in X$ and $n \geq 1$. There is a fully faithful embedding

$$\mathcal{F}\mathrm{in}_p^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_{\overline{\mathbb{F}}_p}, \qquad X \mapsto \overline{\mathbb{F}}_p^{X_+},$$

and this extends to a fully faithful embedding $\mathrm{Pro}(\mathcal{F}\mathrm{in}_p)^{\mathrm{op}} \simeq \mathrm{Ind}(\mathcal{F}\mathrm{in}_p^{\mathrm{op}}) \to \mathcal{C}\mathrm{Alg}_{\overline{\mathbb{F}}_p}$. In particular, given $X, Y \in \mathcal{G}\mathrm{pd}_\infty$ which are simply connected and of finite type, we can identify

$$\mathrm{Map}(X, Y_p^\wedge) \simeq \mathcal{C}\mathrm{Alg}_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p^{Y_+}, \overline{\mathbb{F}}_p^{X_+}).$$

This is a theorem of Mandell [Man01]; see [Lur11] for the p-profinite interpretation. Now the obstruction theory of Theorem 5.6.1, combined with Proposition 5.4.3, can be viewed as an unstable Adams spectral sequence. To note a special case, observe that the construction at the end of Subsection 5.3 readily translates to describe a free functor $S_{\mathcal{U}} \colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\mathrm{aug}}$. We find that in the Massey-Peterson case, where X and Y are pointed simply connected spaces and $H^*Y \cong S_{\mathcal{U}} M$ for some $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{U}}$, we have a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}_{\mathcal{U}}^{p-q}(M; \widetilde{H}^{*-p} X) \Rightarrow \pi_q \mathrm{Map}_*(X, Y_p^\wedge), \qquad d_r^{p,q} \colon E_r^{p,q} \colon E_r^{p+r,q-1}.$$

The Koszul complexes guaranteed by Proposition 5.5.6 recover the lambda complexes for computing the E_1 page of this unstable Adams spectral sequence. ◁

5.6.3. **Example.** Recall from Example 2.3.7 the plethystic functor $U \colon \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_p} \to \mathcal{C}\mathrm{Alg}^{\mathrm{cn}}_{\mathbb{F}_p}$. Write the right adjoint as $\mathbb{A}^1$; then for $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{cn}}_{\mathbb{F}_p}$ we can identify

$$\mathbb{A}^1(R) \simeq \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(\mathbb{F}_p[t], R),$$

where $t \in \pi_0(\mathbb{F}_p[t])$. More generally, for any $a \in \mathbb{Z}$, we can consider the $\mathbb{F}_{p*}$-ring $S(e_a)$ as a differential graded algebra with trivial differential, in this way upgrade it to an object of $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$, and for $A \in \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ consider the space $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(S(e_a), A)$. The following are some comments about what Theorem 5.6.1 gives in this situation.

Observe first that the $\mathbb{F}_{p*}$-plethory DL is augmented over the initial $\mathbb{F}_{p*}$-plethory; this is not a purely formal fact, but is easily seen from the structure of DL. Restriction along the augmentation is itself a plethystic functor $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_{p*}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$. More generally, we have plethystic functors $\mathrm{C}\mathcal{R}\mathrm{ing}_{R/S} \to \mathcal{R}\mathrm{ing}_{R \otimes \mathrm{DL}/S}$ for $R \in \mathcal{R}\mathrm{ing}_{\mathrm{DL}}^{\heartsuit}$ and $S \in \mathcal{R}\mathrm{ing}_{R\otimes\mathrm{DL}}^{\heartsuit}$; write G for the right adjoints to these. The filtration of $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(S(e_a), R)$ guaranteed by Theorem 5.6.1 has layers that can now be identified as

$$\mathcal{H}^{n+1}_{\mathrm{DL}/R_*}(S(e_a), \pi_* \Omega^n R) \simeq \mathrm{Map}_{\mathrm{DL}/R_*}(S(e_a), B^{n+1}_{R_*} \pi_* R^{S^n_+}) \simeq G(B^{n+1}_{R_*} \pi_* R^{S^n_+})_a,$$

where $B^n_{R_*}$ denotes n-fold delooping in the slice category over R_*. Consider now for simplicity the case where R is augmented. We can then identify the resulting spectral sequence for computing $\pi_* \mathcal{C}\mathrm{Alg}^{\mathrm{aug}}_{\mathbb{F}_p}(S(e_a), R)$ as

$$E_1^{p,q} = \mathcal{H}^{p-q}_{\mathrm{DL}/R_*}(S(e_a); \pi_* \Omega^p R) \simeq \mathrm{Ext}^{p-q}_{\Delta}(e_a, s^{-p} R_*) \Rightarrow \pi_q \mathcal{C}\mathrm{Alg}^{\mathrm{aug}}_{\mathbb{F}_p}(S(e_a), R),$$

where Δ acts trivially on e_a. For further simplicity, consider the $p = 2$ case, and write $M = R_*$; we can describe the Koszul complex $K_\Delta(e_a, s^{-*} M)$ computing the above E_1 page as follows. Consider the space of tensors $\lambda_I \otimes m$ where $m \in M$ and $I = (r_1, \ldots, r_k)$ is a coadmissible sequence satisfying $r_1 < -a - 1$ and

$r_1 + \cdots + r_k + k \geq -m$. Now, complete this space to allow for infinite sums of the form $\sum_i \lambda_{I_i} \otimes m_i$ so long as for any fixed $n \in \mathbb{Z}$, there are finitely many nonzero terms involving m_i with $|m_i| = n$. This is $K_\Delta(e_a, s^{-*}M)$. The differential is given by

$$\delta(\lambda_I \otimes m) = \sum_{r\in\mathbb{Z}} \lambda_I \lambda_{-r-1} \otimes Q^r(m).$$

Return now to the special case of $\mathbb{A}^1$. Possibly more well-known is the subspace

$$\mathbb{G}_m(R) = \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(\mathbb{F}_p[t^{\pm 1}], R) \simeq \mathbb{A}^1(R)^\times \subset \mathbb{A}^1(R)$$

of $\mathbb{A}^1(R)$ given by the strict units of R. All path components of $\mathbb{A}^1(R)$ and $\mathbb{G}_m(R)$ are equivalent, so the only difference between these objects is in π_0. The Goerss-Hopkins spectral sequence computing $\pi_*\mathbb{G}_m(R)$, and relevant Koszul complex, has been studied by Fung [Fun19]. The perspective on $\mathbb{G}_m(R)$ afforded by viewing it as the spectrum of units of $\mathbb{A}^1(R)$ extends [Fun19, Proposition 3.11] to show that $\pi_n\mathbb{G}_m(R)$ is always an $\mathbb{F}_p$-vector space for $n > 0$. ◁

5.6.4. **Remark.** In addition to Theorem 5.6.1, there is an obstruction theory for realizing DL-rings as $\mathbb{E}_\infty$ rings. This can be read off [Bal20, Theorem 5.4.7]. ◁

5.7. **André-Quillen-Goodwillie towers.** Fix $R \in \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$, and consider the resulting category $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}$ of augmented $\mathbb{E}_\infty$ algebras over R. We will write $\mathbb{P}\colon \mathcal{M}\mathrm{od}_R \to \mathcal{C}\mathrm{Alg}_R$ for the free functor. For $A \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}$, we then have the topological André-Quillen homology and cohomology of A; write these as $\mathrm{TAQ}^R(A)$ and $\mathrm{TAQ}_R(A)$. Here, $\mathrm{TAQ}_R(A) = \mathcal{M}\mathrm{od}_R(\mathrm{TAQ}^R(A), R)$, and $\mathrm{TAQ}^R\colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R$ can be identified as the unique functor which preserves geometric realizations and sends $\mathbb{P}M$ to M for $M \in \mathcal{M}\mathrm{od}_R$. Put another way, we can identify $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}$ with the category of algebras over the monad $\widetilde{\mathbb{P}}$ on $\mathcal{M}\mathrm{od}_R$, and TAQ^R is left adjoint to the functor obtained by restriction along $\widetilde{\mathbb{P}} \to I$.

We can do something a little more general. Consider the augmentation ideal functor

$$U\colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R.$$

This has a Goodwillie tower, which we will write as

$$U \to \cdots \to P_n^R \to P_{n-1}^R \to \cdots \to P_1^R \to 0.$$

The n'th layer $P_n^R\colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R$ can be identified as the unique functor which preserves geometric realizations and satisfies

$$P_n^R(\mathbb{P}M) = \bigoplus_{1\leq k\leq n} \mathbb{P}_k M.$$

In particular, $P_1^R = \mathrm{TAQ}^R$. See [Kuh06, Section 3] for more on this tower. Now, by restricting P_n^R to $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug,free}}$ and taking homotopy groups, we obtain a functor $\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug,free}} = \mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}}^{\mathrm{aug},\heartsuit,\mathrm{free}} \to \mathcal{M}\mathrm{od}_{R_*}^{\heartsuit}$. By left Kan extension, this extends to a functor

$$\overline{P}_n^{R_*}\colon \mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}}^{\mathrm{aug},\heartsuit} \to \mathcal{M}\mathrm{od}_{R_*}^{\heartsuit}.$$

For the sake of avoiding a conflict of notation, we will denote the total left-derived functor of this by $\mathbb{L}\overline{P}_n^{R_*}$. When $n = 1$, we can identify this as the left adjoint to

$$\mathcal{M}\mathrm{od}_{R_*} \to \mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}}^{\mathrm{aug}}, \qquad M_* \mapsto R_* \ltimes \overline{M}_*.$$

Thus, where $\epsilon\colon R_* \otimes \Delta \to R_*$ is the augmentation and $Q_{R_*}\colon \mathcal{R}\mathrm{ing}^{\mathrm{aug}}_{R_*\otimes \mathrm{DL}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{R_*\otimes\Delta}$ is the functor of derived indecomposables, we have

$$\mathbb{L}\overline{P}_1^{R_*} \simeq \epsilon_! Q_{R_*},$$

which can be computed via a bar construction as in Subsection 3.3.

5.7.1. **Theorem.** Fix notation as above, and fix $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{aug}}_R$. Then there is a convergent spectral sequence in $\mathcal{M}\mathrm{od}^{\heartsuit}_{R_*}$ of signature

$$E^1_{p,q} = s^q \mathbb{L}_{p+q}\overline{P}_n^{R_*}(A_*) \Rightarrow \pi_{*+p} P_n^R(A), \qquad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

There is, in addition, a conditionally convergent spectral sequence of signature

$$E_1^{p,q} = H^{p+q}_{R_*\otimes\mathrm{DL}/R_*}(A_*;\pi_*\Omega^{-p}\overline{R}) \cong \mathrm{Ext}^{p+q}_{R_*\otimes\Delta}(Q_{R_*}(A_*), \overline{R}_{*+p}) \Rightarrow \mathrm{TAQ}^q_R(A),$$

with differential $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q+1}$.

Proof. Given the preceding discussion, the first spectral sequence is a special case of [Bal20, Theorem 4.2.2]. The second spectral sequence can be obtained, for instance, by patching together the filtrations of $\mathcal{C}\mathrm{Alg}^{\mathrm{aug}}_R(A, R \ltimes \Sigma^n \overline{R})$ for various n given by Theorem 5.6.1. ☐

A form of this spectral sequence for topological André-Quillen homology and cohomology was constructed by Basterra [Bas99], and for this reason it is sometimes called the Basterra spectral sequence.

5.7.2. **Example.** Work of Miller [Mil78] produces a spectral sequence converging to H_*X for a connective spectrum X, with initial page depending on $H_*\Omega^\infty X$, and moreover constructs the Koszul complexes relevant for its computation. Although we will not prove that the spectral sequences agree, let us note how some of this story looks from the perspective of Theorem 5.7.1. The main point is that

$$\mathrm{TAQ}^{\mathbb{F}_p}(\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^\infty X) \simeq \mathbb{F}_p \otimes X$$

when X is connective [Kuh06, Example 3.9]. One thus obtains a spectral sequence

$$E^1_{p,q} = s^q \mathbb{L}_{p+q}\overline{P}_1^{\mathbb{F}_{p*}}(H_*\Omega^\infty X) = s^q \pi_{p+q}\epsilon_! Q(H_*\Omega^\infty X) \Rightarrow H_{*+p}X.$$

The underlying ring of $H_*\Omega^\infty X$ can be identified as

$$H_*\Omega^\infty X = \mathbb{F}_p[\pi_0 X] \otimes H_*\Omega^\infty_0 X,$$

where $\mathbb{F}_p[\pi_0 X]$ is an abelian group ring and $H_*\Omega_0^\infty X$ is a connected Hopf algebra. By writing $\pi_0 X$ as a union of its finitely generated abelian subgroups, and by appealing to the structure theory of connected bicommutative Hopf algebras, we find that $Q(H_*\Omega^\infty X)$ is 1-truncated, and so the E_1-page of this spectral sequence somewhat accessible even when $H_*\Omega^\infty X$ is not smooth. ◁

6. Lubin-Tate spectra

This section applies the machinery we have developed to the study of $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum. Before this, in Subsection 6.1, we introduce the notion of an even-periodic plethory, and in Subsection 6.2, we show how a cobialgebroid over an ordinary ring, under some niceness assumptions, gives rise to a formal category scheme. Parts of these stories are present in Rezk's work [Rez09], which also describes the general structure of power operations for Lubin-Tate spectra, and in Subsection 6.3 we recall the relevant parts of this story.

Subsection 6.4 is a technical section, indicating how to deal with some of the issues that arise when incorporating completions into the story. With all of this algebra in place, we reap the benefits in Subsection 6.5 and Subsection 6.6; in particular, we use it to describe an obstruction theory for mapping spaces whose obstruction groups often vanish at heights $h \leq 2$, and give some applications of this.

6.1. **Even-periodic plethories.** Fix an ordinary $\mathbb{Z}$-graded commutative ring $R = R_*$, with associated category $\mathcal{M}\mathrm{od}_R = \mathcal{M}\mathrm{od}_{R_*}$ of $\mathbb{Z}$-graded modules. We consider this as a symmetric monoidal category with symmetrizer employing the Koszul sign rule, and write the tensor product as just $\otimes$. Write E for a copy of R generated in degree 1. The work in this subsection should be viewed as taking place entirely inside various 1-categories, only with the obvious extensions to their derived categories.

6.1.1. **Definition.**

(1) R is *even-periodic* if $R_1 = 0$, and for all $k \in \mathbb{Z}$ the map $R_k \otimes_{R_0} R_2 \to R_{k+2}$ is an isomorphism. The following definitions will be made with this assumption.
(2) An R-cobialgebroid Γ is *even-periodic* if
  (a) As a functor, Γ preserves the category of R-modules which are concentrated in even degrees;
  (b) Γ is equipped with a suspension map $\Gamma^E \to \Gamma$ (Definition 4.5.1) which is an isomorphism on the full subcategory of even modules.
(3) An R-plethory Λ is *even-periodic* if
  (a) Λ preserves the full subcategory of even modules;
  (b) Λ is equipped with an isomorphism $\Delta(\Lambda)^E \to \Gamma(\Lambda)$ which equips Λ with suspensions (Definition 4.5.1) and makes $\Gamma(\Lambda)$ into an even-periodic cobialgebroid. ◁

We suppose for the rest of this subsection that R is even-periodic. Because R is even-periodic, we can identify $R_2 = L$ for some invertible R_0-module L. Let $\mathcal{M}\mathrm{od}_{R_\star} = \mathcal{M}\mathrm{od}_{R_0}^{\times 2}$ be the category of $\mathbb{Z}/(2)$-graded R_0-modules. Then the functor

$$\mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_{R_\star}, \qquad M_* \mapsto (M_0, M_{-1})$$

is an equivalence of categories, for it has essential inverse

$$(M_0, M_{-1}) \mapsto M_*, \qquad M_{2n-\epsilon} = L^n \otimes M_{-\epsilon}.$$

In particular, we can identify $\mathcal{M}\mathrm{od}_{R_0}$ with the full subcategory of $\mathcal{M}\mathrm{od}_R$ spanned by those modules which are concentrated in even degrees. Under the above equivalence, $\mathcal{M}\mathrm{od}_{R_\star}$ is made into a symmetric monoidal category with tensor product

$$\begin{aligned}(M_0, M_{-1}) \otimes (M_0', M_{-1}') = (&M_0 \otimes_{R_0} M_0' \oplus L \otimes_{R_0} M_{-1} \otimes_{R_0} M_{-1}', \\ &M_0 \otimes_{R_0} M_{-1}' \oplus M_{-1} \otimes_{R_0} M_0'),\end{aligned}$$

where the symmetrizer acts with a sign on L. We will abbreviate $\otimes = \otimes_{R_0}$, as no confusion should occur. Now let $L^{1/2} = E^{-1}$, considered as either an object of $\mathcal{M}\mathrm{od}_R$ or $\mathcal{M}\mathrm{od}_{R_\star}$. Then

$$L^{1/2} = (0, R_0), \qquad L^{1/2} \otimes L^{1/2} = L^1 = (L, 0).$$

So for every $M \in \mathcal{M}\mathrm{od}_R$ there are unique R_0-modules, or even R-modules, M_0 and M_{-1} such that

$$M \cong M_0 \oplus L^{1/2} \otimes M_{-1}.$$

Fix next an even-periodic cobialgebroid Γ. We abbreviate $\Gamma_{n,m} = \Gamma(E^n R)(E^m R)$. Even-periodicity of R tells us

$$L \otimes \Gamma_{n,m} = \Gamma_{n,m+2}, \qquad \Gamma_{n,m} \otimes L = \Gamma_{n-2,m}.$$

The assumption that Γ preserves even objects tells us that $\Gamma_{n,m} = 0$ if n and m are of different parity. The suspension maps for Γ are maps $\Gamma_{n-1,m-1} \to \Gamma_{n,m}$, and even-periodicity of Γ tells us that these are isomorphisms when n and m are even. The algebra structure on Γ is now essentially determined by $\Gamma_{0,0}$, and there is an equivalence of categories $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star}$ overlying the equivalence $\mathcal{M}\mathrm{od}_R \simeq \mathcal{M}\mathrm{od}_{R_\star}$, where $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star} = \mathrm{L}\mathcal{M}\mathrm{od}^{\times 2}_{\Gamma_{0,0}}$ is the category of $\mathbb{Z}/(2)$-graded $\Gamma_{0,0}$-modules.

The coproduct on Γ is encoded by maps

$$\Delta^\times \colon \Gamma_{n+n',m+m'} \to \Gamma_{n,m} \otimes_{R_0} \Gamma_{n',m'}.$$

which for $n = n' = 0 = m = m'$ contribute to the R_0-cobialgebroid structure of $\Gamma_{0,0}$. As $\mathcal{M}\mathrm{od}_{\Gamma_\star}$ is strongly monoidal over $\mathcal{M}\mathrm{od}_{R_\star}$, we know that its tensor product takes the form

$$\begin{aligned}(M_0, M_{-1}) \otimes (M'_0, M'_{-1}) = (&M_0 \otimes M'_0 \oplus L \otimes M_{-1} \otimes M'_{-1},\\ &M_0 \otimes M'_{-1} \oplus M_{-1} \otimes M'_0).\end{aligned}$$

But this does not fully describe the tensor product; missing is a description of the $\Gamma_{0,0}$-module structure. On the summands $M'_0 \otimes M''_0$, $M'_0 \otimes M''_{-1}$, and $M'_{-1} \otimes M''_0$, this action is obtained just from the coproduct on $\Gamma_{0,0}$ and isomorphism $\Gamma_{-1,-1} \cong \Gamma_{0,0}$, so consider the remaining summand $L \otimes M'_{-1} \otimes M''_{-1}$. Here by definition the action arises via the map

$$\Gamma_{0,0} \cong L \otimes \Gamma_{-2,-2} \otimes L^{-1} \to L \otimes \Gamma_{-1,-1} \otimes \Gamma_{-1,-1} \otimes L^{-1} \cong L \otimes \Gamma_{0,0} \otimes \Gamma_{0,0} \otimes L^{-1}.$$

On the other hand, there is an action of $\Gamma_{0,0}$ obtained from the coproduct of $\Gamma_{0,0}$ and its action on L by way of the double suspension $\Gamma_{0,0} \to \Gamma_{2,2}$. These actions agree; the definition of a cobialgebroid with suspensions was chosen in order to make this so. Thus we have a full understanding of $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star}$ as a symmetric monoidal category. We can summarize the situation as follows.

6.1.2. **Proposition.** Let Γ be an even-periodic R-cobialgebroid. In particular, $\Gamma_{0,0}$ is an R_0-cobialgebroid, and there is a chosen $\Gamma_{0,0}$-module structure on $L = R_2$. Then there is an equivalence of symmetric monoidal categories

$$\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star},$$

where $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star} = \mathrm{L}\mathcal{M}\mathrm{od}^{\times 2}_{\Gamma_{0,0}}$ is the category of $\mathbb{Z}/(2)$-graded $\Gamma_{0,0}$-modules, with symmetric monoidal product given by

$$\begin{aligned}(M_0, M_{-1}) \otimes (M'_0, M'_{-1}) = (&M_0 \otimes M'_0 \oplus L \otimes M_{-1} \otimes M'_{-1},\\ &M_0 \otimes M'_{-1} \oplus M_{-1} \otimes M'_0),\end{aligned}$$

where the symmetrizer acts on L with a sign. □

6.1.3. **Remark.** Note that although L is an invertible R_0-module, it is generally not invertible as a $\Gamma_{0,0}$-module. ◁

6.1.4. **Remark.** Let A be an object of $\mathcal{R}\mathrm{ing}^\heartsuit_{\Gamma_\star}$. Given $x, y \in A_{-1}$, one may wish to form their product xy, and to consider how the elements of $\Gamma_{0,0}$ act on this product. However $xy \in A_{-2}$, i.e. this product takes us outside the $\mathbb{Z}/(2)$-graded setting. To remain in the $\mathbb{Z}/(2)$-graded setting, it is more correct to say that the product of elements of A_{-1} is given by a map

$$L \otimes A_{-1} \otimes A_{-1} \to A_0$$

of $\Gamma_{0,0}$-modules. If L is trivializable, then by choosing a trivialization we can treat this as a map $A_{-1} \otimes A_{-1} \to A_0$, but one must not forget the presence of L in considering the interaction of this map with the Γ-module structure. $\triangleleft$

6.1.5. **Example.** Let Γ be the $R = \mathbb{Z}_p[[a]]$-cobialgebroid of Example 3.6.5 and Example 4.2.6. Then Γ upgrades to an even-periodic cobialgebroid over $R[u^{\pm 1}]$, where $|u| = 2$. The Γ-module structure on $\omega = L = R\{u\}$ encoding this is given by

$$Q_0 u = 0, \qquad Q_1 u = -u, \qquad Q_2 u = 0.$$

If M is a Γ-module, then $\omega \otimes M$ consists of elements um for $m \in M$ and has Γ-action

$$Q_0(um) = -2uQ_2(m), \quad Q_1(um) = -uQ_0(m) - auQ_2(m), \quad Q_2(um) = -uQ_1(m).$$

A $\mathbb{Z}/(2)$-graded Γ-ring is then a $\mathbb{Z}/(2)$-graded R-ring A equipped with an action of Γ such that if either $x \in A_0$ or $y \in A_0$, then Γ acts on xy via Example 4.2.6; and if $x, y \in A_{-1}$, then Γ acts on $xy \in A_0$ by

$$\begin{aligned}
Q_0(xy) = &- 2Q_0(x)Q_2(y) - 2Q_2(x)Q_0(y) - 2Q_1(x)Q_1(y) - 2aQ_2(x)Q_2(y),\\
Q_1(xy) = &- Q_0(x)Q_0(y) - 2Q_1(x)Q_2(y) - 2Q_2(x)Q_1(y)\\
&- aQ_0(x)Q_2(y) - aQ_2(x)Q_0(y) - aQ_1(x)Q_1(y) - a^2Q_2(x)Q_2(y),\\
Q_2(xy) = &- Q_0(x)Q_1(y) - Q_1(x)Q_0(y)\\
&- aQ_1(x)Q_2(y) - aQ_2(x)Q_1(y) - 2Q_2(x)Q_2(y).
\end{aligned}$$

$\triangleleft$

Now fix an even-periodic R-plethory Λ. Abbreviate $\Gamma = \Gamma(\Lambda)$ and $\Delta = \Delta(\Lambda)$. We can picture the relevant suspension maps as fitting into a diagram

$$\begin{array}{ccccccccccc}
 & & \Gamma_{-1,-1} & & \Gamma_{0,0} & & \Gamma_{1,1} & & \Gamma_{2,2} & & \Gamma_{3,3}\\
 & \overset{\simeq}{\nearrow} & \downarrow{\scriptstyle\simeq} & \overset{\simeq}{\nearrow} & \downarrow & \overset{\simeq}{\nearrow} & \downarrow{\scriptstyle\simeq} & \overset{\simeq}{\nearrow} & \downarrow & \overset{\simeq}{\nearrow} & \\
\Delta_{-2,-2} & & \Delta_{-1,-1} & & \Delta_{0,0} & & \Delta_{1,1} & & \Delta_{2,2} & &
\end{array}.$$

As R is even-periodic, tensoring with L yields canonical equivalences of categories $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_{n,n}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_{n+2,n+2}}$ and $\mathrm{L}\mathcal{M}\mathrm{od}_{\Delta_{n,n}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Delta_{n+2,n+2}}$ for each $n \in \mathbb{Z}$. Coupling these with the isomorphisms in the above diagram yields the following.

6.1.6. **Proposition.** There is a canonical Morita equivalence

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\Delta} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma},$$

and the composite

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Delta} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma}$$

is given by $L^{1/2} \otimes -$. $\square$

6.2. **Quasicoherent sheaves.** This subsection takes place in the 1-categorical setting. Fix an ordinary commutative ring R and cobialgebroid Γ. If we forget the algebra and right R-module structures on Γ, then we are left with nothing more than a (counital, coassociative, cocommutative) R-coalgebra. Under suitable niceness assumptions, R-coalgebras give one approach to the theory of formal schemes over R; this is best known when R is a field, see for instance [Dem72, Section I.6]. It turns out we can understand the rest of the structure of Γ in this way, under suitable niceness assumptions. We will freely use the language of formal schemes as developed in [Str99], and in particular the technical notions of solid formal schemes and coalgebraic formal schemes; however, we will write $\mathrm{Sp}^\vee$ for what is written there as sch. We will abbreviate "coalgebra with good basis" to "good coalgebra".

6.2.1. **Proposition.** Let Γ be an R-cobialgebroid which is good as an R-coalgebra. Then the pair $(\operatorname{Spec} R, \mathrm{Sp}^\vee \Gamma)$ naturally carries the structure of a formal category scheme.

Proof. Fix Γ. We must describe the structure of a category object on the pair $(\operatorname{Spec} R, \mathrm{Sp}^\vee \Gamma)$. The source map $s\colon \mathrm{Sp}^\vee \Gamma \to \operatorname{Spec} R$ is simply the map arising from the definition of $\mathrm{Sp}^\vee \Gamma$ as a formal R-scheme. In describing the remaining maps, we make use of the fact that $\mathrm{Sp}^\vee \Gamma$ is a solid formal scheme, so it suffices to work with $\Gamma^\vee = \mathrm{L}\mathcal{M}\mathrm{od}_R(\Gamma, R)$ as a topological ring. The target map $t\colon \mathrm{Sp}^\vee \Gamma \to \operatorname{Spec}(R)$ is dual to the map of formal rings

$$t\colon R \to \Gamma^\vee, \qquad t(r)(\gamma) = \epsilon(\gamma r).$$

The unit map $\iota\colon \operatorname{Spec} R \to \mathrm{Sp}^\vee \Gamma$ is dual to the map of formal rings

$$\iota\colon \Gamma^\vee \to R, \qquad \iota(f) = f(1).$$

To define the composition map $c\colon \mathrm{Sp}^\vee \Gamma_s \times_{\operatorname{Spec} R, t} \mathrm{Sp}^\vee \Gamma \to \Gamma$, observe first that $\mathrm{Sp}^\vee \Gamma\,{}_s{\times}_{\operatorname{Spec} R,t} \mathrm{Sp}^\vee \Gamma$ is a solid formal scheme represented by

$$\Gamma^\vee{}_s \mathbin{\widehat{\otimes}}_{R,t} \Gamma^\vee = \mathrm{L}\mathcal{M}\mathrm{od}_R(\Gamma, R) \mathbin{\widehat{\otimes}}_R \mathrm{L}\mathcal{M}\mathrm{od}_R(\Gamma, R).$$

As Γ admits a good basis, we may write $\Gamma \cong \operatorname{colim}_\alpha \Gamma_\alpha$ where $\Gamma_\alpha \subset \Gamma$ is a standard coalgebra, in which case $\Gamma^\vee \cong \lim_\alpha \Gamma_\alpha^\vee$ as a topological ring with each $\Gamma_\alpha^\vee$ discrete and finitely generated free as a right R-module. So we have

$$\begin{aligned}
\Gamma^\vee{}_s \mathbin{\widehat{\otimes}}_{R,t} \Gamma^\vee &\cong (\lim_\alpha \Gamma_\alpha^\vee) \mathbin{\widehat{\otimes}} \Gamma^\vee \\
&\cong \lim_\alpha (\Gamma_\alpha \otimes_R \Gamma^\vee) \\
&\cong \lim_\alpha (\Gamma \otimes_R \Gamma_\alpha)^\vee \cong (\Gamma\,{}_r{\otimes}_{R,l} \Gamma)^\vee.
\end{aligned}$$

A similar argument can be used to show that also $\Gamma\,{}_r{\otimes}_{R,l} \Gamma$ is itself a good R-coalgebra, and the above then gives an isomorphism

$$\mathrm{Sp}^\vee \Gamma\,{}_s{\times}_{\operatorname{Spec} R,t} \mathrm{Sp}^\vee \Gamma \cong \mathrm{Sp}^\vee(\Gamma\,{}_r{\otimes}_l \Gamma)$$

of formal R-schemes. The composition map is now dual to the product on Γ. That $(\operatorname{Spec} R, \mathrm{Sp}^\vee \Gamma)$ is a category scheme with this structure amounts to a straightforward translation between definitions. ☐

6.2.2. **Remark.** Fix a good R-cobialgebroid Γ. A Γ-module can be given by an R-module M together with a suitable coaction $M \to \Gamma^\vee(M) \cong \Gamma^\vee{}_s \mathbin{\widehat{\otimes}}_R M$. By definition, an object of $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit$ is given by an R-ring A equipped with a Γ-module structure satisfying certain compatibility conditions; these can now be nicely summarized as simply asking that the map $A \to \Gamma^\vee{}_s \widehat{\otimes}_R A$ is a homomorphism of rings. ◁

6.2.3. **Example.** Let Γ be the $R = \mathbb{Z}_p[[a]]$-cobialgebroid of Example 3.6.5 and Example 4.2.6. The length grading $\Gamma = \bigoplus_{n\geq 0}\Gamma[n]$ is a decomposition of R-coalgebras, so as each $\Gamma[n]$ is finitely generated and free over R, the coalgebra Γ is good, and

$$\mathrm{Sp}^\vee \Gamma \cong \coprod_{n\geq 0} \mathrm{Spec}\,\Gamma[n]^\vee.$$

As Γ is quadratic, we find that a Γ-module is equivalently an R-module M equipped with an R-linear map $P\colon M \to \Gamma[1]^\vee{}_s\otimes_R M$ such that there exists a factorization through the dashed arrow in the diagram

$$\begin{array}{ccc} M & \xrightarrow{P} & \Gamma[1]^\vee{}_s\otimes_R M \\ \downarrow & & \downarrow{\scriptstyle \Gamma[1]^\vee\otimes P} \\ \Gamma[2]^\vee{}_s\otimes_R M & \xrightarrow{c\otimes M} & \Gamma[1]^\vee{}_s\otimes_{R,t}\Gamma[1]^\vee{}_s\otimes_R M \end{array}.$$

Where $d \in \Gamma[1]^\vee$ is dual to Q_1, we can identify

$$\Gamma[1]^\vee \cong R[[d]]/(d^3 = ad+2); \qquad t\colon R \to \Gamma[1]^\vee, \quad t(a) = -ad^2 + 3d + a^2,$$

and R-linearity of P is with respect to the action of R on $\Gamma[1]^\vee$ through t. The category $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit$ is more pleasantly described from this perspective: if A is an R-ring, then a Γ-module structure map $P\colon A \to \Gamma[1]^\vee{}_s\otimes_R A \cong A[[d]]/(d^3 = ad+2)$ makes A into a Γ-ring precisely when P is a ring homomorphism. Additional utility of this viewpoint arises from the fact that various Γ-rings are most naturally described via such ring homomorphisms, and extracting coefficients to translate into the action $\Gamma[1]\,{}_r\otimes_R A \to A$ need not be straightforward. Understanding the relations in Γ from this perspective is a little less pleasant; given Γ, the easiest way to construct $\Gamma[2]^\vee$ together with the ring map

$$c\colon \Gamma[2]^\vee \to \Gamma[1]^\vee{}_s\otimes_{R,t}\Gamma[1]^\vee \cong R[[d',d]]/(d^3 = ad+2, d'^3 = (-ad^2+3d+a^2)+2)$$

is to realize c as the kernel of the map $\Gamma[1]^\vee{}_s\otimes_{R,t}\Gamma[1]^\vee \to R^\vee$ dual to the inclusion of the bimodule of relations R of Γ. It turns out one can do a bit better: there is a Cartesian square

$$\begin{array}{ccc} \Gamma[2]^\vee & \xrightarrow{c} & \Gamma[1]^\vee{}_s\otimes_{R,t}\Gamma[1]^\vee \\ \downarrow & & \downarrow{\scriptstyle f} \\ R & \xrightarrow{s} & \Gamma[1]^\vee \end{array},$$

where $f(d) = d$ and $f(d') = a - d^2$, so one can avoid dealing directly with $\Gamma[2]^\vee$. We note that although the identifications of this example can be readily computed given the structure of Γ, doing so is backwards, as Γ is itself computed from $\Gamma[1]^\vee$ and the above square, which are themselves computed using the interpretation of the category scheme $(\mathrm{Spec}\,R, \mathrm{Sp}^\vee\Gamma)$ that we recall in Subsection 6.3. ◁

Let $\mathcal{M}\mathrm{od}^\heartsuit\colon \mathcal{C}\mathcal{R}\mathrm{ing}^\heartsuit \to \mathcal{C}\mathrm{at}$ denote the pseudofunctor

$$R \mapsto \mathcal{M}\mathrm{od}_R^\heartsuit, \qquad (f\colon R \to S) \mapsto S\otimes_R -.$$

Given some other pseudofunctor $\mathfrak{X}\colon \mathcal{C}\mathcal{R}\mathrm{ing}^\heartsuit \to \mathcal{C}\mathrm{at}$, define $\mathfrak{QCoh}(\mathfrak{X})^\heartsuit$ to be the category of pseudonatural transformations $\mathfrak{X}^{\mathrm{op}} \to \mathcal{M}\mathrm{od}^\heartsuit$. This is an additive symmetric monoidal category.

6.2.4. **Lemma.** Let Γ be a good R-cobialgebroid, and $\mathcal{X} = (\operatorname{Spec} R, \operatorname{Sp}^\vee \Gamma)$. Then there is an equivalence of symmetric monoidal categories $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit \simeq \mathcal{QC}\mathrm{oh}(\mathcal{X})^\heartsuit$.

Proof. There is a well-known symmetric monoidal equivalence between the category of comodules for a commutative Hopf algebroid and the category of quasicoherent sheaves on the associated presheaf of groupoids [Hov01]. The claim at hand is no different from this, so we will indicate the construction but omit detailed verifications of naturality.

The functor $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit \to \mathcal{QC}\mathrm{oh}(\mathcal{X})^\heartsuit$ is constructed as follows. Fix $M \in \mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit$, so we wish to define $\mathcal{F}_M \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{M}\mathrm{od}^\heartsuit$; to that end, we define the functor $\mathcal{F}_M^S \colon \mathcal{X}(S)^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_S^\heartsuit$ as follows. Fix $g \in \mathcal{X}(S)$, realized by a map $g \colon R \to S$. Then $\mathcal{F}_M(g) = S \otimes_R M$. Fix $\alpha \colon g' \to g$ in $\mathcal{X}(S)$, realized by a diagram

$$\begin{array}{ccc} & & \operatorname{Spec} R \\ & \overset{g}{\nearrow} & \uparrow t \\ \operatorname{Spec} S & \xrightarrow{\alpha} & \operatorname{Sp}^\vee \Gamma \\ & \underset{g'}{\searrow} & \downarrow s \\ & & \operatorname{Spec} R \end{array} .$$

Then α is dual to a map $\alpha \colon \Gamma^\vee \to S$ of formal rings, i.e. one that factors through some discrete quotient $\Gamma_\alpha^\vee$, where $\Gamma_\alpha \subset \Gamma$ is a standard coalgebra. Then we have the map

$$\begin{aligned} \mathcal{F}_M(g) = S\,{}_g{\otimes_R}\, M \to S\,{}_g{\otimes_R}\, \Gamma^\vee(M) &\cong S\,{}_g{\otimes_{R,t}}\, \Gamma^\vee{}_s\widehat{\otimes}_R M \\ &\to S\,{}_g{\otimes_{R,g}}\, S\,{}_{g'}{\otimes_R}\, M \to S\,{}_{g'}{\otimes_R}\, M = \mathcal{F}_M(g'). \end{aligned}$$

Conversely, the functor $\mathcal{QC}\mathrm{oh}(\mathcal{X})^\heartsuit \to \mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit$ is constructed as follows. Fix $\mathcal{F} \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{M}\mathrm{od}^\heartsuit$. Let $i \in \mathcal{X}(R)$ be classified by the identity of R, and let $M = \mathcal{F}_R(i)$. A $\Gamma^\vee$-comodule structure on M can be defined as follows. Note first that $\mathcal{X}$ extends to a functor on pro-rings in the evident way, so we have a category $\mathcal{X}(\Gamma^\vee) = \operatorname{colim}_\alpha \mathcal{X}(\Gamma_\alpha^\vee)$ and can let $s, t \in \mathcal{X}(\Gamma^\vee)$ be classified by the source and target maps. Then the identity map of $\Gamma^\vee$ corresponds to a map $c \colon s \to t$ in $\mathcal{X}(\Gamma^\vee)$. This gives a $\Gamma^\vee$-linear map

$$\Gamma^\vee{}_t \,\widehat{\otimes}\, M \to \Gamma^\vee{}_s \,\widehat{\otimes}\, M \cong \Gamma^\vee(M)$$

adjoint to the desired coaction $M \to \Gamma^\vee(M)$. □

6.2.5. **Example.** Let $\sigma \colon R \to R$ be a ring homomorphism, and consider the R-cobialgebroid

$$\Gamma = R\langle\psi\rangle/(\psi \cdot r = \sigma(r) \cdot \sigma), \qquad \Delta^\times(\psi) = \psi \otimes \psi, \qquad \epsilon(\psi) = 1;$$

compare Example 4.4.4. Then

$$\Gamma \cong \bigoplus_{n \geq 0} R\{\psi^n\}$$

with ψ^n grouplike, so

$$\operatorname{Sp}^\vee \Gamma \cong \coprod_{n \geq 0} \operatorname{Spec} R.$$

The functor $\mathcal{X}\colon \mathrm{C}\mathcal{R}\mathrm{ing}^{\heartsuit} \to \mathcal{C}\mathrm{at}$ obtained from this R-cobialgebroid sends a ring S to the category $\mathcal{X}(S)$ identified as follows. Objects of $\mathcal{X}(S)$ are maps $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$. Given $f, f' \in \mathcal{X}(S)$, a morphism $\alpha\colon f \to f'$ is given by a decomposition $\operatorname{Spec} S = \coprod_{0\leq n\ll\infty} \operatorname{Spec}(S_n)$ such that $f'|_{\operatorname{Spec}(S_n)} = (\sigma^n)^* f$. ◁

6.3. **Power operations for Morava E-theory.** Let κ be a perfect field of positive characteristic p, and $\mathbb{G}_0 \to \operatorname{Spec}(\kappa) = X_0$ a formal group of finite height h. Let $\mathbb{G} \to X$ be the universal Lubin-Tate deformation [LT66] of this formal group, and E the associated Lubin-Tate spectrum, also referred to as a spectrum of Morava E-theory. We will write $\mathfrak{m} \subset E_0 = \mathcal{O}_X$ for the maximal ideal. By the Goerss-Hopkins-Miller theorem, E is a $K(h)$-local even-periodic $\mathbb{E}_\infty$ ring spectrum, and this assignment is functorial in the input $(\kappa, \mathbb{G}_0)$ and fiberwise isomorphisms. There results a theory of E-power operations acting on the homotopy groups of $K(h)$-local $\mathbb{E}_\infty$ algebras over E, and these are now well-understood conceptually owing to work of Ando, Hopkins, Strickland, and Rezk. The formulation in [Rez09], building on the computation of [Str98], is the most convenient approach for our purposes. It seems easiest, both for the writer and the reader, to collect what we need in one place, so we will summarize some of the structure of these operations in one big statement. Write $\widehat{\mathbb{P}}$ for the free $K(h)$-local $\mathbb{E}_\infty$ algebra monad on $\mathcal{M}\mathrm{od}_E$, so there is a decomposition $\widehat{\mathbb{P}} = L_{K(h)} \bigoplus_{n\geq 0} \widehat{\mathbb{P}}_n$ with $\widehat{\mathbb{P}}_n M = L_{K(h)} M^{\otimes_E n}_{\Sigma_n}$. Write $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ for the category of $K(h)$-local $\mathbb{E}_\infty$ algebras over E; we will abuse terminology and refer to these just as *E-algebras*.

6.3.1. **Theorem** ([Rez09], [Rez12])**.** There is a monad $\mathbb{T}$ on the category of E_*-modules satisfying and determined by the following three points:

(1) The functor $\mathbb{T}$ preserves sifted colimits;
(2) There are natural maps $\mathbb{T}(M_*) \to \pi_* \widehat{\mathbb{P}} M$ for $M \in \mathcal{M}\mathrm{od}_E$ compatible with the monad structures on $\mathbb{T}$ and $\widehat{\mathbb{P}}$, and in particular the homotopy groups of any $A \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ naturally form a $\mathbb{T}$-algebra;
(3) There is a decomposition $\mathbb{T} \cong \bigoplus_{n\geq 0} \mathbb{T}_n$ compatible with the summands $\widehat{\mathbb{P}}_n \subset \widehat{\mathbb{P}}$ such that if M is a finitely generated free E-module then the resulting map $\mathbb{T}_n(M_*) \to \pi_* \widehat{\mathbb{P}}_n M$ is an isomorphism.

In addition,

(4) $\mathbb{T}$ is an exponential monad, thus an E_*-plethory, with exponential structure obtained from the natural isomorphisms $\widehat{\mathbb{P}}_n(M \oplus N) \simeq \bigoplus_{i+j=n} \widehat{\mathbb{P}}_i M \otimes \widehat{\mathbb{P}}_j N$;
(5) $\mathbb{T}$ is an even-periodic plethory, with suspension maps obtained from the natural maps $\Sigma \widehat{\mathbb{P}}_{>0} M \to \widehat{\mathbb{P}}_{>0} \Sigma M$;
(6) $\mathbb{T}$ is smooth, in fact free, relative to alternating E_0-algebras (Example 4.4.5).

Now write $\Gamma = \Gamma(\mathbb{T})_{0,0} \subset \mathbb{T}(E_*)_0$ for the ordinary E_0-cobialgebroid underlying the even-periodic cobialgebroid of $\mathbb{T}$.

(7) Let $\Gamma[n]$ denote the intersection of Γ with $\mathbb{T}(E_*)_{p^n}$. Then $\Gamma = \bigoplus_{n\geq 0} \Gamma[n]$ is a graded algebra. Moreover, this is a decomposition of coalgebras, and each $\Gamma[n]$ is finitely generated free as a left and right E_0-module, so in particular Γ is a good E_0-cobialgebroid. In addition, each $\Gamma[n]^\vee$ is a complete local ring with residue field κ.
(8) Let $\mathcal{X} = (\operatorname{Spec} E_0, \operatorname{Sp}^\vee \Gamma)$ be the formal category scheme associated to Γ, and let $\mathrm{Def} \subset \mathcal{X}$ be the full subcategory spanned by $\operatorname{Spf} E_0$. In other words, Def is the formal category scheme with objects $\operatorname{Spf} E_0$ and morphisms

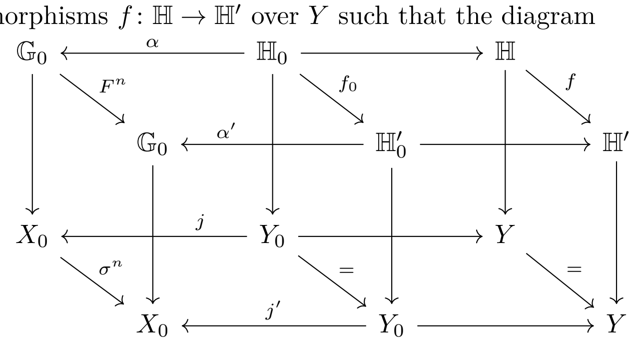
$\coprod_{n\geq 0} \operatorname{Spf}\Gamma[n]^\vee$, where $\Gamma[n]^\vee$ is given its adic topology. Consider Def as a presheaf of categories on the category of formal schemes Y such that $\mathcal{O}_Y$ is a complete local ring equipped with its adic topology. Then $\mathrm{Def}(Y)$ is the category with

(a) Objects: Deformations of $\mathbb{G}_0$ to Y. These can be summarized as diagrams of Cartesian squares

$$\begin{array}{ccccc} \mathbb{G}_0 & \xleftarrow{\alpha} & \mathbb{H}_0 & \longrightarrow & \mathbb{H} \\ \downarrow & & \downarrow & & \downarrow \\ X_0 & \xleftarrow{j} & Y_0 & \longrightarrow & Y \end{array},$$

where $Y_0 \subset Y$ is the special fiber and α is a group homomorphism. We will write these as $\langle \mathbb{H}, j, \alpha\rangle$, or just $\mathbb{H}$ when the remaining structure is understood.

(b) Morphisms: A morphism $f\colon \langle \mathbb{H}, j, \alpha\rangle \to \langle \mathbb{H}', j', \alpha'\rangle$ in $\mathrm{Def}(Y)$ classified by a map landing in the connected component $\operatorname{Spf}\Gamma[n]^\vee$ is a deformation of the n-fold Frobenius of $\mathbb{G}_0$. These can be summarized as homomorphisms $f\colon \mathbb{H} \to \mathbb{H}'$ over Y such that the diagram



commutes. Here, σ^n is the n-fold algebraic Frobenius and F^n is the n-fold absolute Frobenius homomorphism.

(9) Equivalently, $\operatorname{Spf}\Gamma[n]^\vee$ is the formal scheme $\mathrm{Sub}^n_{\mathbb{G}}$ classifying degree p^n subgroups of $\mathbb{G}$. The target map $t\colon \mathrm{Sub}^n_{\mathbb{G}} \to X$ sends a degree p^n subgroup $K \subset \mathbb{H}$, where $\mathbb{H}$ is a deformation of $\mathbb{G}_0$, to the quotient $\mathbb{H}/K$, considered as a deformation of $\mathbb{G}_0$ via the isomorphism $(\mathbb{H}/K)_0 \simeq \mathbb{G}_0/(\mathbb{G}_0[F^n]) \simeq (\sigma^n)^*\mathbb{G}_0$, where $\mathbb{G}_0[F^n]$ is the kernel of the n-fold Frobenius and the second equivalence is given by the n-fold relative Frobenius.

Now,

(10) Write $\omega = \pi_2 E$, so that $\omega = \omega_{\mathbb{G}}$ is the module of invariant differentials on $\mathbb{G}$. Then the Γ-module structure on ω, encoding what is necessary to recover the full even-periodic cobialgebroid $\Gamma(\mathbb{T})$, is given by the quasicoherent sheaf on Def sending a deformation $\mathbb{H}$ to the module of invariant differentials $\omega_{\mathbb{H}}$ on $\mathbb{H}$.

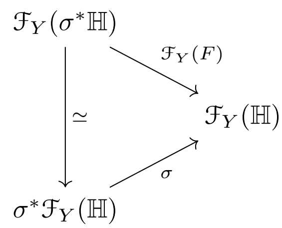
Slightly modifying notation used in Subsection 6.1, write $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit$ for the category of graded modules over Γ and $\mathcal{QC}\mathrm{oh}(\mathrm{Def})^\heartsuit$ for the equivalent category of quasicoherent sheaves of graded modules on Def. Likewise write $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit$ for the category of alternating Γ-rings, and $\mathcal{QC}\mathrm{oh}(\mathrm{Def}, \mathcal{R}\mathrm{ing}^\heartsuit)$ for the equivalent category of alternating rings in $\mathcal{QC}\mathrm{oh}(\mathrm{Def})^\heartsuit$.

(11) The restriction $\mathcal{R}\mathrm{ing}_{\mathbb{T}}^\heartsuit \to \mathcal{R}\mathrm{ing}_\Gamma^\heartsuit$ is fully faithful when restricted to the full subcategory of p-torsion free $\mathbb{T}$-rings. The essential image is spanned by

those p-torsion free Γ-rings B whose underlying ungraded Γ-ring B_0 satisfies either of the following equivalent congruence criteria:

(a) Let $X_1 = \operatorname{Spf}(E_0/(p))$ and $\mathbb{G}_1 = X_1 \times_X \mathbb{G}$. Consider the map $X_1 \to \operatorname{Spf}\Gamma[1]^\vee \cong \operatorname{Sub}^1_{\mathbb{G}}$ classifying $\mathbb{G}_1[F]$, and choose a lift of this to an E_0-linear map $\Gamma[1]^\vee \to E_0$. Dualize this to obtain an element $Q \in \Gamma[1]$ which is well defined in $\Gamma[1]/(p)$. Then $Qx \equiv x^p \pmod{p}$ for all $x \in B_0$.

(b) Let $\mathcal{F}$ be the quasicoherent sheaf of rings associated to B_0. Then for every deformation $\mathbb{H}$ of $\mathbb{G}_0$ to Y where Y lives over $\mathbb{F}_p$, the diagram



commutes. Here, $F\colon \mathbb{H} \to \sigma^*\mathbb{H}$ is the relative Frobenius on $\mathbb{H}$, the left vertical isomorphism arises from pseudonaturality of $\mathcal{F}$, and the bottom diagonal map σ is the algebraic Frobenius on the $\mathcal{O}_Y$-ring $\mathcal{F}_Y(\mathbb{H})$.

Write $\Delta = \Delta(\mathbb{T})_{0,0}$ and, slightly modifying notation used in Subsection 6.1, write $\mathrm{L}\mathcal{M}\mathrm{od}_\Delta$ for the category of graded modules over Δ. So $\mathrm{L}\mathcal{M}\mathrm{od}_\Delta \simeq \mathrm{L}\mathcal{M}\mathrm{od}_\Gamma$ in the manner described in Proposition 6.1.6, and the choice of a coordinate on $\mathbb{G}$ gives an isomorphism of algebras $\Delta \cong \Gamma$. Then

(12) The algebra Δ is graded compatibly with Γ, and both Γ and Δ are Koszul E_0-algebras. Moreover, $H^n(\Delta) = 0$ for $n > h$. In particular, every Δ-module which is projective over E_0 admits a length h projective Koszul resolution, and $\mathrm{L}\mathcal{M}\mathrm{od}_\Delta^\heartsuit$ itself has projective dimension $2h$. □

6.3.2. **Example.** The basic example is given when $\kappa = \mathbb{F}_p$ and $\mathbb{G}_0$ is the formal multiplicative group. In this case, $E = KU_p$ can be identified as the p-completion of complex K-theory, and the full subcategory of $\mathcal{R}\mathrm{ing}_{\mathbb{T}}^\heartsuit$ spanned by the even objects is equivalent to the category of θ-rings over $\mathbb{Z}_p$.

We extend this to identify the full category $\mathcal{R}\mathrm{ing}_{\mathbb{T}}^\heartsuit$ as follows. The $\Gamma = \mathbb{Z}_p[\psi]$-module $\omega = \pi_2 E$ can be identified as $\mathbb{Z}_p\{u\}$, with action $\psi(u) = pu$. Following the discussion of Remark 6.1.4, if $A \in \mathcal{R}\mathrm{ing}_{\mathbb{T}}^\heartsuit$ and $x, y \in A_{-1}$, then $\psi(xy) = p\psi(x)\psi(y) \in A_0$. By considering the generic case, we can factor out this p, and so learn that $\mathcal{R}\mathrm{ing}_{\mathbb{T}}^\heartsuit$ can be identified as the category of $\mathbb{Z}/(2)$-graded alternating rings A over $\mathbb{Z}_p$ together with a θ-ring structure on A_0 and an additive map $\psi\colon A_{-1} \to A_{-1}$ such that if $x \in A_0$ or $y \in A_0$, then $\psi(xy) = \psi(x)\psi(y)$, and if $x, y \in A_{-1}$, then $\theta(xy) = \psi(x)\psi(y)$. ◁

6.3.3. **Example.** Let C_0 be the elliptic curve over a perfect field κ of characteristic 2 with affine equation $v^2 + v = u^3$ and identity $(u, v) = (0, 0)$. This is a supersingular elliptic curve with formal group $\mathbb{G}_0$. The universal deformation $\mathbb{G}$ of $\mathbb{G}_0$ can be identified as the formal group associated to the elliptic curve over $R = W(\kappa)[[a]] = E_0$ with affine equation $v^2 + auv + v = u^3$ and identity $(u, v) = (0, 0)$, and u is a coordinate on this formal group. The structure of power operations for the resulting Lubin-Tate spectrum have been calculated by Rezk [Rez08], and we have recalled the structure of the associated cobialgebroid Γ in Examples 3.6.5, 4.2.6, 6.1.5, and 6.2.3. A congruence element of $\Gamma[1]$ allowing us to recover the full category of

$\mathbb{T}$-rings is given by Q_0. Thus if A is a p-torsion free Γ-ring, with Γ-ring structure on A_0 encoded by a map $P\colon A_0 \to A_0[[d]]/(d^3 = ad + 2)$, we can view A as a $\mathbb{T}$-algebra precisely when $P(a) \equiv a^2 \pmod{d, 2}$ for all $a \in A_0$.

We can generically decompose the operation Q_0 as $Q_0(x) = x^2 + 2\theta(x)$ with $\theta \in \mathbb{T}(E_*)_0$. Thus Δ is generated by θ, Q_1, Q_2, with relations

$$\begin{aligned} \theta a &= a^2\theta - aQ_1 + 3Q_2 \\ Q_1\theta &= Q_2Q_1 - 2\theta Q_2, \\ Q_2\theta &= \theta Q_1 + a\theta Q_2 - Q_1Q_2; \end{aligned}$$

the right action on Q_1 and Q_2 is as before. The map $\Gamma \to \Delta$ is

$$Q_0 \mapsto 2\theta, \qquad Q_1 \mapsto Q_1, \qquad Q_2 \mapsto Q_2.$$

The suspension isomorphism $\Delta \to \Gamma$ is

$$\theta \mapsto -Q_2, \qquad Q_1 \mapsto -Q_0 - aQ_2, \qquad Q_2 \mapsto -Q_1.$$

$\triangleleft$

We would like to apply our understanding of algebraic structures such as $\mathbb{T}$ to obstruction-theoretic machinery for computing with E-algebras. Here, one runs into the subtlety that $\mathbb{T}$ does not perfectly encode the structure of all operations that act on the homotopy groups of E-algebras; in particular, $\mathbb{T}(\pi_*F) \to \pi_*\widehat{\mathbb{P}}F$ is not an isomorphism for F free. Algebraically, the missing piece is the presence of suitable completeness conditions on our $\mathbb{T}$-algebras. Write $\overline{\mathcal{A}}_{\mathfrak{m}}$ for the 0'th left-derived functor of $\mathfrak{m}$-adic completion on the category of E_*-modules. This is a localization, and we will call the $\overline{\mathcal{A}}_{\mathfrak{m}}$-local objects $\mathfrak{m}$-*complete*, and denote the resulting category as $\mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$. Note that this is distinct from the classic notion of $\mathfrak{m}$-adic completeness; these concepts are studied in [HS99, Appendix A] under the name of L-completion, in [Rez18] under the name of analytic completion, and in other places by other names. We will not make any use of the classic notion of $\mathfrak{m}$-adic completeness, so minimal confusion should arise. We review the definitions in Subsection 6.4.

The main theorem of [BF15] gives $\overline{\mathcal{A}}_{\mathfrak{m}}\mathbb{T}$ the structure of a monad; we will not use this theorem, but instead show how it follows easily from the use of algebraic theories and the general philosophy that constructing the category of algebras for a monad can be easier than constructing the monad itself. Write $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$ for the category of E-algebras which are free on a free E-module. Then $\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$ is a discrete theory whose category of discrete models is monadic over $\mathcal{M}\mathrm{od}_{E_*}^{\heartsuit}$; write the associated monad as $\widehat{\mathbb{T}}$.

6.3.4. **Proposition.** The forgetful functor $\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}} \to \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\heartsuit}$ is fully faithful, with essential image spanned by those $\mathbb{T}$-rings whose underlying E_*-module is $\mathfrak{m}$-complete. In particular, $\widehat{\mathbb{T}}$ is a plethory for the theory of $\mathfrak{m}$-complete E_*-modules.

Proof. There is by construction a map $\mathbb{T} \to \widehat{\mathbb{T}}$ of monads on $\mathcal{M}\mathrm{od}_{E_*}^{\heartsuit}$. As $\widehat{\mathbb{T}}$ takes values in $\mathfrak{m}$-complete modules, as a map of functors this factors as $\mathbb{T} \to \overline{\mathcal{A}}_{\mathfrak{m}}\mathbb{T} \to \widehat{\mathbb{T}}$. By general nonsense about localizations of monads [Bal20, Lemmas 6.1.1, 6.1.2], it is sufficient to verify that the map $\overline{\mathcal{A}}_{\mathfrak{m}}\mathbb{T} \to \widehat{\mathbb{T}}$ is an isomorphism of functors. As both source and target preserve geometric realizations, it is sufficient to verify that $\overline{\mathcal{A}}_{\mathfrak{m}}\mathbb{T}(F_*) \to \widehat{\mathbb{T}}(F_*)$ is an isomorphism when F_* is a free E_*-module. Fix such F_*, and write $F_* = \pi_* F$ for a free E-module F. By the construction of the monads $\mathbb{T}$ and $\widehat{\mathbb{T}}$, we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{T}(F_*) & \longrightarrow & \overline{\mathcal{A}}_{\mathfrak{m}}\mathbb{T}(F_*) & \longrightarrow & \widehat{\mathbb{T}}(F_*) \\ \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ \pi_*\widehat{\mathbb{P}}F & \xrightarrow{\simeq} & \pi_*\widehat{\mathbb{P}}F & \xrightarrow{\simeq} & \pi_*\widehat{\mathbb{P}}F \end{array}.$$

Here, the right vertical map is an equivalence by construction, and the middle vertical map is an equivalence as $\mathbb{T}(F_*)$ is free [Rez09, Proposition 4.17]. This proves the proposition. ☐

6.3.5. **Remark.** The abstract construction of $\widehat{\mathbb{T}}$ certainly does not rely on E being a Lubin-Tate spectrum, and with some work the algebraic constructions of Theorem 6.3.1 can also be extended to more general $K(h)$-local even-periodic $\mathbb{E}_\infty$ ring spectra. To set this up correctly would take us too far afield, so we will not do so here. However, let us note how the algebraic story plays out at height $h = 1$. As discussed in [Hop14], there is an equivalence $L_{K(1)}\Sigma^\infty B\Sigma_p \simeq \mathbb{S}_{K(1)}$, using this one can define an operation $\theta \in \pi_0 L_{K(1)}\Sigma^\infty B\Sigma_p$ making π_0 of an arbitrary $K(1)$-local $\mathbb{E}_\infty$ ring spectrum into a θ-ring, and in fact $\pi_0 L_{K(1)}\mathbb{P}_R R$ is the free Ext-p-complete θ-ring on $\pi_0 R$. If R is even-periodic, then this splitting and identification extends to nonzero degrees. Thus $\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,free}}$ is the theory of Ext-p-complete $\mathbb{Z}/(2)$-graded θ-rings equipped with a map from $R_\star$. To be precise, the correct notion of a "$\mathbb{Z}/(2)$-graded θ-ring" must incorporate the $\mathbb{Z}_p[\psi]$-module structure on $\pi_2 R$, in the same manner as it was incorporated in Example 6.3.2. ◁

We end this subsection by pointing to where one can find some computations of the structure of E-power operations. The height $h = 1$ case is more or less simple, and explicit computations at heights $h \geq 3$ are not currently feasible, so we are left with height $h = 2$, where computations are made possible by the theory of elliptic curves. The first full explicit computation in this setting is the computation at $p = 2$ of Rezk [Rez08] recalled in Example 6.3.3. Further computations at $p = 2$ have been carried out by Schumann [Sch14], allowing for elliptic curves with more general Weierstrasse equations. Notably, this work gives a concise description of the total power operation on $E^0 BU(1) \cong R[[u]]$; for the Weierstrasse equation of Example 6.3.3, this is the ring map

$$P\colon R[[u]] \to R[[u]][[d]]/(d^3 - ad - 2), \qquad P(u) = \frac{u^2 - du}{1 + d^2 u}.$$

At $p = 3$, the structure of power operations has been computed by Nendorf [Nen12] and by Zhu [Zhu14]. The latter also describes the power operation structure on $L_{K(1)}E$ for the height $h = 2$ Lubin-Tate spectrum E in question. Further work of Zhu in [Zhu19] gives a recipe that works for arbitrary primes. We point also to [Rez13], which contains a wealth of information at heights $h \leq 2$, and in particular a number of cohomology computations.

6.4. **Completions.** Fix notation as in the preceding section. Following Proposition 6.3.4, we are interested in the homotopy theory of certain completed contexts. We will generally denote completed or $K(h)$-localized constructions with hats, such as we have done with $\widehat{\mathbb{P}} = L_{K(h)}\mathbb{P}$. In [Bal20, Section 6], we studied in general some of the interaction between theories and completions. And in this paper, we have taken care to set things up so that they are already applicable in completed settings. So in this subsection we will content ourselves with just recalling the main definitions and recording some facts we will need.

We use the formulation of completions studied in [Lur18, Chapter 7]. Given $M \in \mathcal{M}\mathrm{od}_{E_*}$, we say that M is $\mathfrak{m}$-*nilpotent* if $M[x^{-1}] = 0$ for all $x \in \mathfrak{m}$, is $\mathfrak{m}$-*local* if $\mathrm{Map}(N, M) \simeq *$ for all $\mathfrak{m}$-nilpotent N, and is $\mathfrak{m}$-*complete* if $\mathrm{Map}(N, M) \simeq *$ for all $\mathfrak{m}$-local N. The full subcategory $\mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m})} \subset \mathcal{M}\mathrm{od}_{E_*}$ of $\mathfrak{m}$-complete modules is a reflective subcategory, and we can give an explicit formula for the reflection $(-)^\wedge_\mathfrak{m}$ as follows. Choose generators $u_0, \ldots, u_{h-1} \in \mathfrak{m}$ and fix $M \in \mathcal{M}\mathrm{od}_{E_*}$. Then $M^\wedge_\mathfrak{m}$ is the total cofiber of the h-cube obtained as the external product of the 1-cubes $T_i - u_i \colon M[[T_i]] \to M[[T_i]]$.

Observe that the preceding definitions can be applied equally well in any linear setting in which there is an action by the elements of $\mathfrak{m}$. In particular, they apply to $\mathcal{M}\mathrm{od}_E$, where $\mathfrak{m}$-completion coincides with $K(h)$-localization. In general, if $\mathcal{M}$ is some category over $\mathcal{M}\mathrm{od}_E$ or $\mathcal{M}\mathrm{od}_{E_*}$, we will write $\mathcal{M}^{\mathrm{Cpl}(\mathfrak{m})} \subset \mathcal{M}$ for the full subcategory spanned by those objects which are $\mathfrak{m}$-complete. Thus for instance Proposition 6.3.4 tells us that $\mathcal{M}\mathrm{odel}^\heartsuit_{\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}} \simeq \mathcal{R}\mathrm{ing}_\mathbb{T}^{\heartsuit,\mathrm{Cpl}(\mathfrak{m})}$. The main reason that this perspective on completions is useful is the fact that this extends to the derived setting.

6.4.1. **Lemma.** We can identify $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}} \simeq \mathcal{R}\mathrm{ing}_\mathbb{T}^{\mathrm{Cpl}(\mathfrak{m})}$.

Proof. This follows from [Bal20, Theorem 6.1.3] and Proposition 6.3.4, as $\overline{\mathcal{A}}_\mathfrak{m}\mathbb{T}(F_*) \simeq \mathbb{T}(F_*)^\wedge_\mathfrak{m}$ for $F_* \in \mathcal{M}\mathrm{od}_{E_*}^{\mathrm{free}}$. ☐

Because of this, we will generally write things in terms of $\mathbb{T}$, although there is a sense in which $\widehat{\mathbb{T}}$ is more fundamental. We note in particular that if $R \in \mathcal{R}\mathrm{ing}_\mathbb{T}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, $S \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, $M \in \mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}/S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, and $A \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}/S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, then

$$\begin{aligned}\mathrm{Map}_{R/\mathcal{R}\mathrm{ing}_\mathbb{T}^{\mathrm{Cpl}(\mathfrak{m})}/S}(A, S \ltimes B^n M) &\simeq \mathrm{Map}_{R\otimes\mathbb{T}/S}(A, S \ltimes B^n M) \simeq \mathcal{H}^n_{R\otimes\mathbb{T}/S}(A; M) \\ &\simeq \mathcal{E}\mathrm{xt}^n_{S\otimes\Delta}(S \otimes_A \Omega_{A|R}, M) \simeq \mathcal{E}\mathrm{xt}^n_{S\otimes\Delta}(S \mathbin{\widehat{\otimes}}_A \widehat{\Omega}_{A|R}, M).\end{aligned}$$

The primary subtlety of completions relevant to us is that Lemma 6.4.1 does not extend to all settings. For example, if R is an E_*-ring, then we can form the category $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})}$ of $\mathfrak{m}$-complete R-modules, and can identify $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_\mathcal{P}$ where $\mathcal{P} \subset \mathcal{M}\mathrm{od}_R$ is the full subcategory spanned by the $\mathfrak{m}$-completions of free R-modules. But the failure of coproducts to be exact in general [Bak09, Appendix B] can force this theory $\mathcal{P}$ to be non-discrete, and in particular $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})}$ need not be the derived category of $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$. Call an object M *tame* if $(M^{\oplus I})^\wedge_\mathfrak{m}$ is discrete for any set I; it is sufficient to consider the case $I = \omega$. Then most of these subtleties vanish so long as we build on tame objects. For example, if $R \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ with R_* tame, then $\pi_*\widehat{\mathbb{P}}_R(R \mathbin{\widehat{\otimes}} F) \simeq \overline{\mathcal{A}}_\mathfrak{m}(R_* \otimes \mathbb{T}(F_*))$ for $F \in \mathcal{M}\mathrm{od}_E^{\mathrm{free}}$, and one can identify $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,free}}} \simeq \mathcal{R}\mathrm{ing}_{R_*\otimes\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m})}$.

We end by noting the following.

6.4.2. **Lemma.** Let $\mathbb{T}$ be as formed at height h. Fix $R \in \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, $S \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}}^{\heartsuit}$, $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, and $A \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}/S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$. If A is smooth as an $\mathfrak{m}$-complete R-ring, then $H^n_{R\otimes\mathbb{T}/S}(A;M) = 0$ for $n > h$.

Proof. More generally, choose $N \in \mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m})}$ which is the completion of a projective S-module; we claim that $\mathrm{Ext}^n_{S\otimes\Delta}(N,M) = 0$ for $n > h$. In the case where $S = E_*$ and N is finitely generated, this is a consequence of Theorem 6.3.1(12). In general, consider the diagram

$$\begin{array}{ccc}
\mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit} & \longrightarrow & \mathcal{M}\mathrm{od}_{S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit} \\
\downarrow & & \downarrow \\
\mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\heartsuit} & \longrightarrow & \mathcal{M}\mathrm{od}_{S}^{\heartsuit} \\
\downarrow & & \downarrow \\
\mathrm{L}\mathcal{M}\mathrm{od}_{\Delta}^{\heartsuit} & \longrightarrow & \mathcal{M}\mathrm{od}_{E_*}^{\heartsuit}
\end{array}.$$

Write $\mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{F}^{\heartsuit}$, where F is an algebra over $\mathcal{M}\mathrm{od}_{S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$. Each square in the above is distributive, so by Lemma 3.5.7 the algebra F is Koszul, with length h Koszul resolutions. Though N may not be discrete if S is not tame, we can nonetheless by Lemma 3.2.1 identify $\mathcal{E}\mathrm{xt}_{S\otimes\Delta}(N,M)$ as the totalization of $\mathcal{E}\mathrm{xt}_{S\otimes\Delta}(B(S\otimes\Delta, S\otimes\Delta, N), M)$. As M is $\mathfrak{m}$-complete and discrete, we can identify $B_{S\otimes\Delta}(N,M) \simeq B_F(\pi_0 N, M)$. As $\pi_0 N$ is a projective object of $\mathcal{M}\mathrm{od}_{S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, we have $B_F(\pi_0 N, M) \simeq K_F(\pi_0 N, M)$ by Koszulity, proving the lemma. □

6.5. **Mapping spaces and highly structured orientations.** We can now describe some applications of the preceding theory. We continue to take E to be a Lubin-Tate spectrum as in Subsection 6.3.

6.5.1. **Theorem.** Fix $R \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$, and choose $S \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{loc}}$ such that $R_* \to S_*$ is surjective (such as $S = 0$ or $S = R$). Let $A, B \in \mathcal{C}\mathrm{Alg}_{R/S}^{\mathrm{loc}}$, and choose a map $\phi\colon A_* \to B_*$ in $\mathcal{R}\mathrm{ing}_{R_*\otimes\mathbb{T}/S_*}$. Let $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A,B)$ be the space of lifts of ϕ to a map in $\mathcal{C}\mathrm{Alg}_{R/S}$. Then there is a decomposition

$$\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A,B) \simeq \lim_{n\to\infty} \mathcal{C}\mathrm{Alg}^{\phi,\leq n}_{R/S}(A,B),$$

with layers fitting into fiber sequences

$$\mathcal{C}\mathrm{Alg}^{\phi,\leq n}_{R/S}(A,B) \to \mathcal{C}\mathrm{Alg}^{\phi,\leq n-1}_{R/S}(A,B) \to \mathcal{H}^{n+1}_{R_*\otimes\mathbb{T}/B_*}(A_*;\pi_*\Omega^n F),$$

where $F = \mathrm{Fib}(B \to S)$. In particular,

(1) There are successively defined obstructions in $H^{n+1}_{R_*\otimes\mathbb{T}/B_*}(A_*;\pi_*\Omega^n F)$ for $n \geq 1$ to exhibiting a point of $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A,B)$;

(2) Once a point of $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A,B)$ is chosen, there is a fringed spectral sequence of signature

$$E_1^{p,q} = H^{p-q}_{R_*\otimes\mathbb{T}/B_*}(A_*;\pi_*\Omega^p F) \Rightarrow \pi_q(\mathcal{C}\mathrm{Alg}_{R/S}(A,B), f), \quad d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-1}.$$

Specializing further, if A_* is smooth as an $\mathfrak{m}$-complete alternating R_*-ring, then

(3) If $h = 1$, then $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$ is nonempty. Moreover, $\pi_0\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B) \cong H^1_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega F)$, and if we choose $f \in \mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$, then there are short exact sequences

$$0 \to H^1_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^{n+1}F) \to \pi_n(\mathcal{C}\mathrm{Alg}_{R/S}(A, B), f) \to H^0_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^n F) \to 0$$

for $n \geq 1$.

(4) If $h = 1$ and each of the rings in question is concentrated in even degrees, then $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$ is connected. Moreover,

$$\pi_n\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B) \cong \begin{cases} H^0_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^n F), & n \text{ even;} \\ H^1_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^{n+1}F), & n \text{ odd.} \end{cases}$$

(5) If $h = 2$ and each of the rings in question is concentrated in even degrees, then $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$ is nonempty. Moreover, $\pi_0\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B) \cong H^2_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^2 F)$, and if we choose $f \in \mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$, then there are short exact sequences

$$0 \to H^2_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^{2(n+1)}F) \to \pi_{2n}(\mathcal{C}\mathrm{Alg}_{R/S}(A, B), f) \to H^0_{R*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^{2n}F) \to 0$$

and isomorphisms

$$\pi_{2n-1}(\mathcal{C}\mathrm{Alg}_{R/S}(A, B), f) \cong H^1_{R_*\otimes\mathbb{T}/B_*}(A_*; \pi_*\Omega^{2n}F)$$

for $n \geq 1$.

Proof. The obstruction theory is a special case of [Bal20, Theorem 5.3.1]. The final statements then follow using Lemma 6.4.2. □

6.5.2. **Remark.** Following Remark 6.3.5, the preceding theorem applies when E is instead taken to be an arbitrary $K(1)$-local even-periodic $\mathbb{E}_\infty$ ring spectrum. ◁

Our main application of Theorem 6.5.1 is to the theory of $\mathbb{E}_\infty$ orientations. We first recall some history. Power operations for Lubin-Tate spectra were first studied by Ando [And95] precisely in the context of producing highly structured complex orientations. In particular, there it is shown that the Honda formal group law refines to a unique $\mathbb{H}_\infty$ orientation of the associated Lubin-Tate spectrum. The characterization of $\mathbb{H}_\infty$ orientations is described in a more general setting in Ando-Hopkins-Strickland [AHS04], which in addition transitions to explicitly considering MUP orientations, where MUP is the Thom spectrum of the tautological bundle over $\mathbb{Z} \times BU$. In brief, homotopy ring maps $MUP \to E$ correspond to coordinates on $\mathbb{G}_E = \mathbb{G}$, and the conditions necessary for this coordinate to correspond to an $\mathbb{H}_\infty$-ring map are determined; we will call these coordinates *norm-coherent*, and will very briefly review the characterization in the proof of Theorem 6.5.3. Work of Zhu [Zhu20] extends the existence and uniqueness of norm-coherent coordinates to an arbitrary Lubin-Tate spectrum so long as $\kappa \subset \overline{\mathbb{F}}_p$; in our language, this says that there is an isomorphism

$$\mathcal{R}\mathrm{ing}_{\mathbb{T}}(E_0^\wedge MUP, E_0) \cong \mathcal{R}\mathrm{ing}_{E_0}(E_0^\wedge MUP, \kappa) \cong \mathrm{Coord}(\mathbb{G}_0).$$

Recall that at height $h = 1$, the category of even $\mathbb{T}$-rings is exactly the category of θ-rings sliced under E_0. Here it is classical that $E_0 = W(\kappa)$ is in fact the cofree

θ-ring on the E_0-ring κ, and so the above isomorphism is immediate, and we need not assume $\kappa \subset \overline{\mathbb{F}}_p$. An unpublished theorem of Rezk extends this to arbitrary heights, showing that E_0 is the cofree $\mathbb{T}$-ring on the E_0-ring κ.

Given the preceding, we can safely say that $\mathbb{H}_\infty$ orientations are well-understood. By contrast, significantly less is known about $\mathbb{E}_\infty$ orientations. The exception to this is $\mathbb{E}_\infty$ orientations by MU at height $h = 1$; the case of p-adic K-theory has been studied by Walker [Wal09], and the more general $K(1)$-local case by Möllers [MÏ0], using methods similar to those employed in [AHR10]. In Hopkins-Lawson [HL16], a general obstruction theory for $\mathbb{E}_\infty$ orientations by MU is constructed that recovers the known $h = 1$ story. Even less is known about $\mathbb{E}_\infty$ orientations by MUP. The only work in this direction we are aware of is [HY19], which demonstrates their existence when $h = 1$ and $\kappa = \mathbb{F}_2$. We now give the following.

6.5.3. **Theorem.**

(1) Let R be a $K(1)$-local even-periodic $\mathbb{E}_\infty$ ring spectrum. Then every norm-coherent coordinate on the formal group $\mathbb{G}_R$ associated to R refines uniquely to an $\mathbb{E}_\infty$ orientation $MUP \to R$.
(2) The multiplicative formal group law $x + y - xy$ refines uniquely to an $\mathbb{E}_\infty$ orientation $MUP \to KU$.
(3) Let E be a Lubin-Tate spectrum at height $h = 2$. Then every norm-coherent coordinate on $\mathbb{G}_E$ refines to an $\mathbb{E}_\infty$ orientation $MUP \to E$.

Proof. Claims (1) and (3) are immediate consequences of Theorem 6.5.1, as $E_0 MUP$ is smooth. Claim (2) follows directly from (1), the arithmetic fracture square, and the fact that $x + y - xy$ is a norm-coherent coordinate of the multiplicative formal group at all primes; for completeness, we give the details. First, arithmetic fracture gives a Cartesian square of the form

$$\begin{array}{ccc} \mathcal{C}\mathrm{Alg}(MUP, KU) & \longrightarrow & \prod_p \mathcal{C}\mathrm{Alg}_{KU_p}(KU_p \mathbin{\widehat{\otimes}} MUP, KU_p) \\ \downarrow & & \downarrow \\ \mathcal{C}\mathrm{Alg}(MUP_{\mathbb{Q}}, KU_{\mathbb{Q}}) & \longrightarrow & \mathcal{C}\mathrm{Alg}(MUP_{\mathbb{Q}}, (\prod_p KU_p)_{\mathbb{Q}}) \end{array} .$$

As $MUP_{\mathbb{Q}}$ is free as a rational $\mathbb{E}_\infty$ ring, the coordinate $x + y - xy$ gives points of the bottom two spaces, and $\pi_1 \mathcal{C}\mathrm{Alg}(MUP_{\mathbb{Q}}, (\prod_p KU_p)_{\mathbb{Q}}) = 0$. So it is sufficient to verify that for each prime p, the homotopy orientation associated to the formal group law $x + y - xy$ refines to a map $KU_p \mathbin{\widehat{\otimes}} MUP \to KU_p$ of KU_p-algebras. By Theorem 6.5.1, it is sufficient to verify that the coordinate associated to the formal group law $x + y - xy$ is norm-coherent at all primes.

The description of norm-coherent coordinates given in [AHS04, Section 4], in the case of a Lubin-Tate spectrum E, can be summarized as follows. A norm-coherent coordinate x on $\mathbb{G}$ is a coordinate such that for every formal scheme Y, map $f\colon Y \to X$, and finite subgroup $K \subset f^*\mathbb{G}$, we have $N_\pi \mu^* f^*(x) = q^* \alpha^*(x)$, where $\pi\colon K \times_Y f^*\mathbb{G} \to f^*\mathbb{G}$ is the projection, N_π is the associated norm map, $\mu\colon K \times_Y f^*\mathbb{G} \to f^*\mathbb{G}$ is the multiplication, and $\alpha\colon (f^*\mathbb{G})/K \to \mathbb{G}$ identifies $(f^*\mathbb{G})/K$ as a deformation of $\mathbb{G}_0$. To be precise, [AHS04] works in the context of level structures rather than finite subgroups, but the translation follows readily from the fact that $\coprod_{|A|=p^n} \mathrm{Level}(A, \mathbb{G}) \to \mathrm{Sub}^n_{\mathbb{G}}$ is surjective [Str97, Theorem 12.4]. In addition, it is sufficient to restrict to the case where K is a subgroup of rank p.

When $h = 1$, the kernel of formal multiplication by p is the unique subgroup of rank p. The p-series associated to the formal group law $x + y - xy$ is given by $[p](x) = 1 - (1 - x)^p$, so the above condition translates to asking that multiplication by $x + y - xy$ on the free $\mathbb{Z}_p[[x]]$-module $\mathbb{Z}_p[[x, y]]/(1 - (1 - y)^p)$ has determinant of $1 - (1 - x)^p$. This itself can be checked by direct computation, proving the theorem. □

We do not know whether the uniqueness statement of Theorem 6.5.3 can be extended to height $h = 2$. This is equivalent to whether $\mathrm{Ext}^2_\Delta(\widehat{Q}(E_0 MUP), \omega) = 0$.

We now give a few more examples illustrating Theorem 6.5.1.

6.5.4. **Example.** In Chatham-Hahn-Yuan [CHY19], an interesting family of $\mathbb{E}_\infty$ ring spectra R_{h-1} at chromatic height h are constructed, and left open is the question of whether there exists an $\mathbb{E}_\infty$ map $R_{h-1} \to E$, where E is a Lubin-Tate spectrum of height h. Combining [CHY19, Theorem 7.6] with the preceding, we learn that there are $\mathbb{E}_\infty$ maps $R_1 \to E$ whenever E is a Lubin-Tate spectrum of height 2 associated to a supersingular elliptic curve. ◁

6.5.5. **Example.** Given an arbitrary $\mathbb{E}_\infty$ ring spectrum R, we can define

$$\mathbb{A}^1(R) = \mathcal{C}\mathrm{Alg}(\Sigma^\infty_+ \mathbb{N}, R), \qquad \mathbb{G}_m(R) = \mathcal{C}\mathrm{Alg}(\Sigma^\infty_+ \mathbb{Z}, R).$$

We considered the case where R is of characteristic p in Example 5.6.3; in the general case, $\mathbb{A}^1(R)$ carries only the structure of a strictly commutative multiplicative monoid. We note that $\mathbb{G}_m(R) \subset \mathbb{A}^1(R)$ is a collection of path components, and $\mathbb{G}_m(R)$ can be regarded as a $\mathbb{Z}$-module. We can describe $\mathbb{A}^1(KU)$ as a simple example illustrating the use of Theorem 6.5.1. Write $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ for the profinite integers. Then we claim that $\pi_0 \mathbb{A}^1(KU) = \{-1, 0, 1\}$, that we can identify

$$\pi_n \mathbb{G}_m(KU) = \begin{cases} \widehat{\mathbb{Z}}, & n = 1; \\ \mathbb{Z}, & n = 2; \\ \widehat{\mathbb{Z}}/\mathbb{Z}, & n > 2 \text{ odd}; \end{cases}$$

and that there are short exact sequences

$$0 \to \widehat{\mathbb{Z}}/\mathbb{Z} \to \pi_{2n+1}(\mathbb{A}^1(KU), 0) \to \prod_p \mathbb{Z}/(p^n) \to 0,$$

which are necessarily split as $\widehat{\mathbb{Z}}/\mathbb{Z}$ is injective. Here, the above product ranges over primes p, and all unspecified groups are zero. This is computed as follows. Arithmetic fracture gives a Cartesian square

$$\begin{array}{ccc} \mathbb{A}^1(KU) & \longrightarrow & \prod_p \mathbb{A}^1(KU_p) \\ \downarrow & & \downarrow \\ \mathbb{A}^1(KU_{\mathbb{Q}}) & \longrightarrow & \mathbb{A}^1\left(\left(\prod_p KU_p\right)_{\mathbb{Q}}\right) \end{array}.$$

If R is a rational $\mathbb{E}_\infty$ ring, then $\mathbb{A}^1(R) = \Omega^\infty R$, and this determines the bottom two spaces. The claimed structure of $\mathbb{A}^1(KU)$ will follow easily by inspecting this square as soon as we understand $\mathbb{A}^1(KU_p)$. We claim that $\pi_0 \mathbb{A}^1(KU_p) \cong \mathbb{F}_p \subset \mathbb{Z}_p = \pi_0 KU_p$ is given by the image of the Teichmüller character, and that

$$\pi_1 \mathbb{G}_m(KU_p) \cong \mathbb{Z}_p \cong \pi_2 \mathbb{G}_m(KU_p), \qquad \pi_{2n+1}(\mathbb{A}^1(KU_p), 0) \cong \mathbb{Z}/(p^n),$$

all other groups being zero. One method of computing this proceeds by observing that $\mathbb{A}^1(KU_p)$ fits into a fiber sequence

$$\mathbb{A}^1(KU_p) \to \Omega^\infty KU_p \to \Omega^\infty KU_p,$$

with second map corresponding to the KU_p-cohomology operation θ^p. As we wish to illustrate the use of Theorem 6.5.1, we will proceed in a different way, although the two approaches are not really any different. The goal is to compute $\pi_*\mathbb{A}^1(KU_p) = \pi_*\mathcal{C}\mathrm{Alg}_{KU_p}(KU_p \mathbin{\widehat{\otimes}} \Sigma^\infty_+\mathbb{N}, KU_p)$. As θ-rings, we can identify

$$\pi_0 KU_p \cong \mathbb{Z}_p, \qquad \psi(\lambda) = \lambda;$$
$$\pi_0 KU_p \otimes \Sigma^\infty_+\mathbb{N} \cong \mathbb{Z}_p[t], \qquad \psi(t) = t^p.$$

So by Theorem 6.5.1, we can identify

$$\pi_0\mathbb{A}^1(KU_p) \cong \mathcal{R}\mathrm{ing}_{\mathbb{T}}(\mathbb{Z}_p[t], \mathbb{Z}_p) \cong \{\lambda \in \mathbb{Z}_p : \lambda^p = \lambda\}.$$

This is the image of the Teichmüller character as claimed. Given an element ϕ of the above, we need to compute

$$H^*_{\mathbb{T}/\mathbb{Z}_p}(\mathbb{Z}_p[t], \pi_*\Omega^{2n}KU_p) \cong \mathrm{Ext}^*_\Delta(Q(\mathbb{Z}_p[t]), \omega^n)$$

for $n \geq 1$. Here, $\omega^n = \mathbb{Z}_p\{u_n\}$ has $\Gamma = \mathbb{Z}_p[\psi]$-module structure $\psi(u_n) = p^n u_n$, and thus $\Delta = \mathbb{Z}_p[\theta]$-module structure $\theta(u_n) = p^{n-1}u_n$. Consider first the case where ϕ is in a path component corresponding to an element of $\mathbb{F}_p^\times$. These are all equivalent, so we reduce to considering the map

$$\phi\colon \mathbb{Z}_p[t] \to \mathbb{Z}_p, \qquad \phi(t) = 1.$$

With this augmentation, we can identify $Q(\mathbb{Z}_p[t]) = \mathbb{Z}_p\{s\}$ with Γ-module structure $\psi(s) = ps$, and thus Δ-module structure $\theta(s) = s$. We find that the Koszul complex for $\mathrm{Ext}_\Delta(\mathbb{Z}_p\{s\}, \omega^n)$ takes the form

$$p^{n-1} - 1\colon \mathbb{Z}_p \to \mathbb{Z}_p.$$

As $p^{n-1} - 1$ is a unit unless $n = 1$, the only nonzero groups are

$$\mathrm{Ext}^0_\Delta(\mathbb{Z}_p\{s\}, \omega) = \mathbb{Z}_p = \mathrm{Ext}^1_\Delta(\mathbb{Z}_p\{s\}, \omega).$$

By Theorem 6.5.1, we learn $\pi_1\mathbb{G}_m(KU_p) = \mathbb{Z}_p = \pi_2\mathbb{G}_m(KU_p)$ as claimed. Next consider the map

$$\phi\colon \mathbb{Z}_p[t] \to \mathbb{Z}_p, \qquad \phi(t) = 0.$$

With this augmentation, we have $Q(\mathbb{Z}_p[t]) = \mathbb{Z}_p\{t\}$ with Δ-module structure $\theta(t) = 0$, and the Koszul complex for $\mathrm{Ext}_\Delta(\mathbb{Z}_p\{t\}, \omega^n)$ takes the form

$$p^{n-1}\colon \mathbb{Z}_p \to \mathbb{Z}_p.$$

The resulting nonzero groups are

$$\mathrm{Ext}^1_\Delta(\mathbb{Z}_p\{t\}, \omega^n) = \mathbb{Z}/(p^{n-1}).$$

Again by Theorem 6.5.1, we learn $\pi_{2n+1}\mathbb{A}^1(KU_p) = \mathbb{Z}/(p^n)$ as claimed. ◁

6.5.6. **Example.** A conjecture of Hopkins-Lurie [HL13, Conjecture 5.4.14], readily verified at height $h = 1$, in particular gives the following. Let E be the Lubin-Tate spectrum associated to the formal multiplicative group over an algebraically closed field κ of characteristic p. Let G be a finite p-group. Then

$$BG \simeq \mathcal{C}\mathrm{Alg}_E(E^{BG_+}, E).$$

By the p-adic Atiyah-Segal completion theorem [AT69, Proposition III.1.1], we can identify $E^1BG = 0$ and $E^0BG = W(\kappa) \otimes R(G)$, where $R(G)$ is the complex representation ring of G. By Theorem 6.5.1, we obtain a curious filtration of the space BG; to phrase it in terms of a fringed spectral sequence, this is

$$E_1^{2p,q} = \mathrm{Ext}^{2p-q}_{\mathbb{Z}_p[\theta]}(Q(R(G)); \omega^p) \Rightarrow \pi_q BG, \qquad d_r^{p,q} \colon E_{2r}^{2p,q} \to E_{2r}^{2(p+r),q-1},$$

where $Q(R(G))$ is the derived indecomposables of $R(G)$ and $\omega^p = \pi_{2p}E$. ◁

6.5.7. **Remark.** There is an obstruction theory for realizing $\widehat{\mathbb{T}}$-rings as the homotopy groups of E-algebras. This can be read off [Bal20, Theorem 5.4.7], and the obstruction groups vanish in ranges similar to what one sees in Theorem 6.5.1. ◁

6.6. **Topological André-Quillen cohomology.** We now consider the $K(h)$-local topological André-Quillen homology and cohomology of E-algebras, where E is a Lubin-Tate spectrum as in the preceding subsections. This is of particular interest due to work of Behrens-Rezk [BR17] [BR20], which constructs for a space X a natural comparison map $\Phi_h X \to \mathrm{TAQ}_{\mathbb{S}_{K(h)}}(\mathbb{S}_{K(h)}^{X_+})$, showing it to be an isomorphism in some nice cases. Here, Φ_h is the $K(h)$-local Bousfield-Kuhn functor [Kuh08]. This gives rise to a comparison map $E \mathbin{\widehat{\otimes}} \Phi_h X \to \mathrm{TAQ}_E(E^{X_+})$, again an isomorphism in some nice cases. This gives an approach to computing $E_* \Phi_h X$, which in turn gives an approach to computing $\pi_* \Phi_h X$ by a suitable descent spectral sequence.

We must introduce some notation. Given a $K(h)$-local $\mathbb{E}_\infty$ ring spectrum R, and $M \in \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$, we will write $\widehat{\mathrm{TAQ}}^R(A; M) = L_{K(h)} \mathrm{TAQ}^R(A; M)$ for the $K(h)$-local André-Quillen homology of $A \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,aug}}$ with coefficients in M. We can describe $\widehat{\mathrm{TAQ}}^R(-; M)$ as the unique functor

$$\widehat{\mathrm{TAQ}}^R(-; M) \colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,aug}} \to \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$$

which preserves geometric realizations and satisfies

$$\widehat{\mathrm{TAQ}}^R(\widehat{\mathbb{P}}_R N; M) \cong M \mathbin{\widehat{\otimes}}_R N$$

for $N \in \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$. In addition, we write $\mathrm{TAQ}_R(A; M) = \mathcal{M}\mathrm{od}_R(\widehat{\mathrm{TAQ}}^R(A, R), M)$. When $M = R$, we omit it from the notation.

On the algebraic side, given $R \in \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$ and $M \in \mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, define

$$\widehat{\mathbb{T}\mathrm{AQ}}^R(-; M) \colon \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})}$$

to be the unique functor preserving geometric realizations and satisfying

$$\widehat{\mathbb{T}\mathrm{AQ}}^R(R \mathbin{\widehat{\otimes}} \widehat{\mathbb{T}}(P)) = M \mathbin{\widehat{\otimes}}_{E_*} P$$

for $P \in \mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m}),\mathrm{free}}$. In addition, we set

$$\mathbb{T}\mathrm{AQ}_R(A; M) = \mathcal{E}\mathrm{xt}_R(\widehat{\mathbb{T}\mathrm{AQ}}^R(A; R), M).$$

Observe now that

$$\widehat{\mathbb{T}\mathrm{AQ}}^R(A; M) \simeq M \mathbin{\widehat{\otimes}}_R \epsilon_! \widehat{Q}_R(A) \simeq M \mathbin{\widehat{\otimes}}_{R\otimes\Delta} \epsilon_! \widehat{Q}_R(A),$$

where $\epsilon \colon R \otimes \Delta \to R$ is the augmentation, and that

$$\mathbb{T}\mathrm{AQ}_R(A; M) \simeq \mathcal{H}_{R\otimes\mathbb{T}/R}(A; \overline{M}).$$

6.6.1. **Theorem.** Fix $R \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ and $M \in \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$, and choose $A \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,aug}}$. Then there is a conditionally convergent spectral sequence of signature

$$E_1^{p,q} = \mathbb{T}\mathrm{AQ}_{R_*}^{p+q}(A_*; \omega^{-p/2} \otimes \overline{M}_*) \Rightarrow \mathrm{TAQ}_R^q(A; M), \qquad d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q+1}.$$

If R_* is tame, then there is a spectral sequence of signature

$$E_{p,q}^1 = \widehat{\mathbb{T}\mathrm{AQ}}_{p+q}^{R_*}(A_*; \omega^{p/2} \otimes \overline{M}_*) \Rightarrow \widehat{\mathrm{TAQ}}_q^R(A; M), \qquad d_{p,q}^r \colon E_{p,q}^r \to E_{p-r,q-1}^r,$$

which is convergent if each $\widehat{\mathbb{T}\mathrm{AQ}}^{R_*}(A_*; \omega^{p/2} \otimes \overline{M}_*)$ is truncated.

Proof. The first spectral sequence can be obtained, for instance, by patching together the filtrations of $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}(A; R \ltimes \Sigma^n \overline{M})$ for various n given by Theorem 6.5.1. For the second, tameness of R guarantees that $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,aug,free}}} \simeq \mathcal{R}\mathrm{ing}_{R_* \otimes \mathbb{T}/R_*}^{\mathrm{Cpl}(\mathfrak{m})}$ and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{M}\mathrm{od}_R^{\mathrm{loc,free}}} \simeq \mathcal{M}\mathrm{od}_{R_*}^{\mathrm{Cpl}(\mathfrak{m})}$, and the spectral sequence can be obtained as a special case of [Bal20, Theorem 4.2.2]. □

The most common situation has $R = E = M$, and we will write $\overline{E}_* = \mathrm{nul}$ and $\mathbb{T}\mathrm{AQ}_{E_*} = \mathbb{T}\mathrm{AQ}$.

6.6.2. **Remark.** The action of Δ on $\omega^{p/2} \otimes \overline{M}_*$ is through the augmentation on Δ; the presence of $\omega^{p/2}$ only serves to shift degrees, i.e. $\omega^{p/2} \otimes \overline{M}_* = s^{-p}\overline{M}_*$. In particular, the presence of $\omega^{p/2}$ in the spectral sequence of Theorem 6.6.1 effectively disappears once a trivialization of ω is chosen. However, these terms should not be completely ignored; for instance, they are relevant in situations where one has an action of the Morava stabilizer group. ◁

In [Rez13] and [Zhu18], $E_*^\wedge \Phi_h S^{2n+1}$ is computed for $h \leq 2$, proceeding by computing the cohomology groups $\mathbb{T}\mathrm{AQ}^n(E^* S^{2n+1}; \omega^{m/2} \otimes \mathrm{nul})$. In particular, it is shown in these $h \leq 2$ cases that this group vanishes for $n \neq h$. In general, let us say that E satisfies the weak algebraicity condition if $\mathbb{T}\mathrm{AQ}^n(E^* S^{2n+1}; \omega^{m/2} \otimes \mathrm{nul}) = 0$ for $n \neq h$.

6.6.3. **Example.** Let E be the Lubin-Tate spectrum of Example 6.3.3. Then combining Example 3.6.5, Theorem 3.7.1, and Example 6.2.3 gives the following description of $\mathbb{T}\mathrm{AQ}^2(E^* S^{2n+1}; \omega^{1/2} \otimes \mathrm{nul})$, and thus of $E_{-1}^\wedge \Phi_2 S^{2n+1}$. First, we can identify

$$\mathbb{T}\mathrm{AQ}^2(E^* S^{2n+1}; \omega^{1/2} \otimes \mathrm{nul}) \cong \mathrm{Ext}_\Delta^2(\omega^{(2n+1)/2}, \omega^{1/2} \otimes \mathrm{nul}) \cong \mathrm{Ext}_\Gamma^2(\omega^n; \mathrm{nul}).$$

Now write $d^n = f_0 + f_1 d + f_2 d^2$ in $R[[d]]/(d^3 = ad + 2)$ with $f_0, f_1, f_2 \in R$, and let $Q = Q^0 f_0 + Q^1 f_1 + Q^2 f_2 \in H^*(\Gamma)$. Then $\mathrm{Ext}_\Gamma^2(\omega^n; \mathrm{nul})$ is isomorphic to the cokernel of the map $H^1(\Gamma) \to H^2(\Gamma)$ given by left multiplication by Q. ◁

Work of Bousfield [Bou99] [Bou07] describes the v_1-periodic homotopy groups of nice spaces in terms of their K-theory. One obstruction to extending this to higher heights using TAQ is in determining when $\Phi_h X \simeq \mathrm{TAQ}_{\mathbb{S}_{K(h)}}(\mathbb{S}_{K(h)}^{X_+})$; we will not consider this issue here. Provided one takes this as known, Bousfield's description of $\pi_* \Phi_1 X$ for nice spaces X at primes $p \geq 3$ can be reinterpreted as a description of $KU_{p,*}^\wedge \Phi_1 X$ that can be obtained from Theorem 6.6.1, combined with the standard fiber sequence $\Phi_1 X \to KU_p \mathbin{\widehat{\otimes}} \Phi_1 X \to KU_p \mathbin{\widehat{\otimes}} \Phi_1 X$. We view the following observation as an indication of a higher height version of this.

6.6.4. **Proposition.** Suppose E satisfies the weak algebraicity condition defined above. Let X be a simply connected space such that $H^*(X)$ is a finitely generated exterior algebra on odd-dimensional classes. Then

$$\mathrm{TAQ}^q_E(E^{X_+}) \cong \mathbb{T}\mathrm{AQ}^h(E^*X; \omega^{(q-h)/2} \otimes \mathrm{nul}) \cong \mathrm{Ext}^h_\Delta(\overline{Q}(E^*X); \omega^{(q-h)/2} \otimes \mathrm{nul}),$$

where $\overline{Q}$ is the non-derived functor of indecomposables.

Proof. Write $H^*X \simeq \Lambda(t_1, \ldots, t_n)$ with $|t_i| = m_i$. We learn that the Atiyah-Hirzebruch spectral sequence collapses to give $E^*X \simeq \Lambda_{E_*}(x_1, \ldots, x_n)$. More precisely, we may find a cell structure on X with the following property. Write $X_{\leq n}$ for the n-skeleton of X. Then the cofibering $X_{\leq n-1} \to X_{\leq n} \to \bigvee S^n$ induces a short exact sequence on H^* and E^*, and the restriction of x_i to $X_{\leq m_i}$ is in the image of a generator of some $E^*S^{m_i}$. As E^*X is smooth, we can identify $Q(E^*X) = \overline{Q}(E^*X) = E_*\{x_1, \ldots, x_n\}$ as an E_*-module. By the preceding, we find that $E_*\{x_i, \ldots, x_n\} \subset Q(E^*X)$ is a sub-Δ-module, and that $E_*\{x_i, x_{i+1}, \ldots, x_n\}/E_*\{x_{i+1}, \ldots, x_n\} \cong \omega^{m_i/2}$. We obtain a finite filtration of the Δ-module $Q(E^*X)$ with filtration quotients given by various $\omega^{m_i/2}$, and the associated spectral sequence for $\mathrm{Ext}^*_\Delta(Q(E^*X); \omega^{p/2} \otimes \mathrm{nul})$ collapses by the weak algebraicity condition, implying that it is concentrated in degree h. This implies that the spectral sequence of Theorem 6.6.1 collapses on a single line into the claimed isomorphism. □

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