APPROXIMATING HIGHER ALGEBRA BY DERIVED ALGEBRA (DRAFT)

WILLIAM BALDERRAMA

Abstract. We develop a form of the theory of product-preserving presheaves of ∞-groupoids as a natural setting for the construction of obstruction theories and spectral sequences approximating higher, or spectral, algebra by mere derived algebra. We account for infinitary phenomena, allowing for the inclusion of completed settings into the general theory.

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1. Introduction

Consider an example from spectral algebra. If R is a $\mathbb{E}_\infty$-ring, and M and N are R-modules, then there are Universal Coefficient and Künneth spectral sequences

$$\operatorname{Ext}^{p+q}_{R_*}(M_*, N_{*+p}) \Rightarrow \pi_{*-q}\mathcal{M}\mathrm{od}_R(M, N),$$
$$\operatorname{Tor}^{R_*}_{p+q}(M_*, N_{*-p}) \Rightarrow \pi_{*+q} M \otimes_R N.$$

We can view the existence of these spectral sequences as saying that the homotopy theory of R-modules is, in some sense, approximated by the homological algebra of R_*-modules. We might summarize the general situation, which occurs frequently in homotopy theory, as follows: to a homotopy theory $\mathcal{M}$ (here, $\mathcal{M}\mathrm{od}_R$), we associate an algebraic category $\mathcal{A}$ (here, $\mathcal{M}\mathrm{od}_{R_*}$), with the property that questions in $\mathcal{M}$ admit filtrations with filtration quotients being questions in a suitable derived category of $\mathcal{A}$.

The goal of this paper is to describe a certain conceptual ∞-categorical framework in which the above heuristic is made precise. Note that the case of R-modules is as nice as possible: $\mathcal{M}\mathrm{od}_R$ is a stable category with compact generator R, and $\mathcal{M}\mathrm{od}_{R_*}$ is a compactly generated abelian category. There are a number of contexts not satisfying these, and we develop a framework that allows for more exotic situations that can arise, with an emphasis on the ease of applicability to new situations.

To give an example of such an exotic situation, this paper initially arose out of the following question: if E is a Lubin-Tate spectrum of height h, in what sense is the category of $K(h)$-local commutative E-algebras approximated by the category of commutative E_*-algebras equipped with E-power operations? For instance, one expects an obstruction theory for computing maps in the former in terms of Quillen cohomology for the latter. The $K(h)$-local condition, which corresponds to a completeness condition with respect to the maximal ideal of E_0, forces one to consider suitably completed E_*-modules, and so produces categories that are not compactly generated. The framework we develop allows us to cleanly deal with this sort of complication, incorporating such categories in the same general theory.

The setting in which we work, to be described below, is very closely related to the setting of finite product-preserving presheaves studied by Pstrągowski [Pst17]. We have attempted to exposit the theory in such a way as to facilitate its application to examples beyond just those given, and we provide a number of tools for producing and recognizing examples.

1.1. **Conventions.**

We will freely use the theory of ∞-categories, which we will refer to just as categories, as developed by Lurie in [Lur17b], and all of our constructions are to be interpreted in this sense. We write $\mathcal{G}\mathrm{pd}_\infty$ for the category of ∞-groupoids, also commonly known as the ∞-category of spaces, and for a small category $\mathcal{C}$ we will write $\mathrm{Psh}(\mathcal{C})$ for the category of presheaves of ∞-groupoids on $\mathcal{C}$, writing instead $\mathrm{Psh}(\mathcal{C}, \mathcal{S}\mathrm{et})$ when we mean presheaves of sets, and similarly for presheaves valued in other categories.

We follow the standard convention of fixing a small universe of ∞-groupoids, with respect to which everything in sight will be at least locally small, contained in a universe of large ∞-groupoids, with respect to which everything in sight is small, unless otherwise specified. For a (locally small) category $\mathcal{C}$, we will write $\mathrm{Psh}(\mathcal{C})$ for the category of presheaves on $\mathcal{C}$ that arise as small colimits of representable presheaves; this is the cocompletion of $\mathcal{C}$ under small colimits. We write

$h\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ for the Yoneda embedding, and write the same for various restricted Yoneda embeddings. When $f\colon \mathcal{C} \to \mathcal{D}$ is a functor, we write $f_!\colon \mathrm{Psh}(\mathcal{C}) \to \mathrm{Psh}(\mathcal{D})$ for the functor obtained from f by left Kan extension, and write the same for similar situations, such as $f_!\colon \mathrm{Psh}(\mathcal{C}) \to \mathcal{D}$ when $\mathcal{D}$ admits sufficiently many colimits. For a category $\mathcal{C}$, we will write $\mathcal{C}^{\simeq}$ for the maximal sub-∞-groupoid of $\mathcal{C}$, and $\mathrm{h}\mathcal{C}$ for the homotopy category of $\mathcal{C}$.

1.2. **Overview.** This subsection outlines the framework developed in the paper and describes, in a form that does not require familiarity with the details of the general theory, a few of the obstruction theories and spectral sequences that it can be used to produce. The basis of our framework is a variant of the classic notion of an algebraic theory, going back to ideas of Lawvere [Law04]. Some familiarity with this story is useful for understanding the rest of the paper, so we give a very brief review here. A *finitary theory* is a small category $\mathcal{C}$ admitting finite coproducts, and the *category of models* of $\mathcal{C}$ is the category $\mathrm{Psh}^{\times}(\mathcal{C})$ of finite product-preserving presheaves on $\mathcal{C}$, i.e. those presheaves X such that $X(\coprod_{i\in F} C_i) \simeq \prod_{i\in F} X(C_i)$ for any finite collection $\{C_i : i \in F\}$ of objects of $\mathcal{C}$. Taking $\mathcal{C}$ to be a 1-category—say that $\mathcal{C}$ is a *discrete* theory—and considering the category of discrete models $\mathrm{Psh}^{\times}(\mathcal{C}, \mathcal{S}\mathrm{et})$, one can obtain in this manner a large number of naturally occurring algebraic categories. Taking $\mathcal{C}$ to still be a discrete finitary theory, the ∞-category $\mathrm{Psh}^{\times}(\mathcal{C})$ is a familiar homotopy theory: $\mathrm{Psh}^{\times}(\mathcal{C})$ is the underlying ∞-category of the category $\mathrm{Psh}^{\times}(\mathcal{C}, \mathrm{s}\mathcal{S}\mathrm{et})$ of simplicial discrete models of $\mathcal{C}$ equipped with the model structure constructed by Quillen [Qui67, Section II.4], as can be seen starting with work of Badzioch [Bad02], generalized by Bergner [Ber06], and put into the ∞-categorical context by Lurie [Lur17b, Section 5.5.9].

One can view the categories arising in this manner as exactly the categories of models of multisorted finite product theories, and this is useful for understanding various examples; for instance, if $\mathcal{C}$ is the category of finitely generated free abelian groups, then $\mathrm{Psh}^{\times}(\mathcal{C}, \mathcal{S}\mathrm{et}) \simeq \mathcal{A}\mathrm{b}$, roughly as an abelian group M is determined by its addition map, which is recovered as restriction along the diagonal $\Delta^*\colon \mathcal{A}\mathrm{b}(\mathbb{Z}, M) \times \mathcal{A}\mathrm{b}(\mathbb{Z}, M) \cong \mathcal{A}\mathrm{b}(\mathbb{Z} \oplus \mathbb{Z}, M) \to \mathcal{A}\mathrm{b}(\mathbb{Z}, M)$. For our purposes, it is more useful to view the categories arising in this manner as categories admitting a family of compact projective generators; from this viewpoint, the category $\mathrm{Psh}^{\times}(\mathcal{C})$ is best characterized as the free cocompletion of $\mathcal{C}$ under sifted colimits [Lur17b, Section 5.5.8], or what is in this case equivalent, under filtered colimits and geometric realizations.

We are interested in certain categories that admit a family of projective but not necessarily compact generators, and the associated notion of an infinitary theory. Infinitary algebraic theories can be studied in general; the classic reference is [Wra70], although some issues of size are not treated there. However, in order to obtain a story mimicking the finitary case, we cannot work with arbitrary infinitary theories. Instead, the class of theories suitable for our purposes are those which are Mal'cev; see for instance [Smi76] and [Lam92], though we require very little of the general theory. Concretely, a Mal'cev operation is a ternary operation t satisfying $t(x, x, y) = y$ and $t(x, y, y) = x$; a set equipped with a Mal'cev operation is called a herd. The motivating example is when working with groups, where we have the Mal'cev operation $t(x, y, z) = xy^{-1}z$, and all of our examples are ultimately derived from this. Herds are the models of a single-sorted finitary theory, so we can speak of herds in arbitrary categories with finite products.

1.2.1. **Definition.** A *Mal'cev theory* is a category $\mathcal{P}$ such that

(1) $\mathcal{P}$ admits small coproducts;
(2) All objects of $\mathcal{P}$ admit the structure of a coherd.

The *category of models* of $\mathcal{P}$ is the category $\mathrm{Psh}^{\Pi}(\mathcal{P})$ of small product-preserving presheaves on $\mathcal{P}$. $\triangleleft$

The Mal'cev condition can be motivated by the following classical fact: if $\mathcal{C}$ is a discrete finitary theory, then $\mathcal{C}$ is Mal'cev if and only if every simplicial object in $\mathrm{Psh}^{\times}(\mathcal{C}, \mathcal{S}\mathrm{et})$ satisfies the Kan fibrancy condition. It is plausible that there is a similarly elegant characterization of (not necessarily discrete) Mal'cev theories, but the definition given is sufficient for our purposes.

We will only be concerned with Mal'cev theories, and so will refer to them simply as *theories*. Throughout the entire paper, we will use $\mathcal{P}$ to refer to some theory. We note that a similar notion was studied in [Lur11, Section 4.2]. If $\mathcal{C}$ is a finitary theory and $\mathcal{P} \subset \mathrm{Psh}^{\times}(\mathcal{C})$ is generated by $\mathcal{C}$ under coproducts, then $\mathrm{Psh}^{\Pi}(\mathcal{P}) \simeq \mathrm{Psh}^{\times}(\mathcal{C})$ (Proposition 2.1.9). So, theories indeed generalize finitary theories. Throughout the paper, we will make some minor size assumptions, assuming that our theories are generated in a similar way by a small, but not necessary countable, amount of data (Remark 2.1.10); we will ignore this point in this introduction.

1.2.2. **Example.** Although the notion of a theory is fairly general, all of our examples of discrete theories ultimately derive from the following:

(1) If $\mathcal{A}$ is a cocomplete abelian category, and $\mathcal{P} \subset \mathcal{A}$ is a full subcategory consisting of projective objects and closed under coproducts such that every $M \in \mathcal{A}$ is resolved by objects of $\mathcal{P}$, then $\mathcal{A} \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ (Proposition 3.3.3);
(2) If $\mathcal{P}$ is a discrete theory, T is a monad on $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ preserving reflexive coequalizers, and $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ is the full subcategory spanned by the image of $\mathcal{P}$ under T, then $T\mathcal{P}$ is a theory and $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi}(T\mathcal{P}, \mathcal{S}\mathrm{et})$;
(3) If $\mathcal{P}$ is a discrete theory and $X \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$, then $\mathcal{P}/X$ is a theory and $\mathrm{Psh}^{\Pi}(\mathcal{P}/X, \mathcal{S}\mathrm{et}) \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})/X$. $\triangleleft$

In Section 2, we show that (Mal'cev) theories and their categories of models indeed behave similar to finitary theories. Specifically, in Subsection 2.1, we show that $\mathrm{Psh}^{\Pi}(\mathcal{P})$ is the free cocompletion of $\mathcal{P}$ under geometric realizations, and is presentable under our mild size conditions on $\mathcal{P}$. In Subsection 2.2, we verify that that if $\mathcal{P}$ is a discrete theory then $\mathrm{Psh}^{\Pi}(\mathcal{P})$ is the underlying ∞-category of the category $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$ of simplicial discrete models of $\mathcal{P}$ with model structure constructed by Quillen [Qui67, Section II.4]. Moreover, we review the notion of left-derived functors available in this context.

The categories we are most interested in developing tools for are not of the form $\mathrm{Psh}^{\Pi}(\mathcal{P})$. For instance, those arising in spectral algebra, at least in nonconnective settings, are never of this form. Heuristically, whereas theories allow for operations with arities indexed by sets, we require operations with arities indexed over higher dimensional objects. To incorporate these, we are led to the following definition.

1.2.3. **Definition.** A theory $\mathcal{P}$ is a *resolution theory* if for any finite wedge of spheres F and $P \in \mathcal{P}$, the tensor $F \otimes P = \operatorname{colim}_{x \in F} P$ exists in $\mathcal{P}$. If $\mathcal{P}$ is a resolution theory, then $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P})$ is the full subcategory of models X such that $X(F \otimes P) \simeq X(P)^F$ for all $P \in \mathcal{P}$ and finite wedge of spheres F. $\triangleleft$

We can now describe the general philosophy of this paper. For a great many categories $\mathcal{M}$ that arise in homotopy theory, one can find (possibly many) full subcategories $\mathcal{P} \subset \mathcal{M}$ which are resolution theories theories such that $\mathcal{M} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. By then embedding $\mathcal{M}$ into the larger category $\mathrm{Psh}^{\Pi}(\mathcal{P})$, one is able to give various questions in $\mathcal{M}$ new filtrations, with filtration quotients computed as questions in the essentially algebraic category $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$.

In the finitary case, where the objects of $\mathcal{P}$ are instead asked to be homotopy cogroups, this context was first studied in general by Pstrągowski [Pst17], where it was used to give a conceptual approach to the realization problem for Π-algebras of Blanc-Dwyer-Goerss [BDG04]. The first instance we are aware of where a particular case of this context is used is in work of Hopkins-Lurie [HL17]. This paper adds to these in considering a different class of $\mathcal{P}$, giving tools for detecting examples, and performing distinct constructions with the categories at hand. In addition to the pleasant formal properties of these constructions, we view as one of the primary benefits of the framework the ease in which it is adapted to new situations, making the resulting computational tools more readily acessible.

We can now describe some of the rest of the paper. In Section 3, we develop the basic properties of resolution theories. Most important for the general theory is Pstrągowski's interpretation of the spiral sequence [Pst17, Section 2.5], which holds equally well in our setting, implying in particular the following: if $\tau\colon \mathcal{P} \to \mathrm{h}\mathcal{P}$ is the truncation and $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, then $\tau_! X = \pi_0 X$ in $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$. In Subsection 3.3, we give tools for constructing and identifying examples. Heuristically, $\mathcal{M} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ whenever $\mathcal{P} \subset \mathcal{M}$ is a reasonable collection of free objects closed under coproducts and S^1-tensors. We refer to Subsection 3.3 for precise statements; let us just give some examples to illustrate this.

1.2.4. **Example.** The following are examples of $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ and their associated $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$.

(1) Let G be a finite group (such as $G = e$), let $\mathcal{S}\mathrm{p}^G$ be the category of genuine G-equivariant spectra, and let R be an $\mathbb{A}_\infty$-ring in $\mathcal{S}\mathrm{p}^G$. Then $\mathrm{L}\mathcal{M}\mathrm{od}_R \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}')$ where $\mathcal{P} \subset \mathrm{L}\mathcal{M}\mathrm{od}_R$ is the full subcategory generated under coproducts by objects of the form $\Sigma^\alpha R \otimes S_+$ for $\alpha \in RO(G)$ and S a finite G-set, and $\mathcal{P}' \subset \mathcal{P}$ is the full subcategory where α is restricted to $\mathbb{Z} \subset RO(G)$. We can identify $\mathrm{h}\mathcal{P}$ as the category of free $RO(G)$-graded left modules over the Green functor $R_\star$, and so identify $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ as the category of all $RO(G)$-graded left modules over $R_\star$. Likewise, $\mathrm{Psh}^{\Pi}(\mathcal{P}', \mathcal{S}\mathrm{et})$ is the category of $\mathbb{Z}$-graded left modules over R_*.

(2) Let $\mathcal{P}$ be a resolution theory and $\mathcal{J}$ a 1-category. Then $\mathrm{Fun}(\mathcal{J}, \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})) \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}^{\mathcal{J}})$, with $\mathrm{Fun}(\mathcal{J}, \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})) \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}^{\mathcal{J}}, \mathcal{S}\mathrm{et})$, for the resolution theory $\mathcal{P}^{\mathcal{J}}$ obtained as the image of the representable functors under the composite $\mathrm{Fun}(\mathcal{J} \times \mathcal{P}^{\mathrm{op}}, \mathcal{G}\mathrm{pd}_\infty) \simeq \mathrm{Fun}(\mathcal{J}, \mathrm{Psh}(\mathcal{P})) \to \mathrm{Fun}(\mathcal{J}, \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}))$, where the latter map is obtained from a localization $\mathrm{Psh}(\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. Explicitly, for $P \in \mathcal{P}$ and $i \in \mathcal{J}$, define $H_{P,i}\colon \mathcal{J} \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ by $H_{P,i}(j) = \mathcal{J}(i,j) \otimes P = \coprod_{x \in \mathcal{J}(i,j)} P$; then $\mathcal{P}^{\mathcal{J}}$ is generated under coproducts in the category $\mathrm{Fun}(\mathcal{J}, \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}))$ by functors of the form $H_{P,i}$.

(3) Let R be an $\mathbb{E}_2$-ring and $I \subset \pi_0 R$ a finitely generated ideal. There results a category $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$ of I-complete left R-modules, and a category

$\mathrm{L}\mathcal{M}\mathrm{od}_{R_*}^{I\text{-An}}$ of I-analytic, or "derived I-complete", R_*-modules; we will review these notions in Subsection 6.2. For example, taking $R = E$ to be a Lubin-Tate spectrum at height h and $I = \mathfrak{m} \subset E_0$ to be the maximal ideal returns the category of $K(h)$-local E-modules and category of L-complete E_*-modules in the sense of [HS99, Appendix A.3]. We can identify $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}})$, and under a minor algebraic tameness condition on I, we have $\mathrm{L}\mathcal{M}\mathrm{od}_{R_*}^{I\text{-An}} \simeq \mathrm{Psh}^{\Pi}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}}, \mathcal{S}\mathrm{et})$.

(4) If $\mathcal{P}$ is a resolution theory and T is a monad on $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ which preserves geometric realizations, then $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P})$, where $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ is the full subcategory spanned by the image of $\mathcal{P}$ under T. In this case, $\mathrm{Psh}^{\Pi}(T\mathcal{P}, \mathcal{S}\mathrm{et})$ is monadic over $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$, and heuristically consists of objects of $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ equipped with the additional operations that act naturally on the homotopy of T-algebras; compare the treatments of algebraic theories of power operations in [Rez06] and [Law19]. In particular,
   (a) If $\mathcal{P} = \mathcal{M}\mathrm{od}_{HR}^{\mathrm{free}}$ for a discrete ring R of positive characteristic and T is the free commutative algebra functor, then $\mathrm{Psh}^{\Pi}(T\mathcal{P}, \mathcal{S}\mathrm{et})$ is the category of graded commutative R-algebras equipped with Dyer-Lashof operations satisfying the Cartan formula and subject to certain instability conditions;
   (b) If $\mathcal{P} = \mathcal{M}\mathrm{od}_E^{\mathrm{loc},\mathrm{free}}$ is the category of $K(h)$-localizations of free E-modules, where E is a height h Lubin-Tate spectrum, and T is the free $K(h)$-local commutative algebra functor, then $\mathrm{Psh}^{\Pi}(T\mathcal{P}, \mathcal{S}\mathrm{et})$ is the category of $\mathfrak{m}$-analytic $\mathbb{T}$-algebras, where $\mathbb{T}$ is the monad on E_*-modules considered in [Rez09].

The preceding two examples are algebraically well-behaved, making the resulting obstruction theories relatively computable. We will treat this in detail in separate work. ◁

1.2.5. **Remark.** It is worth pointing out the following notational subtlety. By way of example, let R be an $\mathbb{A}_\infty$-ring, so that $\mathrm{Psh}^{\Pi,\Omega}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_R$ and $\mathrm{Psh}^{\Pi}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}}, \mathcal{S}\mathrm{et}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{R_*}$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{L}\mathcal{M}\mathrm{od}_R & \xrightarrow{\pi_*} & \mathrm{L}\mathcal{M}\mathrm{od}_{R_*} \\ \downarrow h & & \simeq \downarrow h \\ \mathrm{Psh}^{\Pi}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}}) & \xrightarrow{\pi_0} & \mathrm{Psh}^{\Pi}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}}, \mathcal{S}\mathrm{et}) \end{array},$$

i.e.

$$\pi_* M = \pi_0 h(M).$$

In general, if $\mathcal{P}$ is a resolution theory and $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, then $\pi_0 X$ is the entire algebraic avatar of X, encoding all of its homotopy groups. ◁

In Sections 4 and 5, which are entirely independent of each other, we use the categories $\mathrm{Psh}^{\Pi}(\mathcal{P})$ to produce computational tools for categories of the form $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. We begin by describing a sample of Section 4.

1.2.6. **Definition.** $\mathcal{P}$ is said to be *stable* if

(1) $\mathcal{P}$ is pointed and admits suspensions;
(2) $\Sigma\colon \mathcal{P} \to \mathcal{P}$ is an equivalence. ◁

Fix a stable resolution theory $\mathcal{P}$. As we will see, in this case $\mathrm{Psh}^{\Pi}(\mathcal{P})$ is additive and thus embeds into its stabilization, which we identify as $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$; moreover, $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is a stable category and $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})$ is an abelian category with enough projectives. For $X, Y \in \mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$, there is a mapping spectrum $\mathbf{Map}(X,Y)$, so that $\Omega^\infty\mathbf{Map}(X,Y) = \mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})}(X,Y)$. For $X \in \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P},\mathcal{S}\mathrm{p})$ and $n \in \mathbb{Z}$, write $X[n] = X \circ \Sigma^n$. In Subsection 4.4, we show that for $M, N \in \mathcal{M}$, Postnikov towers in $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ give a filtration of $\mathbf{Map}(M,N)$ yielding the following universal coefficient-type spectral sequence.

1.2.7. **Theorem** (4.4.1)**.** For $M, N \in \mathcal{M}$, there is a conditionally convergent spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})}(\pi_0 M; \pi_0 N[p]) \Rightarrow \pi_{-q}\mathbf{Map}(M,N), \quad d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q+1}.$$

◁

Part of the strength here is the freedom in choice of $\mathcal{P}$. For example, in addition to the classical universal coefficient spectral sequences, one can recover homotopy limit and Bockstein-type spectral sequences in this way.

Fix now another resolution theory $\mathcal{P}'$, not necessarily stable, together with a functor $F\colon \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}') \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ which preserves geometric realizations. The composite $\pi_0 \circ F \circ h\colon \mathcal{P}' \to \mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})$ factors uniquely through $\mathrm{h}\mathcal{P}'$, yielding a functor $\overline{f}\colon \mathrm{h}\mathcal{P}' \to \mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})$, and thus by left Kan extension a functor $\overline{F}\colon \mathrm{Psh}^{\Pi}(\mathcal{P}',\mathcal{S}\mathrm{et}) \to \mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})$ preserving reflexive coequalizers as well as a total left-derived functor $\mathbb{L}\overline{F}\colon \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}') \to \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$.

1.2.8. **Theorem** (4.2.2)**.** In the situation of the previous paragraph, there is for $R \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}')$ a spectral sequence in $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})$ of the form

$$E^1_{p,q} = (\mathbb{L}_{p+q}\overline{F}\pi_0 R)[-p] \Rightarrow (\pi_0 FR)[q], \qquad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

If π_0 preserves filtered colimits, or else if $\mathbb{L}\overline{F}\pi_0 R$ is truncated, then this spectral sequence is convergent. ◁

This gives rise, for instance, to Künneth-type spectral sequences, homotopy colimit spectral sequences, spectral sequences for topological André-Quillen homology, and other variants of these. Theorem 1.2.8 is proved by studying an explicit localization $L\colon \mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})$, the existence of which is one of the key features of the stable setting. Moreover, this construction has good monoidal properties, allowing for the introduction of pairings on the resulting spectral sequences in the cases one would expect (Theorem 4.3.3). This gives rise, for instance, to pairings on Künneth-type spectral sequences; such pairings have a history of being difficult to construct by hand, see [Til16]. The construction relies on the relation between pairings of towers and pairings on their associated spectral sequences. We could not locate this relation in the literature, so we provide an overview in Appendix A, which is independent of the rest of the paper.

We now move on to describing some of Section 5, where we do not require $\mathcal{P}$ to be stable. This section is concerned with Postnikov decompositions in $\mathrm{Psh}^{\Pi}(\mathcal{P})$. The constructions we give here might be compared to earlier simplicial work of Blanc-Dwyer-Goerss [BDG04] and Goerss-Hopkins [GH05], and build on Pstrągowski [Pst17]. One consequence of the theory is the following.

1.2.9. **Theorem** (5.3.1)**.** Fix $A, C \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, together with a map $\phi\colon \pi_0 A \to \pi_0 C$ in $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$. Let $\mathrm{Map}^{\phi}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(A, C) \subset \mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(A, C)$ be the space of maps f such that $\pi_0 f = \phi$. Then the Postnikov tower of C in $\mathrm{Psh}^{\Pi}(\mathcal{P})$ gives a decomposition

$$\mathrm{Map}^{\phi}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(A, C) \simeq \lim_{n\to\infty} \mathrm{Map}^{\phi}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(A, C_{\leq n}),$$

where $\mathrm{Map}^{\phi,\leq 0}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(A, C) = \{\phi\}$ and for each $n \geq 1$ there is a canonical fiber sequence

$$\mathrm{Map}^{\phi}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(A, C_{\leq n}) \to \mathrm{Map}^{\phi}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(A, C_{\leq n-1}) \to \mathrm{Map}_{\mathrm{h}\mathcal{P}/\pi_0 C}(\pi_0 A; B^{n+1}_{\pi_0 C}\Pi_n C),$$

where we have written $\Pi_n C = \pi_0 C^{S^n}$. In particular, there are successively defined obstructions in the Quillen cohomology groups $H^{n+1}_{\mathrm{h}\mathcal{P}/\pi_0 C}(\pi_0 A; \Pi_n C) = \pi_0 \mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})/\pi_0 C}(\pi_0 A; B^{n+1}_{\pi_0 C}\Pi_n C)$ to realizing ϕ as arising from a map $A \to C$. ◁

In fact, we prove something a little stronger, giving a similar decomposition for mapping spaces in slice categories of $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. In Subsection 5.4, we verify that the obstruction theory of [Pst17] for realizing an object $\Lambda \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ as $\Lambda = \pi_0 R$ with $R \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ holds in our setting.

In Section 6, we deal with completions in detail; here, it is essential that we have allowed for infinitary theories. In Subsection 6.1, we study in general some interactions between localizations and theories. In Subsection 6.2, we introduce R-linear theories for a connective $\mathbb{E}_2$-ring R, and study the corresponding notions of I-completions for a finitely generated ideal $I \subset R_0$. In particular, we give conditions under which the associated algebraic categories, and thus associated obstruction theories and spectral sequences, can be understood in terms of more familiar algebra.

2. Mal'cev theories

This section covers the general theory of Mal'cev theories. In particular, in Subsection 2.1, we show that if $\mathcal{P}$ is a Mal'cev theory, then the category $\mathrm{Psh}^{\Pi}(\mathcal{P})$ of models of $\mathcal{P}$ is obtained by freely adjoining geometric realizations to $\mathcal{P}$. Moreover, we verify that it is presentable under some minor smallness conditions on $\mathcal{P}$. In Subsection 2.2, we restrict to the case where $\mathcal{P}$ is a discrete Mal'cev theory, verify that $\mathrm{Psh}^{\Pi}(\mathcal{P})$ is the underlying ∞-category of Quillen's model structure on simplicial objects in $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$, and review the resulting notions of co/homology.

2.1. Definitions and universal properties.

Recall that the structure of a *herd* on a set X is a ternary operation t satisfying $t(x, x, y) = y$ and $t(x, y, y) = x$, and write $\mathcal{H}\mathrm{rd}$ for the category of herds. Herds are the models of an algebraic theory, so we can interpret herd objects in an arbitrary category with finite products, allowing for the following definition.

2.1.1. **Definition.** A *Mal'cev theory* is a category $\mathcal{P}$ such that

(1) $\mathcal{P}$ admits small coproducts;
(2) All objects of $\mathcal{P}$ admit the structure of a coherd.

For a regular cardinal κ, a Mal'cev theory $\mathcal{P}$ is said to be κ*-bounded* if there exists an essentially small full subcategory $\mathcal{P}_0 \subset \mathcal{P}$ such that

(3) $\mathcal{P}_0$ is closed under κ-small coproducts;
(4) Every object of $\mathcal{P}$ is a retract of a small coproduct of objects of $\mathcal{P}_0$;
(5) For every $P \in \mathcal{P}_0$ and set of objects $\{P'_i : i \in I\}$ in $\mathcal{P}$, the canonical map $\mathrm{colim}_{F\subset I,\, |F|<\kappa} \mathrm{Map}_{\mathcal{P}}(P, \coprod_{i\in F} P'_i) \to \mathrm{Map}_{\mathcal{P}}(P, \coprod_{i\in I} P'_i)$ is an equivalence.

We say that $\mathcal{P}$ is *bounded* if it is κ-bounded for some κ. ◁

As in the introduction, we will refer to Mal'cev theories as just *theories*. Beyond this subsection, we will moreover assume that all of our theories are bounded; see Remark 2.1.10. Throughout this subsection, and everywhere else in the paper, $\mathcal{P}$ will always refer to a theory. For the sake of brevity, we will generally assume that the objects of $\mathcal{P}$ have been equipped with the structure of a coherd; the main point is that maps in $\mathcal{P}$ need not respect this structure.

For us, herds appear as an unpointed generalization of groups, with which they share a number of niceness properties. We can summarize the properties of herds that we will use in this paper as follows.

2.1.2. **Lemma.** We have the following.

(1) Every map of simplicial herds which is a degreewise surjection onto path components in its image is a Kan fibration;
(2) Every group object in $\mathcal{H}\mathrm{rd}$ is abelian;
(3) Every covering map in the category of herd objects in $\mathcal{G}\mathrm{pd}_\infty$ which admits a section has trivial monodromy.

Proof. Assertion (1) is well known should we replace herds with groups, and the same proof applies. In brief, suppose given a surjection $p\colon E \to B$ of simplicial herds, elements $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in E_n$ such that $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$ and $i \neq k$, and $y \in B_{n+1}$ such that $d_i(y) = p(x_i)$ for $i \neq k$. Then we can inductively define $w_r \in E_{n+1}$ such that $p(w_r) = y$ and $d_i w_r = x_i$ for $i \leq r$ and $i \neq k$ by choosing w_{-1} to be any element in the preimage of y, and setting $w_r = t(w_{r-1}, s_r d_r w_{r-1}, s_r x_r)$, except when $r = k$, in which case $w_r = w_{r-1}$. Then $w_{n+1} \in E_{n+1}$ witnesses the Kan condition. For (2), observe that if G is a group object in $\mathcal{H}\mathrm{rd}$ with unit e, then for any $g, h \in G$ we have $gh = t(g,e,e)t(e,e,h) = t(g,e,h) = t(e,e,h)t(g,e,e) = hg$. For (3), let $\pi\colon E \to X$ be a covering map of herd objects in $\mathcal{G}\mathrm{pd}_\infty$ with section $s\colon X \to E$. To show that this cover has trivial monodromy, it is sufficient to verify that for all $x \in X$ the inclusion $\pi^{-1}(x) \to E$ admits a retraction. Such a retraction is given by $e \mapsto t(e, s\pi(e), s(x))$. □

We use notation as in the introduction, so $\mathrm{Psh}^\Pi(\mathcal{P}) \subset \mathrm{Psh}(\mathcal{P})$ is the category of product-preserving presheaves on $\mathcal{P}$. As $\mathcal{P}$ consists of coherds, we find for $X \in \mathrm{Psh}^\Pi(\mathcal{P})$ and $P \in \mathcal{P}$ that $X(P)$ is a herd; this structure is natural in X, though not in P. This is sufficient for the following.

2.1.3. **Proposition.** The subcategory $\mathrm{Psh}^\Pi(\mathcal{P}) \subset \mathrm{Psh}(\mathcal{P})$ is closed under small limits and geometric realizations.

Proof. The assertion regarding small limits is clear, so we must verify that the pointwise geometric realization of a simplicial object in $\mathrm{Psh}^\Pi(\mathcal{P})$ again lives in $\mathrm{Psh}^\Pi(\mathcal{P})$. As $\mathcal{P}$ consists of coherds and the forgetful functor $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty) \to \mathcal{G}\mathrm{pd}_\infty$ preserves geometric realizations, it is sufficient to verify that geometric realizations commute with small products in the category $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$ of herd objects in $\mathcal{G}\mathrm{pd}_\infty$. As $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$ is modeled by the model category of simplicial herds, we can model a simplicial object in $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$ by a bisimplicial herd and its geometric realization by the diagonal of this bisimplicial herd. As all simplicial herds are fibrant by Lemma 2.1.2, small products of simplicial herds are homotopy products, so the result follows as products and diagonals of bisimplicial sets commute. □

We also record the following here.

2.1.4. **Lemma.** Fix a levelwise Cartesian square

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \longrightarrow & Z \end{array}$$

of simplicial objects in $\mathrm{Psh}^{\Pi}(\mathcal{P})$, and suppose that π is levelwise a π_0-surjection. Then the square remains Cartesian after geometric realization.

Proof. By Proposition 2.1.3, we may by evaluating on $P \in \mathcal{P}$ reduce to proving the corresponding statement with $\mathrm{Psh}^{\Pi}(\mathcal{P})$ replaced by $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$. The square can now be modeled as a Cartesian square of bisimplicial herds in which the map π is levelwise a surjection. This square remains Cartesian upon taking diagonals, and remains homotopy Cartesian as π remains a Kan fibration by Lemma 2.1.2. This proves the claim. □

Observe that $\mathrm{Psh}^{\Pi}(\mathcal{P})$ consists of those small presheaves which are local with respect to the class of maps of the form $\coprod_{i\in I} h(P_i) \to h(\coprod_{i\in I} P_i)$ for $\{P_i : i \in I\}$ a set of objects in $\mathcal{P}$.

2.1.5. **Lemma.** The inclusion $R\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathrm{Psh}(\mathcal{P})$ admits a left adjoint.

Proof. By the Yoneda lemma, it is sufficient to verify the pointwise assertion that for all $X \in \mathrm{Psh}(\mathcal{P})$, the functor $\mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(X, R(-))\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathcal{G}\mathrm{pd}_\infty$ is representable; see for instance [Cis19, Proposition 6.1.11]. By definition of $\mathrm{Psh}(\mathcal{P})$, the presheaf X is small, and thus admits a presentation of the form $X \simeq \mathrm{colim}_{n\in\Delta^{\mathrm{op}}} \coprod_{i\in I_n} h(P_{n,i})$ for some sets I_n and $P_{n,i} \in \mathcal{P}$. As a consequence, we have

$$\mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(X, R(-)) \simeq \mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} h(\coprod_{i\in I_n} P_{n,i}), R(-)).$$

We conclude by Proposition 2.1.3, which asserts that $\mathrm{colim}_{n\in\Delta^{\mathrm{op}}} h(\coprod_{i\in I_n} P_{n,i})$ lives in $\mathrm{Psh}^{\Pi}(\mathcal{P})$. □

2.1.6. **Proposition.** For a theory $\mathcal{P}$,

(1) The category $\mathrm{Psh}^{\Pi}(\mathcal{P})$ admits all small limits and colimits;
(2) The subcategory $\mathrm{Psh}^{\Pi}(\mathcal{P}) \subset \mathrm{Psh}(\mathcal{P})$ is the smallest subcategory containing all representables and closed under geometric realizations;
(3) For $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, the functor $\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(X, -)$ preserves geometric realizations if and only if X is a retract of a representable.

Proof. Assertion (1) follows immediately from Lemma 2.1.5, as $\mathrm{Psh}(\mathcal{P})$ admits all small colimits and limits and the reflective subcategory of a category admitting all small limits and colimits admits the same. Assertion (2) follows from the proof of Proposition 2.1.9, which gives a way of writing any $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ as a geometric realization of representables. For (3), note that if X is a retract of a representable, then $\mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(X, -)$ preserves all small colimits, so $\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(X, -)$ preserves all geometric realizations, as those are computed in $\mathrm{Psh}(\mathcal{P})$. Conversely, if $\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(X, -)$ preserves geometric realizations, then upon using (2) to write $X \simeq \mathrm{colim}_{n\in\Delta^{\mathrm{op}}} h(P_n)$, we find $\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(X, X) \simeq \mathrm{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(X, h(P_n))$, so that the identity of X factors through some representable. □

2.1.7. **Theorem.** Let $\mathcal{D}$ be a category admitting geometric realizations, and let $F\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathcal{D}$ be a functor. Write $f = F \circ h\colon \mathcal{P} \to \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathcal{D}$. Then

(1) F preserves geometric realizations if and only if it arises as the left Kan extension of f;
(2) F preserves colimits if and only if it preserves geometric realizations and f preserves coproducts;
(3) If the following hold, then F is fully faithful:
  (a) F preserves geometric realizations,
  (b) f is fully faithful,
  (c) For all $P \in \mathcal{P}$, the functor $\mathcal{D}(f(P), -)$ preserves geometric realizations;
(4) If the following hold, then F is an equivalence:
  (d) F preserves colimits,
  (e) F is fully faithful,
  (f) The right adjoint to F, given by $G(D) = \mathcal{D}(f(-), D)$, is conservative.

Proof. Assertions (1) and (2) follow quickly from Proposition 2.1.6 and the general theory of cocompletions of categories, as from [Lur17b, Section 5.3.6]. For (3), suppose given $F\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathcal{D}$ satisfying conditions (a)-(c). We must show that

$$\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(X, Y) \simeq \mathrm{Map}_{\mathcal{D}}(F(X), F(Y)).$$

As X arises as a geometric realization of representable functors, by (a) we may reduce to showing that

$$\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(h(P), Y) \simeq \mathrm{Map}_{\mathcal{D}}(f(P), F(Y))$$

for $P \in \mathcal{P}$. As Y is a geometric realization of representable functors, by (a) and (c) we may reduce to showing that

$$\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(h(P), h(P')) \simeq \mathrm{Map}_{\mathcal{D}}(f(P), f(P'))$$

for $P, P' \in \mathcal{P}$, which is a consequence of (b). For (4), condition (d) ensures that the functor G described in (f) is right adjoint to F, and the assertion then follows from the general fact that an adjunction $F \dashv G$ with F fully faithful and G conservative is an equivalence. □

Suppose now that $\mathcal{P}$ is κ-bounded, choose a subcategory $\mathcal{P}_0 \subset \mathcal{P}$ realizing this, and let $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0) \subset \mathrm{Psh}(\mathcal{P}_0)$ be the full subcategory of presheaves preserving κ-small products.

2.1.8. **Lemma.** We have the following.

(1) The inclusion $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0) \subset \mathrm{Psh}(\mathcal{P}_0)$ preserves geometric realizations and κ-filtered colimits,
(2) The category $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ consists of those objects of $\mathrm{Psh}(\mathcal{P}_0)$ local with respect to the set of maps of the form $\coprod_{i\in F} h(P_i) \to h(\coprod_{i\in I} P_i)$ with $\{P_i : i \in F\}$ a collection of objects of $\mathcal{P}_0$ with $|F| < \kappa$.

In particular, $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ is a κ-compactly generated presentable category. □

2.1.9. **Proposition.** Restriction $R\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ is an equivalence.

Proof. We verify the conditions of Theorem 2.1.7. As geometric realizations are computed pointwise in either category, they are preserved by R. Next, by our smallness assumption on the objects of $\mathcal{P}_0$, we find that for any collection of objects $\{P_i : i \in I\}$ in $\mathcal{P}_0$ we have

$$R(h(\coprod_{i\in I} P_i)) \simeq \operatorname*{colim}_{\substack{F\subset I\\ |F|<\kappa}} h(\coprod_{i\in F} P_i) \simeq \operatorname*{colim}_{\substack{F\subset I\\ |F|<\kappa}} \coprod_{i\in F} h(P_i) \simeq \coprod_{i\in F} h(P_i),$$

and hence $\mathcal{P} \to \mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ preserves coproducts. It follows that for any $P \in \mathcal{P}$, the functor $\mathrm{Map}_{\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)}(h(P), -)$ preserves geometric realizations. The right adjoint to R is conservative, so we conclude by applying Theorem 2.1.7. □

Everything in this subsection has a fully 1-categorical analogue, where all categories are taken to be 1-categories, $\mathrm{Psh}^\Pi(\mathcal{P})$ is replaced by $\mathrm{Psh}^\Pi(\mathcal{P}, \mathcal{S}\mathrm{et})$, and geometric realizations reduce to reflexive coequalizers.

2.1.10. **Remark.** Throughout the rest of this paper, all theories will be assumed to be bounded. In order to avoid cumbersome notation, we will adopt the following convention: if $\mathcal{P}$ is a κ-bounded theory, choose $\mathcal{P}_0 \subset \mathcal{P}$ realizing this; now, $\mathrm{Psh}(\mathcal{P})$ refers to $\mathrm{Psh}(\mathcal{P}_0)$, $\mathrm{Psh}^\Pi(\mathcal{P})$ refers to $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$, and so forth. In case one should meet a theory which is not bounded, we point out that an arbitrary theory $\mathcal{P}$ is of the form $\mathcal{P} = \mathcal{P}'_0$ where $\mathcal{P}'$ is a bounded theory with respect to a larger universe. ◁

2.2. **Rigidification and left derived functors.** Throughout this subsection, all of our theories are assumed to be discrete theories, that is, to be 1-categories. In this subsection, we briefly review the notion of left-derived functors available for categories of the form $\mathrm{Psh}^\Pi(\mathcal{P}, \mathcal{S}\mathrm{et})$. This story is classical, and goes back to [Qui67] and [DP61]; see also [TV69]. To facilitate comparisons with the classical theory, we begin with an identification of a model of $\mathrm{Psh}^\Pi(\mathcal{P})$.

2.2.1. **Lemma** ([Qui67, Section II.4])**.** There is a simplicial model structure on $\mathrm{Psh}^\Pi(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$ in which a map $f\colon X \to Y$ is a weak equivalence, resp., fibration, if and only if for all $P \in \mathcal{P}$ the map $f(P)\colon X(P) \to Y(P)$ is a weak equivalence, resp., fibration. □

2.2.2. **Theorem.** The (∞-categorical) colimit functor

$$C\colon\ \mathrm{Psh}^\Pi(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et}) \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Psh}^\Pi(\mathcal{P})) \to \mathrm{Psh}^\Pi(\mathcal{P})$$

realizes $\mathrm{Psh}^\Pi(\mathcal{P})$ as the underlying ∞-category of $\mathrm{Psh}^\Pi(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$.

Proof. Let W denote the class of weak equivalences in $\mathrm{Psh}^\Pi(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$. Then we need to show the following:

(1) C inverts W;
(2) C is essentially surjective;
(3) For $X, Y \in \mathrm{Psh}^\Pi(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$ with X cofibrant and Y fibrant, C induces $\underline{\mathrm{Map}}_{\mathrm{Psh}^\Pi(\mathcal{P},\mathrm{s}\mathcal{S}\mathrm{et})}(X, Y) \simeq \mathrm{Map}_{\mathrm{Psh}^\Pi(\mathcal{P})}(CX, CY)$, where $\underline{\mathrm{Map}}_{\mathrm{Psh}^\Pi(\mathcal{P},\mathrm{s}\mathcal{S}\mathrm{et})}$ denotes the simplicial enrichment of $\mathrm{Psh}^\Pi(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$.

These are themselves consequences of the following observations:

(a) For $X \in \mathrm{Psh}^\Pi(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$ and $P \in \mathcal{P}$, we have

$$X(P) \cong \underline{\mathrm{Map}}_{\mathrm{Psh}^\Pi(\mathcal{P},\mathrm{s}\mathcal{S}\mathrm{et})}(h(P), X) \simeq \mathrm{Map}_{\mathrm{Psh}^\Pi(\mathcal{P},\mathrm{s}\mathcal{S}\mathrm{et})[W^{-1}]}(h(P), X);$$

(b) As homotopy geometric realizations in $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$ are modeled as diagonals, we have for $X \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$ that

$$X \simeq \operatorname*{hocolim}_{n\in\Delta^{\mathrm{op}}} X_n;$$

(c) For $X \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$ and $P \in \mathcal{P}$, as $\underline{\mathrm{Map}}_{\mathrm{Psh}^{\Pi}(\mathcal{P},\mathrm{s}\mathcal{S}\mathrm{et})}(h(P), X)_n \cong X_n(P)$, we have

$$\underline{\mathrm{Map}}_{\mathrm{Psh}^{\Pi}(\mathcal{P},\mathrm{s}\mathcal{S}\mathrm{et})}(h(P), X) \simeq \operatorname*{hocolim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})}(h(P), X_n). \qquad \square$$

We now briefly review the relevant notion of left-derived functor. Let $\mathcal{P}$ and $\mathcal{P}'$ be discrete theories, and fix an arbitrary functor $\overline{f}\colon \mathcal{P}' \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$. By left Kan extension of $\overline{f}$, we obtain a functor $\overline{F}\colon \mathrm{Psh}^{\Pi}(\mathcal{P}', \mathcal{S}\mathrm{et}) \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ preserving reflexive coequalizers. By left Kan extension of the composite $f\colon \mathcal{P}' \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P})$, we obtain a functor $f_!\colon \mathrm{Psh}(\mathcal{P}') \to \mathrm{Psh}(\mathcal{P})$ preserving geometric realizations such that $\pi_0 f_! X = \overline{F} X$ for any $X \in \mathrm{Psh}^{\Pi}(\mathcal{P}', \mathcal{S}\mathrm{et})$.

2.2.3. **Proposition.** Fix notation as above. Fix $X' \in \mathrm{Psh}^{\Pi}(\mathcal{P}')$, and choose $X'_\bullet \in \mathrm{Psh}^{\Pi}(\mathcal{P}', \mathrm{s}\mathcal{S}\mathrm{et})$ modeling X. Choose a simplicial object $P'_\bullet$ of $\mathcal{P}'$ together with a weak equivalence $h(P'_\bullet) \to X'_\bullet$. Then $f_! X'$ is modeled by $f P'_\bullet$.

Proof. By Theorem 2.2.2, to say that $X'_\bullet$ models X is to say we have chosen an identification $\mathrm{colim}_{n\in\Delta^{\mathrm{op}}} X'_n = X'$ in $\mathrm{Psh}^{\Pi}(\mathcal{P}')$, and to say $h(P'_\bullet) \to X'_\bullet$ is a weak equivalence is to say it induces $\mathrm{colim}_{n\in\Delta^{\mathrm{op}}} h(P'_n) \simeq \mathrm{colim}_{n\in\Delta^{\mathrm{op}}} X'_n \simeq X'$ in $\mathrm{Psh}^{\Pi}(\mathcal{P}')$. By definition of $f_!$, we learn

$$f_! X' \simeq f_! \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} h(P'_n) \simeq \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} f P'_n,$$

and the result follows as $\mathrm{colim}_{n\in\Delta^{\mathrm{op}}} f P'_n$ is modeled by $f P'_\bullet$. $\square$

This justifies writing $\mathbb{L}\overline{F} = f_!\colon \mathrm{Psh}^{\Pi}(\mathcal{P}') \to \mathrm{Psh}^{\Pi}(\mathcal{P})$ and calling it the total left-derived functor of $\overline{F}$, for by Proposition 2.2.3 this is equivalent to any other correct definition of $\mathbb{L}\overline{F}$.

3. Resolution theories

This section covers some general theory of resolution theories. The basic example is the category $\mathcal{M}\mathrm{od}_R^{\mathrm{free}}$ of free R-modules for some $\mathbb{A}_\infty$-ring R; as R need not be connected, we allow "free R-module" to include suspensions and desuspensions of R. If M is an R-module, then the associated functor $h(M) \in \mathrm{Psh}^{\Pi}(\mathcal{M}\mathrm{od}_R^{\mathrm{free}})$ lives in the full subcategory $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{M}\mathrm{od}_R^{\mathrm{free}})$ consisting of those X with the additional property that $X(\Sigma F) \simeq \Omega X(F)$, and this turns out be a full characterization, i.e. there is an equivalence $\mathcal{M}\mathrm{od}_R \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{M}\mathrm{od}_R^{\mathrm{free}})$; a particular case of this appears in [HL17, Proposition 4.2.5]. Resolution theories axiomatize the general situation, and their utility arises from the ability, that we will demonstrate throughout the rest of the paper, to relate problems in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ to problems in the more algebraic derived category $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$. In particular, in Subsection 3.2, we verify that the spiral exact sequence, as interpreted in [Pst17], holds equally well in our setting; this is the main tool for relating $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ to $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$. In Subsection 3.3, we record some tools for writing categories $\mathcal{M}$ as $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ for some $\mathcal{P} \subset \mathcal{M}$, and for describing the categories $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et})$.

3.1. **Definitions and notation.** Fix a theory $\mathcal{P}$.

3.1.1. **Definition.** The theory $\mathcal{P}$ is

(1) A *resolution theory* if for all finite wedges F of spheres and $P \in \mathcal{P}$, the tensor $F \otimes P = \operatorname{colim}_{x \in F} P$ exists in $\mathcal{P}$;
(2) A *pointed resolution theory* if moreover $\mathcal{P}$ is pointed and admits suspensions;
(3) An *additive resolution theory* if it is pointed and additive;
(4) A *stable resolution theory* if it is pointed and $\Sigma\colon \mathcal{P} \to \mathcal{P}$ is an equivalence. ◁

Suppose now that $\mathcal{P}$ is a resolution theory, and define $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P})$ to consist of those X satisfying the additional condition that $X(F \otimes P) \simeq X(P)^F$ for all $P \in \mathcal{P}$ and finite wedge of spheres F. In other words, $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is the full subcategory of $\mathrm{Psh}^{\Pi}(\mathcal{P})$ local with respect to $F \otimes h(P) \to h(F \otimes P)$ for all $P \in \mathcal{P}$ and finite wedge of spheres F. Because we are assuming that $\mathcal{P}$ is bounded, as laid out in Remark 2.1.10, we obtain the following.

3.1.2. **Lemma.** The category $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is an accessible localization of $\mathrm{Psh}^{\Pi}(\mathcal{P})$. In particular, it is presentable. □

For $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ and a finite wedge of spheres F, write X_F for the presheaf $X_F(P) = X(F \otimes P)$. Then there are canonical maps $X_F \to X^F$, and the condition that $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is equivalent to the condition that these maps be equivalences for all finite wedges of spheres F. It turns out to be sufficient to verify this in the case where $F = S^1$.

3.1.3. **Lemma.** Fix a coCartesian diagram

$$\begin{array}{ccc} F_1 & \longleftarrow & F_2 \\ \uparrow & & \uparrow{\scriptstyle i} \\ F_3 & \longleftarrow & F_4 \end{array}$$

of wedges of spheres. Fix $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$. Then the resulting square

$$\begin{array}{ccc} X_{F_1} & \longrightarrow & X_{F_2} \\ \downarrow & & \downarrow \\ X_{F_3} & \longrightarrow & X_{F_4} \end{array}$$

is Cartesian. In particular, the cogroup structure on S^n gives maps

$$X_{S^n} \to X_{S^n \vee S^n} \simeq X_{S^n} \times_X X_{S^n}$$

making X_{S^n} into a group object in $\mathrm{Psh}^{\Pi}(\mathcal{P})/X$ for $n \geq 1$.

Proof. If X is representable, or more generally if $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, then this is clear. In general, by splitting off the path components not in the image of i, we may reduce to the case where i is an injection on path components. Here the claim follows by writing X as a geometric realization of representables and appealing to Lemma 2.1.4. □

3.1.4. **Proposition.** Fix $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$. Then the following are equivalent:

(1) $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$;
(2) The map $X_{S^1} \to X^{S^1}$ is an equivalence.

Proof. The implication (1) ⇒ (2) is clear, so suppose conversely that $X_{S^1} \to X^{S^1}$ is an equivalence. We must show that $X_F \to X^F$ is an equivalence when F is any wedge of spheres. By Lemma 3.1.3, we have equivalences $X_{F' \vee F''} \simeq X_{F'} \times_X X_{F''}$, and so can reduce to the case where $F = S^n$. When $n = 0$, this is a consequence of the fact that $X \in \mathrm{Psh}^\Pi(\mathcal{P})$. For $n \geq 2$, it follows by an inductive argument using the Cartesian squares

$$\begin{array}{ccc} X_{S^{n+1}} & \longrightarrow & (X_{S^n})_{S^1} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_{S^n} \times_X X_{S^1} \end{array}$$

for $n \geq 1$, obtained by applying Lemma 3.1.3 to the cofibering $S^n \vee S^1 \to S^n \times S^1 \to S^{n+1}$ and identifying $X_{S^n \times S^1} \simeq (X_{S^n})_{S^1}$. □

We end this subsection by introducing some additional notation. When $\mathcal{P}$ is pointed, write X_{Σ^n} for the presheaf $X_{\Sigma^n}(P) = X(\Sigma^n P)$. If, for instance, $\mathcal{P}$ is additive, we can split $X_{S^n} \simeq X \times X_{\Sigma^n}$, so in particular $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ consists of those $X \in \mathrm{Psh}^\Pi(\mathcal{P})$ such that $X_\Sigma \simeq \Omega X$.

The functor $P \mapsto S^n \otimes P$ descends to a functor on $\mathrm{h}\mathcal{P}$; for $X \in \mathrm{Psh}^\Pi(\mathrm{h}\mathcal{P})$, write $X\langle n \rangle$ for the restriction of X along this functor. Thus $\pi_0(X_{S^n}) = (\pi_0 X)\langle n \rangle$ for $X \in \mathrm{Psh}^\Pi(\mathcal{P})$. Similarly write $X[n]$ for the restriction of X along $\Sigma^n \colon \mathcal{P} \to \mathcal{P}$ when $\mathcal{P}$ is pointed. We point out that these constructions rely on structure present in $\mathcal{P}$ that is not present in $\mathrm{h}\mathcal{P}$.

3.2. **The spiral.** Let $\mathcal{P}$ be a resolution theory, and write $\tau \colon \mathcal{P} \to \mathrm{h}\mathcal{P}$ for the canonical map to its homotopy category. To connect $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ with $\mathrm{Psh}^\Pi(\mathrm{h}\mathcal{P})$, we will need to understand $\tau_! \colon \mathrm{Psh}^\Pi(\mathcal{P}) \to \mathrm{Psh}^\Pi(\mathrm{h}\mathcal{P})$. This understanding is achieved via the following.

3.2.1. **Theorem** ([Pst17, Theorem 2.86])**.** For $X \in \mathrm{Psh}^\Pi(\mathcal{P})$,

(1) The map $X \to \tau^*\tau_! X$ is a π_0-equivalence;
(2) There is a canonical Cartesian square

$$\begin{array}{ccc} B_X X_{S^1} & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \tau^*\tau_! X \end{array},$$

where $B_X X_{S^1}$ is the delooping of X_{S^1} in the slice category $\mathrm{Psh}^\Pi(\mathcal{P})/X$.

Proof. In case $X = h(P)$ with $P \in \mathcal{P}$, we can identify $\tau_! h(P) = h(\tau P)$, and thus $\tau^*\tau_! X = \pi_0 X$. In this case, $X \to \tau^*\tau_! X$ is certainly a π_0-equivalence, and the above square becomes

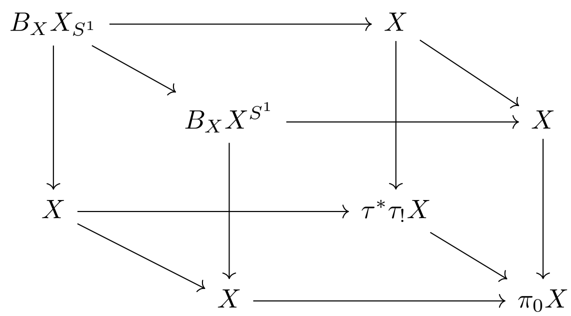
$$\begin{array}{ccc} B_X X^{S^1} & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0 X \end{array},$$

which is Cartesian. Next, observe both that the terms in the original square, as well as the property of $X \to \tau^*\tau_! X$ being a π_0-equivalence, are compatible with the formation of geometric realizations. By writing X as a geometric realization of representables, we may conclude with an application of Lemma 2.1.4. □

For many applications, the following is sufficient.

3.2.2. **Corollary.** For $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, the map $\tau_! X \to \pi_0 X$ is an equivalence if and only if $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$.

Proof. Fix $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, and consider the cube



in which the front and back faces are Cartesian. If $\tau^*\tau_! X \to \pi_0 X$ is an equivalence, then we find that $B_X X_{S^1} \to B_X X^{S^1}$ must be an equivalence. Conversely, if $B_X X_{S^1} \to B_X X^{S^1}$ is an equivalence, then as $\tau^*\tau_! X \to \pi_0 X$ is a π_0-equivalence, the right square is Cartesian and this implies that $\tau^*\tau_! X \simeq \pi_0 X$. □

3.3. **Producing examples.** This subsection is concerned with producing and identifying categories of the form $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, as well as the associated algebraic categories $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$. A simple class of examples is given by the following observation: if $\mathcal{P}$ is a discrete theory, then $\mathcal{P}$ is a resolution theory with $F \otimes P \simeq (\pi_0 F) \otimes P$ for F a wedge of spheres and $P \in \mathcal{P}$. In this case, $\pi_0 \colon \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ is an equivalence.

More interesting examples come from resolution theories with more homotopical structure. In [Pst17, Proposition 3.1], the following example is given: if $\mathcal{P} \subset \mathcal{G}\mathrm{pd}_\infty^*$ is the full subcategory of wedges of positive-dimensional spheres, then $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is the category of pointed connected spaces, and $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ is the category of Π-algebras. We are interested in examples arising from spectral algebra, so we instead begin with the stable case.

3.3.1. **Lemma.** If $\mathcal{P}$ is a stable resolution theory, then $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is a stable category.

Proof. This is a consequence of [Lur17a, Corollary 1.4.2.27], as precomposition with the equivalence $\Sigma \colon \mathcal{P} \to \mathcal{P}$ agrees with Ω on $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. □

3.3.2. **Theorem.** Let $\mathcal{M}$ be a stable category admitting small colimits, and let $\mathcal{P} \subset \mathcal{M}$ be a full subcategory which is a stable resolution theory with coproducts and suspensions computed in $\mathcal{M}$. Then

(1) The restricted Yoneda embedding $h \colon \mathcal{M} \to \mathrm{Psh}(\mathcal{P})$ is fully faithful upon restriction to the thick subcategory generated by $\mathcal{P}$;
(2) The restricted Yoneda embedding yields an equivalence $\mathcal{M} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ provided either of the following is satisfied:

(a) The restricted Yoneda embedding is conservative and $\mathcal{P}$ is generated under coproducts by objects which are compact in $\mathcal{M}$;
(b) There is a fixed finite diagram $\mathcal{J}$ such that every object of $\mathcal{M}$ may be written as a $\mathcal{J}$-shaped colimit of objects of $\mathcal{P}$.

Proof. Write $k\colon \mathcal{M} \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. As k is a limit-preserving functor between stable categories, it preserves finite colimits. If we fix $Y \in \mathcal{M}$, then the collection of X such that

$$\mathrm{Map}_{\mathcal{M}}(X,Y) \to \mathrm{Map}_{\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})}(X,Y)$$

is an equivalence is a thick subcategory of $\mathcal{M}$ containing $\mathcal{P}$, proving (1). For (2), we claim that in either case k preserves all colimits. In case (a), k preserves filtered colimits, so this follows from preservation of finite colimits. In case (b), it is sufficient to verify that k preserves coproducts. Given a collection $\{M_i : i \in I\}$ of objects of $\mathcal{M}$, we may write $M_i \simeq \mathrm{colim}_{j\in\mathcal{J}} M_{i,j}$, and so compute

$$k\left(\bigoplus_{i\in I} M_i\right) \simeq k\left(\operatorname*{colim}_{j\in\mathcal{J}} \bigoplus_{i\in I} P_{i,j}\right) \simeq \operatorname*{colim}_{j\in\mathcal{J}} \bigoplus_{i\in I} k\left(P_{i,j}\right) \simeq \bigoplus_{i\in I} k\left(M_i\right)$$

as k preserves all finite colimits and all small coproducts of objects of $\mathcal{P}$. As $\mathcal{M}$ admits small colimits, k admits a left adjoint L, and the fact that k preserves small colimits readily implies that $X \simeq kLX$ for $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. It is then sufficient to verify that $LkM \simeq M$ for $M \in \mathcal{M}$. This is immediate in case (b), and in case (a) follows as k is conservative and $kM \simeq kLkM$. □

Observe that if $\mathcal{P}$ is stable then it is additive. We can identify the algebraic categories that arise this way as follows.

3.3.3. **Proposition.** The following hold.

(1) If $\mathcal{P}$ is an discrete additive theory, then $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ is a complete and cocomplete abelian category with enough projectives;
(2) If $\mathcal{A}$ is a cocomplete abelian category, and $\mathcal{P} \subset \mathcal{A}$ is a full subcategory consisting of projective objects and closed under coproducts such that every $M \in \mathcal{A}$ is resolved by objects of $\mathcal{P}$, then $\mathcal{A} \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$.

Proof. Consider first (1). Observe that as $\mathcal{P}$ is additive, we have $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{A}\mathrm{b})$. This is a full subcategory of $\mathrm{Psh}(\mathcal{P}, \mathcal{A}\mathrm{b})$ closed under finite limits and colimits, and is therefore abelian; that it is complete and cocomplete with enough projectives is clear from Proposition 2.1.6.

Consider next (2). Fix such $\mathcal{A}$ and $\mathcal{P} \subset \mathcal{A}$. As $\mathcal{A}$ admits small colimits, the restricted Yoneda embedding admits a left adjoint $L\colon \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et}) \to \mathcal{A}$, and by the 1-categorical analogue of Theorem 2.1.7 we must only verify that $h\colon \mathcal{A} \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ is conservative, which follows from the assumption that every $M \in \mathcal{A}$ is resolved by objects of $\mathcal{P}$. □

We now move on to methods that allow for the production of unstable examples.

3.3.4. **Proposition.** Let $\mathcal{P}$ and $\mathcal{P}'$ be theories, and $f\colon \mathcal{P}' \to \mathcal{P}$ an essentially surjective coproduct-preserving functor. Then

(1) The restriction $f^*\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathrm{Psh}^{\Pi}(\mathcal{P}')$ is the forgetful functor of a monadic adjunction;

(2) If $\mathcal{P}$ and $\mathcal{P}'$ are resolution theories and f preserves S^n-tensors, then restriction $f^*\colon \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}')$ is the forgetful functor of a monadic adjunction.

Proof. Observe that both instances of f^* are right adjoints, with left adjoints constructed from $f_!$. Moreover, the assumption that f is essentially surjective implies that each f^* is conservative. By the Barr-Beck theorem [Lur17a, Theorem 4.7.3.5], it is sufficient to verify that f^* creates f^*-split geometric realizations. Indeed, split geometric realizations are in particular pointwise geometric realizations, so this follows from the fact that f^* is essentially surjective. □

3.3.5. **Proposition.** Let $\mathcal{P}$ be a theory, and let $T\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathrm{Psh}^{\Pi}(\mathcal{P})$ be a monad which preserves geometric realizations, so that T is the left Kan extension of its restriction t to $\mathcal{P}$. Let $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ be the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$. Then $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi}(T\mathcal{P})$.

Proof. This follows easily from Theorem 2.1.7. □

3.3.6. **Theorem.** Let $\mathcal{P}$ be a resolution theory, and T an accessible monad on $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. Let $t = Th$ denote the restriction of T to $\mathcal{P}$, and let $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ be the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$. Write $L\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ for the localization. Then the restricted Yoneda embedding yields an equivalence $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P})$ if and only if the canonical map $Lt_!X \to TX$ is an equivalence for $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. In particular, this holds if T preserves geometric realizations.

Proof. Observe that we have a factorization of forgetful functors

$$\mathcal{A}\mathrm{lg}_T \xrightarrow{h} \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P}) \xrightarrow{t^*} \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) .$$

By Proposition 3.3.4, both $\mathcal{A}\mathrm{lg}_T \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ and $\mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ are forgetful functors of monadic adjunctions, with associated monads T and $Lt_!$. The above factorization gives rise to a map $Lt_! \to T$ of monads, which is an equivalence if and only if $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P})$. □

3.3.7. **Remark.** Although we have used the language of monads, as it makes the relevant applications more apparent, this has the downsides of both relying on more technology than is necessary, and obscuring some of the underlying logic. This could be avoided; for example, Theorem 3.3.6 amounts the following statement, which can be proved directly: Fix a resolution theory $\mathcal{P}$, presentable category $\mathcal{D}$, and conservative right adjoint $U\colon \mathcal{D} \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ which preserves U-split geometric realizations. Write T for the left adjoint, $t = Th$ for the restriction of T to $\mathcal{P}$, and $T\mathcal{P} \subset \mathcal{D}$ for the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$. Then the restricted Yoneda embedding $h\colon \mathcal{D} \to \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P})$ is an equivalence if and only if the diagram

$$\begin{array}{ccccc} \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) & \xrightarrow{T} & \mathcal{D} & \xrightarrow{U} & \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \\ \downarrow{\scriptstyle j} & & & & \uparrow{\scriptstyle L} \\ \mathrm{Psh}^{\Pi}(\mathcal{P}) & \xrightarrow{t_!} & \mathrm{Psh}^{\Pi}(T\mathcal{P}) & \xrightarrow{t^*} & \mathrm{Psh}^{\Pi}(\mathcal{P}) \end{array}$$

canonically commutes, i.e. the natural transformation $Lt^*t_!j \to UT$ is an equivalence. This simplifies further if U preserves all geometric realizations. ◁

In the situation of Theorem 3.3.6, we would like to identify the algebraic category $\mathrm{Psh}^{\Pi}(T\mathcal{P}, \mathcal{S}\mathrm{et})$. To that end, we have the following.

3.3.8. **Proposition.** Let $\mathcal{P}$ be a theory, and fix a monad on $\mathrm{Psh}^{\Pi}(\mathcal{P})$ which preserves geometric realizations, and so has underlying functor of the form $t_!$ for some $t\colon \mathcal{P} \to \mathrm{Psh}^{\Pi}(\mathcal{P})$. Let $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_{t_!}$ be the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$, so that $\mathcal{A}\mathrm{lg}_{t_!} \simeq \mathrm{Psh}^{\Pi}(T\mathcal{P})$. Then

(1) $t^*\colon \mathrm{Psh}^{\Pi}(T\mathcal{P}, \mathcal{S}\mathrm{et}) \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ is the forgetful functor of a monadic adjunction; write $\mathbb{T}$ for the associated monad on $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$.
(2) The monad $\mathbb{T}$ is determined by natural isomorphisms $\mathbb{T}(\pi_0 X) \simeq \pi_0 t_! X$ of T-algebras for $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$;
(3) If $\mathcal{P}$ is a resolution theory and $t_!$ is obtained from a monad T on $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, then $\mathbb{T}$ can instead be described in terms of T in the following manner:
   (a) $\mathbb{T}$ preserves reflexive coequalizers;
   (b) There are canonical maps $\mathbb{T}(\pi_0 X) \to \pi_0 T X$ for $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ which are isomorphisms when $X = h(P)$ for some $P \in \mathcal{P}$;
   (c) The diagrams

$$\begin{array}{ccccc} \mathbb{T}\mathbb{T}(\pi_0 X) & \longrightarrow & \mathbb{T}(\pi_0 T X) & \longrightarrow & \pi_0 T T X \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu} \\ \mathbb{T}(\pi_0 X) & & \longrightarrow & & \pi_0 T X \end{array} \qquad \begin{array}{ccc} \pi_0 X & \xrightarrow{\eta} & \mathbb{T}(\pi_0 X) \\ & \searrow^{\eta} & \downarrow \\ & & \pi_0 T X \end{array}$$

   commute.

Proof. Assertion (1) is as in Proposition 3.3.4, and we can identify $\mathrm{Psh}^{\Pi}(T\mathcal{P}, \mathcal{S}\mathrm{et})$ as the category of discrete $t_!$-algebras. In particular, if X is a $t_!$-algebra, then $\pi_0 X$ is a $t_!$-algebra. As a consequence, for $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ the map $X \to t_! X \to \pi_0 t_! X$ extends uniquely to a map $\mathbb{T}(\pi_0 X) \to \pi_0 t_! X$ of $t_!$-algebras, which is evidently an isomorphism when $X = h(P)$ with $P \in \mathcal{P}$. As this is a natural transformation of functors which preserve geometric realizations computed in $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$, it is a natural isomorphism, verifying (2). For (3), as there are maps $t_! X \to T X$ for $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, we can take as our natural transformation the map $\mathbb{T}(\pi_0 X) \simeq \pi_0 t_! X \to \pi_0 T X$; this has the indicated properties as $t(P) = Th(P)$ by assumption, and these evidently determine $\mathbb{T}$. □

In Proposition 3.3.8, the assumption that $t_!$ is obtained from a monad T on $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is, by Proposition 3.3.4, equivalent to the assumption that $t(P) \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ for $P \in \mathcal{P}$. In Section 6, we will consider situations where this can fail, examples where $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P})$ even when T does not preserve geometric realizations, and examples where the hypotheses of Theorem 3.3.2 are not satisfied yet nonetheless $\mathcal{M} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$.

4. Stable resolution theories

This section concerns those resolution theories $\mathcal{P}$ that naturally arise in stable settings. In the stable setting, it is natural to consider the categories of spectrum-valued models $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$, and corresponding full subcategories $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p})$. These behave differently from their unstable versions; the most important aspect is that the inclusion $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$ has an explicitly describable left adjoint given in Theorem 4.1.2. To illustrate what can be done in this context, we construct in Subsection 4.2 a spectral sequence for computing with suitable functors into the categories $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, and verify in Subsection 4.3 that it is multiplicative when one would expect it to be; the latter relies on the material of Appendix A. In Subsection 4.4 we construct a spectral sequence for computing mapping spaces; this can be seen in part as a warm-up for the more involved obstruction theory available in unstable contexts given in Subsection 5.3, though it is not a special case of the latter.

4.1. **Additive and stable theories.** Fix a theory $\mathcal{P}$, and suppose moreover that $\mathcal{P}$ is additive. In this case, it turns out that $\mathrm{Psh}^{\Pi}(\mathcal{P})$ is close to being stable; specifically, it is prestable in the sense of [Lur18, Appendix C]. The properties we need are summarized below. Write $\mathcal{S}\mathrm{p}$ for the category of a spectra, and observe that $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ is presentable.

4.1.1. **Lemma.** Let $\mathcal{P}$ be an additive resolution theory. Then

(1) Postcomposition with Ω^∞ gives an equivalence $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p}_{\geq 0})\simeq \mathrm{Psh}^{\Pi}(\mathcal{P})$;
(2) The embedding $\mathrm{Psh}^{\Pi}(\mathcal{P})\simeq\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p}_{\geq 0})\subset\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ realizes the category $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ as the category of spectrum objects of $\mathrm{Psh}^{\Pi}(\mathcal{P})$.

Proof. If $\mathcal{P}$ is finitary, then we can appeal directly to [Lur18, Remark C.1.5.9]; in general, we can appeal to the same by use of the embedding $\mathrm{Psh}^{\Pi}(\mathcal{P})\subset\mathrm{Psh}^{\times}(\mathcal{P})$. More directly, using the description of colimits in $\mathrm{Psh}^{\Pi}(\mathcal{P})$ given by Lemma 2.1.5, we see that $\mathrm{Psh}^{\Pi}(\mathcal{P})$ is additive, from which it follows, as in the proof of [Lur18, Proposition C.1.5.7], that $\mathrm{Psh}^{\Pi}(\mathcal{P})\simeq\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p}_{\geq 0})$; the second claim follows in turn as $\Omega\colon \mathrm{Psh}^{\Pi}(\mathcal{P})\to\mathrm{Psh}^{\Pi}(\mathcal{P})$ is computed pointwise. □

When $\mathcal{P}$ is additive, we will at times abuse notation by implicitly identifying $\mathrm{Psh}^{\Pi}(\mathcal{P})\subset\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$.

Fix now a stable resolution theory $\mathcal{P}$, i.e. a pointed resolution theory $\mathcal{P}$ for which $\Sigma\colon\mathcal{P}\to\mathcal{P}$ is an equivalence. By Lemma 3.3.1, $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is stable, so $\mathcal{P}$ is additive. Write $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})\subset\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ for the full subcategory of objects X such that $X_\Sigma\simeq\Omega X$; this is distinct from the image of $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ under $\mathrm{Psh}^{\Pi}(\mathcal{P})\subset\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$.

4.1.2. **Theorem.** The inclusion $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})\subset\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ is the inclusion of a reflective subcategory, and the associated localization L on $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ is described as

$$LX=\operatorname*{colim}_{n\to\infty}\Sigma^{-n}X_{\Sigma^{-n}}.$$

Proof. As both $X\mapsto\Sigma^{-1}X$ and $X\mapsto X_{\Sigma^{-1}}$ are self-equivalences of $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$, we can for $X\in\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ and $Y\in\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})$ identify

$$\mathrm{Map}(X,Y)\simeq\mathrm{Map}(\Sigma^{-1}X_{\Sigma^{-1}},\Sigma^{-1}Y_{\Sigma^{-1}})\simeq\mathrm{Map}(\Sigma^{-1}X_{\Sigma^{-1}},Y),$$

and the composite is given by restriction along $\Sigma^{-1}X_{\Sigma^{-1}}\to X$. Hence, writing $LX=\operatorname{colim}_{n\to\infty}\Sigma^{-n}X_{\Sigma^{-n}}$, we have $\mathrm{Map}(X,Y)\simeq\mathrm{Map}(LX,Y)$ and must only verify that $LX\in\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})$. As $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})$ is stable, finite limits commute past arbitrary colimits. Hence we compute

$$\Omega LX\simeq\Omega\operatorname*{colim}_{n\to\infty}\Sigma^{-n}X_{\Sigma^{-n}}\simeq\operatorname*{colim}_{n\to\infty}\Sigma^{-n-1}X_{\Sigma^{-n}}\simeq(\operatorname*{colim}_{n\to\infty}\Sigma^{-n}X_{\Sigma^{-n}})_\Sigma\simeq(LX)_\Sigma,$$

showing $LX\in\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})$. □

4.1.3. **Corollary.** When $\mathcal{P}$ is stable,

(1) If $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, then the tower of Theorem 4.1.2 producing LX is exactly the Whitehead tower of LX;
(2) Postcomposition with Ω^∞ yields an equivalence $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$;
(3) The subcategory $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$ is closed under all small limits and colimits;
(4) The composite $\mathrm{Psh}^{\Pi}(\mathcal{P}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is left adjoint to the inclusion $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P})$.

Proof. Assertion (1) is clear, as the map $X \to \Sigma^{-1} X_{\Sigma^{-1}}$ is an equivalence on (-1)-connected covers. For assertion (2), identify postcomposition with Ω^∞ as $\tau_{\geq 0}$. Observe that if $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ then $(LX)_{\geq 0} \simeq X$ by (1), hence $\tau_{\geq 0}$ is essentially surjective. To see it is fully faithful, fix $X, Y \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p})$ and compute

$$\mathrm{Map}(X, Y) \simeq \mathrm{Map}(L\tau_{\geq 0} X, Y) \simeq \mathrm{Map}(\tau_{\geq 0} X, Y) \simeq \mathrm{Map}(X_{\geq 0}, Y_{\geq 0}).$$

Assertion (3) is clear; compare the proof of Theorem 4.1.2. For (4), observe that if $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ and $Y \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, then

$$\mathrm{Map}(X, Y) \simeq \mathrm{Map}(X, (LY)_{\geq 0}) \simeq \mathrm{Map}(X, LY) \simeq \mathrm{Map}(LX, LY),$$

so the claim follows from (2). □

Because $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p}) \subset \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$ is closed under small colimits, it is also the inclusion of a coreflective subcategory, i.e. the inclusion admits a right adjoint. We will not make use of this observation.

Observe that any fiber sequence $X \to Y \to Z$ in $\mathrm{Psh}^{\Pi}(\mathcal{P})$ with second map a π_0-surjection remains a fiber sequence in $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$. In particular, Theorem 3.2.1 gives a fiber sequence $BX_\Sigma \to X \to \tau^* \tau_! X$ in $\mathrm{Psh}^{\Pi}(\mathcal{P})$, giving rise to the following.

4.1.4. **Lemma.** For $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, there is a fiber sequence

$$\Sigma X_\Sigma \to X \to \tau^* \tau_! X$$

in $\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$. □

4.2. **Left-derived functor spectral sequences.** Throughout this subsection, we fix a stable resolution theory $\mathcal{P}$ and arbitrary resolution theory $\mathcal{P}'$.

Let $f\colon \mathcal{P}' \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ be a functor, and extend this to $F\colon \mathrm{Psh}^{\Pi}(\mathcal{P}') \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ by left Kan extension. Define $\overline{f}\colon \mathrm{h}\mathcal{P}' \to \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et})$ by $\overline{f}([Q]) = \pi_0 f(Q)$, obtain from this $\overline{F}\colon \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}', \mathcal{S}\mathrm{et}) \to \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et})$ by left Kan extension, and recall from Subsection 2.2 our discussion of the total left-derived functor $\mathbb{L}\overline{F}$ obtained as $\mathbb{L}\overline{F} = \overline{f}_!\colon \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}') \to \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$.

4.2.1. **Proposition.** The diagram

$$\begin{array}{ccc} \mathrm{Psh}^{\Pi}(\mathcal{P}') & \xrightarrow{f_!} & \mathrm{Psh}^{\Pi}(\mathcal{P}) \\ \downarrow{\scriptstyle \tau_!} & & \downarrow{\scriptstyle \tau_!} \\ \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}') & \xrightarrow{\mathbb{L}\overline{F}} & \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}) \end{array}$$

commutes.

Proof. As all functors involved preserve geometric realizations, it suffices to check it commutes upon restriction to $\mathcal{P}'$. This itself follows from Corollary 3.2.2 together with the assumption that $f(P') \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ for $P' \in \mathrm{Psh}^{\Pi}(\mathcal{P}')$. □

4.2.2. **Theorem.** Fix notation as above, and fix $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}')$. Then there is a spectral sequence in $\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})$ of the form

$$E^1_{p,q} = (\mathbb{L}_{p+q}\overline{F}\pi_0 X)[-p] \Rightarrow (\pi_0 FX)[q], \qquad d^r\colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

If either π_0 preserves filtered colimits or $\mathbb{L}\overline{F}\pi_0 X$ is truncated, then this spectral sequence is convergent.

Proof. By Theorem 4.1.2, we can identify

$$FX = \operatorname*{colim}_{n\to\infty} \Sigma^{-n}(f_! X)_{\Sigma^{-n}}.$$

By Lemma 4.1.4 and Proposition 4.2.1, this tower has layers described by cofiber sequences

$$\Sigma^{-(n-1)}(f_! X)_{\Sigma^{-(n-1)}} \to \Sigma^{-n}(f_! X)_{\Sigma^{-n}} \to \tau^*\Sigma^{-n}(\mathbb{L}\overline{F}\pi_0 X)[-n].$$

This gives rise to the indicated spectral sequence in the usual way; we will review the construction and convergence in Appendix A.1. □

The method of constructing spectral sequences by analyzing the tower obtained by applying Theorem 4.1.2 is more general than just that given in Theorem 4.2.2. Roughly, given $M \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, to obtain a tool for computing $\pi_* M$ one wants to find some $M' \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ with $LM' = M$ such that $\tau_! M'$ is something computable. In the preceding theorem, we had $M = FX$, and took $M' = f_! X$; another simple case is the following.

4.2.3. **Example.** Observe that each of $\mathrm{Psh}^{\Pi}(\mathcal{P})$, $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, and $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$ are tensored over $\mathcal{S}\mathrm{p}_{\geq 0}$ by additivity. Denote the resulting tensors by $\otimes_!$, $\otimes$, and $\overline{\otimes}^{\mathbb{L}}$. If $X \in \mathcal{S}\mathrm{p}_{\geq 0}$ and $M \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, then we can identify

$$\tau_!(X \otimes_! M) = X \,\overline{\otimes}^{\mathbb{L}} \tau_! M, \qquad L(X \otimes_! M) = X \otimes LM.$$

For $X \in \mathcal{S}\mathrm{p}_{\geq 0}$ and $\Lambda \in \mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et})$, write

$$\pi_*(X \,\overline{\otimes}^{\mathbb{L}} \Lambda) = H_*(X;\Lambda);$$

this is a form of ordinary homology. For $M \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, we obtain an Atiyah-Hirzebruch type spectral sequence

$$E^1_{p,q} = H_{p+q}(X;\pi_0 M)[-p] \Rightarrow \pi_q(X \otimes M), \qquad d^r\colon E^r_{p,q} \to E^r_{p-r,q-1},$$

by analyzing the tower $X \otimes M = \operatorname{colim}_{n\to\infty} \Sigma^{-n}(X \otimes_! M)_{\Sigma^{-n}}$. ◁

4.3. **Monoidal matters.** We would like to introduce monoidal properties of the constructions discussed in Subsections 4.1 and 4.2. Our primary reason for doing so is to introduce pairings into the spectral sequence of Theorem 4.2.2. For the sake of completeness, we will work briefly in a more general setting than is necessary for just the production of pairings, and for this generality we require the theory of ∞-operads as developed in [Lur17a]. However, the cases of the general theory necessary for our primary application, Theorem 4.3.3, are just as easily performed by hand, completely bypassing the theory of ∞-operads.

Fix a single-colored ∞-operad $\mathcal{O}$ in the sense of [Lur17a]. We will implicitly use the fact that every symmetric monoidal category canonically inherits the structure of an $\mathcal{O}$-monoidal category. We say that an $\mathcal{O}$-monoidal structure on a category $\mathcal{D}$ respects some class of colimits in $\mathcal{D}$ if for every $n \geq 0$ and $f \in \mathcal{O}(n)$, the tensor product $\otimes_f$ preserves such colimits in each variable. An $\mathcal{O}$-monoidal category $\mathcal{D}$ is said to be $\mathcal{O}$-monoidally cocomplete if it admits small colimits, and its $\mathcal{O}$-monoidal structure respects these. In [Lur17a, Section 2.2.6], it is shown that if $\mathcal{C}$ is a small $\mathcal{O}$-monoidal category and $\mathcal{D}$ is an $\mathcal{O}$-monoidally cocomplete category, then $\operatorname{Fun}(\mathcal{C},\mathcal{D})$ admits the structure of an $\mathcal{O}$-monoidally cocomplete category under Day convolution, informally described as follows: For $n \geq 0$, $f \in \mathcal{O}(n)$, and $F_1,\ldots,F_n\colon \mathcal{C}\to\mathcal{D}$, the tensor product $\otimes_f(F_1,\ldots,F_n)$ is the left Kan extension of $\otimes_f \circ (F_1\times\cdots\times F_n)\colon \mathcal{C}^{\times n}\to\mathcal{D}^{\times n}\to\mathcal{D}$ along $\otimes_f\colon \mathcal{C}^{\times n}\to\mathcal{C}$. Of interest is the case where $\mathcal{C}$ is the poset $\mathbb{Z}$ with symmetric monoidal structure given by addition, where for towers $X_1,\ldots,X_n$ in $\mathcal{D}$ we identify

$$\otimes_f(X_1,\ldots,X_n)(p) = \operatorname*{colim}_{p_1+\cdots+p_n\leq p} \otimes_f(X_1(p_1),\ldots,X_n(p_n)).$$

4.3.1. **Definition.** A resolution theory $\mathcal{P}$ is an $\mathcal{O}$-*monoidal resolution theory* if it is equipped with an $\mathcal{O}$-monoidal structure compatible with coproducts and tensors by finite wedges of spheres. ◁

Fix an $\mathcal{O}$-monoidal resolution theory $\mathcal{P}$. We obtain by Day convolution, following [Lur17a, Proposition 4.8.1.10], $\mathcal{O}$-monoidal categories, all compatible with colimits, and strong $\mathcal{O}$-monoidal functors, fitting into the diagram

$$\operatorname{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{et}) \longleftarrow \operatorname{Psh}^{\Pi}(\mathcal{P}) \longrightarrow \operatorname{Psh}^{\Pi,\Omega}(\mathcal{P}) \ .$$

When $\mathcal{P}$ is in addition stable, we similarly obtain

$$\begin{array}{ccc} \operatorname{Psh}^{\Pi}(\mathcal{P}) & \longrightarrow & \operatorname{Psh}^{\Pi,\Omega}(\mathcal{P}) \\ \big\downarrow & & \big\downarrow{\simeq} \\ \operatorname{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p}) & \longrightarrow & \operatorname{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p}) \end{array} .$$

One can easily define an $\mathcal{O}$-monoidal theory, and if $\mathcal{P}$ is such then we obtain the same, only with $\operatorname{Psh}^{\Pi,\Omega}(\mathcal{P})$ and $\operatorname{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})$ omitted. The only thing we have to say in this level of generality is the following.

4.3.2. **Proposition.** Suppose $\mathcal{P}$ is an $\mathcal{O}$-monoidal stable resolution theory. Then the functor $\operatorname{Psh}^{\Pi}(\mathcal{P})\to\operatorname{Fun}(\mathbb{Z},\operatorname{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p}))$ sending X to the tower

$$\cdots\to\Sigma X_{\Sigma}\to X\to\Sigma^{-1}X_{\Sigma^{-1}}\to\cdots$$

is canonically lax $\mathcal{O}$-monoidal.

Proof. This functor preserves colimits, hence by the universal property of Day convolution it is sufficient to verify that it is canonically lax $\mathcal{O}$-monoidal upon restriction to $\mathcal{P}$. By Corollary 4.1.3, we can identify its restriction to $\mathcal{P}$ as the composite

$$\mathcal{P}\to\operatorname{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})\subset\operatorname{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})\xrightarrow{W}\operatorname{Fun}(\mathbb{Z},\operatorname{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})),$$

where W is the functor sending an object to its Whitehead tower. We conclude by Proposition A.2.1. □

We restrict now to the case where $\mathcal{O}$ is the nonunital $\mathbb{A}_2$-operad, i.e. where our monoidal structures consist merely of a single pairing subject to no further coherence conditions. Fix nonunital $\mathbb{A}_2$-monoidal resolution theories $\mathcal{P}$ and $\mathcal{P}'$, and suppose that $\mathcal{P}$ is stable. We write the associated pairings on $\operatorname{Psh}^{\Pi,\Omega}(\mathcal{P})$ and $\operatorname{Psh}^{\Pi,\Omega}(\mathcal{P}')$ as $\otimes$, and the associated pairings on $\operatorname{Psh}^{\Pi}(\mathrm{h}\mathcal{P},\mathcal{S}\mathrm{et})$ and $\operatorname{Psh}^{\Pi}(\mathrm{h}\mathcal{P}',\mathcal{S}\mathrm{et})$ as $\overline{\otimes}$.

Fix a functor $F\colon \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}') \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ which preserves geometric realizations; from here, we will use notation as in Subsection 4.2. Suppose that F is lax monoidal; equivalently, that we have chosen a natural transformation $f(P') \otimes f(Q') \to f(P' \otimes Q')$. This gives rise to lax monoidal structures on $f_!$, $\overline{F}$, and $\mathbb{L}\overline{F}$.

By Theorem 2.2.2 and the classical Dold-Kan correspondence, we can model $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$ as $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et}) \simeq \mathrm{Ch}^{+}(\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et}))$. Following Proposition 2.2.3, the pairing on $\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$ induced by that on $\mathrm{h}\mathcal{P}$ can be modeled by the levelwise pairing on $\mathrm{s}\mathcal{P} \subset \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathrm{s}\mathcal{S}\mathrm{et})$, and by the Eilenberg-Zilber theorem this is modeled by the pairing on $\mathrm{Ch}^{+}(\mathrm{h}\mathcal{P}) \subset \mathrm{Ch}^{+}(\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et}))$. To be precise, we choose the pairing on $\mathrm{Ch}^{+}(\mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et}))$ given by $(C' \overline{\otimes} C'')_p = \bigoplus_{p'+p''} C'_{p'} \overline{\otimes} C''_{p''}$, with differential $d(x' \otimes x'') = d(x') \otimes x'' + (-1)^{|x'|} x' \otimes d(x'')$. With this choice, a pairing $C' \overline{\otimes} C'' \to C$ of chain complexes gives Künneth maps $H_{q'}C' \overline{\otimes} H_{q''}C'' \to H_{q'+q''}C$. From this, for $R, S \in \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P}', \mathcal{S}\mathrm{et})$ we obtain $\mathbb{L}_p\overline{F}(R) \overline{\otimes} \mathbb{L}_q\overline{F}(S) \to \mathbb{L}_{p+q}\overline{F}(R \overline{\otimes} S)$.

4.3.3. **Theorem.** Fix notation as above, and for $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}')$ write $E(X)$ for the spectral sequence of Theorem 4.2.2. Then a pairing $X' \otimes X'' \to X$ in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}')$ gives rise to a pairing $E(X') \overline{\otimes} E(X'') \to E(X)$ of spectral sequences, i.e. maps

$$\smile\colon E^r_{p',q'}(X') \overline{\otimes} E^r_{p'',q''}(X'') \to E^r_{p'+p'',q'+q''}(X)$$

satisfying

$$d^r(x' \smile x'') = d^r(x') \smile x'' + (-1)^{q'} x \smile d^r(x''),$$

with the pairing on E^{r+1} induced by that on E^r. When $r = 1$, this pairing is the algebraic pairing on $\mathbb{L}_*\overline{F}$ twisted by $(-1)^{q'p''}$.

Proof. For the construction of the pairings, combine Proposition 4.3.2 and Theorem A.3.2. By the construction of the pairing and the identification of the E^1 page of the spectral sequence, the diagram

$$\begin{array}{ccc}
E^1_{p',q'}(X') \overline{\otimes} E^1_{p'',q''}(X'') & \xrightarrow{=} & (\mathbb{L}_{p'+q'}\overline{F}\pi_0 X')[-p'] \overline{\otimes} (\mathbb{L}_{p''+q''}\overline{F}\pi_0 X'')[-p''] \\
\Big\downarrow \smile & & \Big\downarrow \\
 & & (\mathbb{L}_{p'+q'+p''+q''}\overline{F}\pi_0 X)[-p'-p''] \\
 & & \Big\downarrow \simeq \\
E^1_{p'+q',p''+q''}(X'') & \xrightarrow{=} & (\mathbb{L}_{p'+p''+q'+q''}\overline{F}\pi_0 X)[-(p'+p'')]
\end{array}$$

commutes, where the top right vertical map is the algebraic pairing, and the bottom right vertical map induces a sign of $(-1)^{q'p''}$. □

In the situation of Theorem 4.3.3, the pairings produced behave well with respect to the pairing $F(X') \otimes F(X'') \to F(X)$; see our comments at the end of Appendix A.3.

4.4. **Universal coefficient spectral sequences.** Fix a stable resolution theory $\mathcal{P}$. For $X, Y \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$, we can form a mapping spectrum $\mathbf{Map}(X, Y)$ with $\Omega^\infty \mathbf{Map}(X, Y) = \mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})}(X, Y)$.

4.4.1. **Theorem.** For $X, Y \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p})$, there is a conditionally convergent spectral sequence of abelian groups

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{\mathrm{h}\mathcal{P}}(\pi_0 X; \pi_0 Y[p]) \Rightarrow \pi_{-q}\mathbf{Map}(X, Y), \qquad d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q+1}.$$

Proof. There is a decomposition of spectra

$$\mathbf{Map}(X,Y) \simeq \mathbf{Map}(X_{\geq 0},Y) \simeq \lim_{n\to\infty} \mathbf{Map}(X_{\geq 0},Y_{\leq n}),$$

with layers described by fiber sequences

$$\mathbf{Map}(X_{\geq 0},\Sigma^p \pi_0 Y[p]) \to \mathbf{Map}(X_{\geq 0},Y_{\leq p}) \to \mathbf{Map}(X_{\geq 0},Y_{\leq p-1}),$$

and by Lemma 4.1.4, we can identify

$$\pi_{-q}\mathbf{Map}(X_{\geq 0},\Sigma^p\pi_0 Y[p]) = \operatorname{Ext}^{p+q}_{\mathrm{h}\mathcal{P}}(\pi_0 X,\pi_0 Y[p]).$$

The theorem then follows from the usual construction of the spectral sequence of a tower, which we will review in Appendix A. ☐

We end with a remark concerning the introduction of extra structure into Theorem 4.4.1. Suppose that $\mathcal{P}$ is a nonunital $\mathbb{A}_2$-monoidal stable resolution theory, and write the resulting pairing on $\mathrm{Psh}^\Pi(\mathcal{P},\mathcal{S}\mathrm{p})$ by $\otimes_!$. Then $\mathrm{Psh}^\Pi(\mathcal{P},\mathcal{S}\mathrm{p})$ is closed monoidal, in that for $X,Y\in\mathrm{Psh}^\Pi(\mathcal{P},\mathcal{S}\mathrm{p})$ we can form objects $F_l(X,Y),F_r(X,Y)\in\mathrm{Psh}^\Pi(\mathcal{P},\mathcal{S}\mathrm{p})$ with

$$\mathbf{Map}(X,F_l(Y,Z))\simeq\mathbf{Map}(Y\otimes_! X,Z),\qquad \mathbf{Map}(X,F_r(Y,Z))\simeq\mathbf{Map}(X\otimes_! Y,Z).$$

Consider just F_r. Observe $F_r(X,Y)(P)=\mathbf{Map}(h(P)\otimes_! X,Y)$; in particular, if $\otimes_!$ admits a left unit $I\in\mathcal{P}$, then $F_r(X,Y)(I)\simeq\mathbf{Map}(X,Y)$. The same remarks hold for $\mathrm{h}\mathcal{P}$, so that, at least if $\otimes_!$ admits a left unit, for $X\in\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})$ and $M\in\mathrm{Psh}^\Pi(\mathcal{P},\mathcal{S}\mathrm{et})$ we can view $\pi_{-q}F_r(X_{\geq 0},M)\in\mathrm{Psh}^\Pi(\mathrm{h}\mathcal{P},\mathcal{S}\mathrm{et})$ as an enrichment of $\operatorname{Ext}^q_{\mathrm{h}\mathcal{P}}(\pi_0 X,M)$, and in this manner obtain an enriched form of Theorem 4.4.1.

5. Postnikov decompositions and obstruction theories

Fix a resolution theory $\mathcal{P}$. This section considers the study of $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ via Postnikov decompositions in $\mathrm{Psh}^\Pi(\mathcal{P})$. We begin in Subsection 5.1 with a brief review of the general theory of Postnikov decompositions, available in any ∞-topos, then specialize in Subsection 5.2 to the case of Postnikov towers $\mathrm{Psh}^\Pi(\mathcal{P})$, which can be computed in the ambient ∞-topos $\mathrm{Psh}(\mathcal{P})$; see also [Pst17]. Given the general theory, the construction of an obstruction theory for mapping spaces in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is essentially immediate, and obtained in Subsection 5.3. Finally, Subsection 5.4 contains a verification that the Blanc-Dwyer-Goerss obstruction theory for realizations, as interpreted in [Pst17], holds in our setting.

5.1. **Eilenberg-MacLane objects and Postnikov towers.** Fix a Grothendieck ∞-topos $\mathcal{X}$. Up to size issues, which for our purposes can be safely ignored, $\mathcal{X}$ admits an object classifier Ω; see [Lur17b, Theorem 6.1.6.8]. In other words, there is a universal map $\Omega^*\to\Omega$ in $\mathcal{X}$, pulling back along which induces an equivalence

$$\mathrm{Map}_{\mathcal{X}}(X,\Omega)\simeq(\mathcal{X}/X)^{\simeq}\simeq\coprod_{f\in\mathcal{X}/X} B\,\mathrm{Aut}_{\mathcal{X}/X}(f)$$

for all $X\in\mathcal{X}$. For $n\geq 1$, there is a subobject $\mathcal{EM}_n\subset\Omega$ classifying abelian Eilenberg-MacLane objects concentrated in degree n, with associated universal map $\mathcal{EM}_n^*\to\mathcal{EM}_n$; write $\mathcal{EM}_1'$ for the object classifying arbitrary Eilenberg-MacLane objects concentrated in degree 1. There are also objects $\mathcal{AB}$ and $\mathcal{GP}$ classifying discrete abelian groups and discrete groups in $\mathcal{X}$ respectively, and following [Lur17b, Proposition 7.2.2.12], there are equivalences $\pi_n\colon\mathcal{EM}_n^*\to\mathcal{AB}$ and $\pi_1\colon\mathcal{EM}_1'^*\to\mathcal{GP}$ with inverses $B^n\colon\mathcal{AB}\to\mathcal{EM}_n^*$ and $B\colon\mathcal{GP}\to\mathcal{EM}_1'^*$. If $X\in\mathcal{X}$, then $\mathrm{Map}_{\mathcal{X}}(X, \mathcal{EM}_n)_{\leq 1} \simeq \mathrm{Map}_{\mathcal{X}}(X, \mathcal{AB})$, allowing us to construct $\pi_n \colon \mathcal{EM}_n \to \mathcal{AB}$ splitting $B^n \colon \mathcal{AB} \to \mathcal{EM}_n$. We can summarize some of the relations between these as follows.

5.1.1. **Proposition.** For $n \geq 1$, there are Cartesian squares

$$\begin{array}{ccc} \mathcal{EM}_n & \xrightarrow{\pi_n} & \mathcal{AB} \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle B^{n+1}} \\ \mathcal{AB} & \xrightarrow{B^{n+1}} & \mathcal{EM}_{n+1} \end{array} .$$

In other words, $\pi_n \colon \mathcal{EM}_n \to \mathcal{AB}$ makes $\mathcal{EM}_n$ into an object of $\mathcal{EM}^*_{n+1}(\mathcal{X}/\mathcal{AB})$, with pointing given by B^n.

Proof. For any $X \in \mathcal{X}$, the Cartesian product of this square with X is the original square taken with respect to the slice topos $\mathcal{X}/X$, so it is sufficient to verify that it is Cartesian upon taking global sections. Taking global sections and looking at the path components corresponding to some $M \in \mathcal{AB}(\mathcal{X})$, it is sufficient to verify that

$$\begin{array}{ccc} B\,\mathrm{Aut}_{\mathcal{X}}(B^n M) & \xrightarrow{\pi_n} & B\,\mathrm{Aut}_{\mathcal{A}\mathrm{b}(\mathcal{X})}(M) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle B^{n+1}} \\ B\,\mathrm{Aut}_{\mathcal{A}\mathrm{b}(\mathcal{X})}(M) & \xrightarrow{B^{n+1}} & B\,\mathrm{Aut}_{\mathcal{X}}(B^{n+1} M) \end{array}$$

is Cartesian. The structure of $\mathcal{AB} \simeq \mathcal{EM}^*_n \to \mathcal{EM}_n$ gives a fiber sequence

$$B^n\,\mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, M) \to B\,\mathrm{Aut}_{\mathcal{A}\mathrm{b}(\mathcal{X})}(M) \to B\,\mathrm{Aut}_{\mathcal{X}}(B^n M).$$

Because M is abelian, this is split by π_n. We can thus identify

$$\Omega\,\mathrm{Fib}(\pi_n) \simeq \mathrm{Fib}(B^n) \simeq B^n\,\mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, M), \qquad \mathrm{Fib}(B^{n+1}) \simeq B^{n+1}\,\mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, M),$$

and so find that the above square is Cartesian by comparing fibers. □

A map $X \to \mathcal{AB}$ classifies a discrete abelian group in $\mathcal{X}/X$; call such an object an X*-module*. As $\mathcal{AB}$ is 1-truncated, X-modules correspond to $X_{\leq 1}$-modules. As moreover $X \to \pi_0 X$ is 1-connected, $\mathrm{Map}_{\mathcal{X}}(\pi_0 X, \mathcal{AB}) \to \mathrm{Map}_{\mathcal{X}}(X, \mathcal{AB})$ is (-1)-truncated, i.e. is an inclusion of a collection of path components; call an X-module *simple* if it is in the image of this map. A theory of Postnikov towers arises from the observation that for all $n \geq 2$, we have Cartesian squares

$$\begin{array}{ccc} X_{\leq n} & \longrightarrow & \mathcal{AB} \\ \downarrow & & \downarrow \\ X_{\leq n-1} & \longrightarrow & \mathcal{EM}_n \end{array} ,$$

and for $n = 1$ we have the analogous square with $\mathcal{EM}_1$ replaced by $\mathcal{EM}'_1$ and $\mathcal{AB}$ by $\mathcal{GP}$. If when $n = 1$ such a replacement is not necessary, we say that X has abelian homotopy groups. The top horizontal map of the above square defines an X-module $\Pi_n X$ for $n \geq 2$, or for $n \geq 1$ if X has abelian homotopy groups. If X has abelian homotopy groups and $\Pi_n X$ is simple for $n \geq 1$, we say that X is simple.

For $\Lambda \in \mathcal{X}_{\leq 1}$ and M a Λ-module, we have the Eilenberg-MacLane objects $B^{n+1}_{\Lambda} M$ in $\mathcal{X}/\Lambda$ for all $n \geq 0$. When $n \geq 1$, these fit into Cartesian squares

$$\begin{array}{ccc} B^{n+1}_{\Lambda}M & \longrightarrow & \mathcal{EM}_n \\ \downarrow & & \downarrow{\scriptstyle \pi_n} \\ \Lambda & \xrightarrow{M} & \mathcal{AB} \end{array} .$$

5.1.2. **Proposition.** Fix $X \in \mathcal{X}$. For $n \geq 2$, there is a canonical Cartesian square

$$\begin{array}{ccc} X_{\leq n} & \longrightarrow & X_{\leq 1} \\ \downarrow & & \downarrow \\ X_{\leq n-1} & \longrightarrow & B^{n+1}_{X_{\leq 1}}\Pi_n X \end{array}$$

in $\mathcal{X}$. If X is simple, then for $n \geq 1$ there is a canonical Cartesian square

$$\begin{array}{ccc} X_{\leq n} & \longrightarrow & \pi_0 X \\ \downarrow & & \downarrow \\ X_{\leq n-1} & \longrightarrow & B^{n+1}_{\pi_0 X}\Pi_n X \end{array}$$

in $\mathcal{X}$.

Proof. Both cases are handled the same way. Consider the diagram

$$\begin{array}{ccccc} X_{\leq n} & \longrightarrow & X_{\leq 1} & \longrightarrow & \mathcal{AB} \\ \downarrow & & \downarrow & & \downarrow \\ X_{\leq n-1} & \stackrel{k}{\dashrightarrow} & B^{n+1}_{X_{\leq 1}}\Pi_n X & \xrightarrow{j} & \mathcal{EM}_n \\ & \searrow & \downarrow & & \downarrow \\ & & X_{\leq 1} & \longrightarrow & \mathcal{AB} \end{array} .$$

Here, the map $j \circ k$ exists making the upper half of the diagram Cartesian and the bottom half commute, so the individual map k exists as the bottom right square is Cartesian. As the upper half of the diagram is Cartesian, to show that the upper left square is Cartesian it is sufficient to verify that the upper right square is Cartesian. As the bottom right square is Cartesian, it is sufficient to verify that the right half of the diagram is Cartesian, which is clear. □

5.2. **The Postnikov tower of a model of a theory.** Fix a theory $\mathcal{P}$, and set $\mathcal{X} = \mathrm{Psh}(\mathcal{P})$. Observe that $\mathrm{Psh}^{\Pi}(\mathcal{P}) \subset \mathcal{X}$ is closed under Postnikov towers. Moreover, we have the following.

5.2.1. **Proposition** ([Pst17, Lemma 2.64])**.** Fix $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$. Then any object of $\mathcal{AB}(\mathrm{Psh}^{\Pi}(\mathcal{P})/X)$ is simple as an X-module. In particular, X is simple.

Proof. By the equivalence between X-modules and $X_{\leq 1}$-modules, we may suppose that X is 1-truncated, so that everything is taking place within the bicategory $\mathrm{Psh}(\mathcal{P}, \mathcal{G}\mathrm{pd})$. Let $\pi\colon E \to X$ be an X-module. By Lemma 2.1.2, we see that for all $P \in \mathcal{P}$ the covering map $\pi_P\colon E(P) \to X(P)$ has trivial monodromy. The module E is classified by the map $c\colon X \to \mathcal{AB}$ given for $P \in \mathcal{P}$ by the functor $c_P\colon X(P) \to \mathcal{A}\mathrm{b}(\mathrm{Psh}(\mathcal{P}/P, \mathcal{S}\mathrm{et}))^{\simeq}$ defined on objects by $c_P(x)(f\colon Q \to P) = \pi_Q^{-1}(f^*x)$ and on morphisms by monodromy; as a consequence, each c_P factors through $\pi_0 X(P)$. By replacing $\mathcal{P}$ with its homotopy 2-category, and strictifying X to a 2-functor and c

to a strict natural transformation, it is seen that this pointwise factorization lifts to factor c through $\pi_0 X$. □

Observe that if $\Lambda \in \mathcal{X}_{\leq 0}$, then $(\mathcal{X}/\Lambda)_{\leq 0} \simeq \mathcal{X}_{\leq 0}/\Lambda$. As a consequence, for $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, we can form the abuses of notation $\pi_0 X = \tau^*\pi_0 X$ and $\Pi_n X = \tau^*\Pi_n X$ as $\pi_0 X$-modules. Moreover, if $\Lambda \in \mathrm{Psh}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et})$ and M is a Λ-module, then $\tau^* B^n_\Lambda = B^n_{\tau^*\Lambda}\tau^* M$. Hence we obtain the following.

5.2.2. **Theorem.** For $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ and $n \geq 1$, there is a canonical Cartesian square

$$\begin{array}{ccc} X_{\leq n} & \longrightarrow & \pi_0 X \\ \downarrow & & \downarrow \\ X_{\leq n-1} & \longrightarrow & \tau^* B^{n+1}_{\pi_0 X}\Pi_n X \end{array}$$

in $\mathrm{Psh}^{\Pi}(\mathcal{P})$.

Proof. Immediate from Propositions 5.1.2 and 5.2.1. □

5.3. **An obstruction theory for mapping spaces.** Fix a resolution theory $\mathcal{P}$, and let $\mathcal{X} = \mathrm{Psh}(\mathcal{P})$. For $Y \in \mathcal{X}$, we may identify $\mathcal{X}/Y \simeq \mathrm{Psh}(\mathcal{P}/Y)$, where $\mathcal{P}/Y$ is the slice category of $\mathcal{P}$ over Y. Given a map $f\colon X \to Y$ in $\mathcal{X}$, we may form the objects $\pi_0 f$ and $\Pi_n f$ for $n \geq 1$, considered as objects of $\mathcal{X}/X$. If $X, Y \in \mathrm{Psh}^{\Pi}(\mathcal{P})$, then each $\Pi_n f$ is a simple X-module by Proposition 5.2.1.

5.3.1. **Theorem.** Fix $R, S \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ together with a π_0-surjection $R \to S$. Fix $A, B \in R/\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})/S$, and write $p\colon B \to S$ for the given map. Choose $\phi\colon \pi_0 A \to \pi_0 B$ in $\pi_0 R/\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})/\pi_0 S$, and let $\mathrm{Map}^{\phi}_{R/\mathcal{P}/S}(A, B)$ be the space of lifts of ϕ to a map in $R/\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})/S$. Then there is a decomposition

$$\mathrm{Map}^{\phi}_{R/\mathcal{P}/S}(A, B) \simeq \lim_{n\to\infty} \mathrm{Map}^{\phi,\leq n}_{R/\mathcal{P}/S}(A, B),$$

where $\mathrm{Map}^{\phi,\leq 0}_{R/\mathcal{P}/S}(A, B) \simeq \{\phi\}$ and for each $n \geq 1$ there is a canonical Cartesian square

$$\begin{array}{ccc} \mathrm{Map}^{\phi,\leq n}_{R/\mathcal{P}/S}(A, B) & \longrightarrow & \{\phi\} \\ \downarrow & & \downarrow \\ \mathrm{Map}^{\phi,\leq n-1}_{R/\mathcal{P}/S}(A, B) & \longrightarrow & \mathrm{Map}_{\pi_0 R/\mathrm{Psh}^{\Pi}(h\mathcal{P})/\pi_0 S}(\pi_0 A, B^{n+1}_{\pi_0 B}\Pi_n p) \end{array}.$$

Proof. The decomposition is obtained using

$$\mathrm{Map}_{R/\mathcal{P}/S}(A, B) \simeq \lim_{n\to\infty} \mathrm{Map}_{R/\mathcal{P}/S}(A, p_{\leq n}),$$

where $p_{\leq n}$ is the n'th Postnikov truncation of B in the slice topos $\mathcal{X}/S$. The layers of this fit into Cartesian squares

$$\begin{array}{ccc} \mathrm{Map}_{R/\mathcal{P}/S}(A, p_{\leq n}) & \longrightarrow & \mathrm{Map}_{R/\mathcal{P}/S}(A, \pi_0 p) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{R/\mathcal{P}/S}(A, p_{\leq n-1}) & \longrightarrow & \mathrm{Map}_{R/\mathcal{P}/S}(A, B^{n+1}_{\pi_0 p}\Pi_n p) \end{array},$$

so we claim first that $\mathrm{Map}_{R/\mathcal{P}/S}(A, \pi_0 p) \simeq \mathrm{Map}_{\pi_0 R/h\mathcal{P}/\pi_0 S}(\pi_0 A, \pi_0 B)$. By resolving A, we may reduce to the case where $A = R \amalg P$ for some $q\colon P \to S$. In this case

$$\mathrm{Map}_{R/\mathcal{P}/S}(R \amalg P, \pi_0 p) \simeq \mathrm{Map}_{\mathcal{P}/S}(P, \pi_0 p) \simeq (\pi_0 p)(q\colon P \to S) = \pi_0 F,$$

where F is the fiber of the map $B(P) \to S(P)$ over q. As the composite $R \to B \to S$ is a π_0-surjection, the map $B \to S$ is a π_0-surjection. As $B, S \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, it follows that $\pi_1 B(P) \to \pi_1 S(P)$ is a surjection at all basepoints. Thus the fiber sequence $F \to B(P) \to S(P)$ remains a fiber sequence on taking π_0; as the fiber of $\pi_0 B(P) \to \pi_0 S(P)$ over q is $\mathrm{Map}_{\pi_0 R/h\mathcal{P}/\pi_0 S}(\pi_0(R \amalg P), \pi_0 B)$, this is as claimed. Next, by restricting to path components corresponding to ϕ in the above square, we reduce to identifying the space

$$\mathrm{Map}_{R/\mathcal{P}/\pi_0 p}(A, B^{n+1}_{\pi_0 p}\Pi_n p) \simeq \mathrm{Map}_{R/\mathcal{P}/\pi_0 B}(A, B^{n+1}_{\pi_0 B}\Pi_n p).$$

This space can be identified as

$$\mathrm{Map}_{R/\mathcal{P}/\pi_0 B}(A, B^{n+1}_{\pi_0 B}\Pi_n p) \simeq \mathrm{Map}_{\tau_! R/h\mathcal{P}/\pi_0 B}(\tau_! A, B^{n+1}_{\pi_0 B}\Pi_n p),$$

and we conclude by Corollary 3.2.2. □

5.4. **An obstruction theory for realizations.** Fix a resolution theory $\mathcal{P}$. In the case where $\mathcal{P}$ is pointed and finitary, Pstrągowski [Pst17] set up an obstruction theory for realizing an object $\Lambda \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$ as $\Lambda = \pi_0 X$ for some $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. In this subsection, we verify that the same obstruction theory exists for a general $\mathcal{P}$; the proof is essentially the same.

We begin with a matter that could have been considered in Subsection 5.2. Fix an ∞-topos $\mathcal{X}$; we will soon specialize to $\mathcal{X} = \mathrm{Psh}(\mathcal{P})$. Fix an $(n-1)$-truncated object Y, and set $\pi_0 Y = \Lambda$. Let M be a Λ-module. Every π_0-equivalence $Y \to B^{n+1}_{\Lambda} M$ gives rise, by pulling back along the zero section $\Lambda \to B^{n+1}_{\Lambda} M$, to an n-truncated object X such that $X_{\leq n-1} \simeq Y$ and $\Pi_n X \simeq M$ as Λ-modules; let $\mathcal{M}(Y +_{\Lambda} (M, n)) \subset \mathcal{X}^{\simeq}$ be the space of such X. If we write $\mathrm{Map}^{0\text{-Eq}}_{\mathcal{X}}$ for spaces of π_0-equivalences, then we obtain a map $\mathrm{Map}^{0\text{-Eq}}_{\mathcal{X}}(Y, B^{n+1}_{\Lambda} M) \to \mathcal{M}(Y +_{\Lambda} (M, n))$. Let also $\mathcal{M}(Y) \subset \mathcal{X}^{\simeq}$ be the space of objects equivalent to Y, and let $\mathrm{Aut}(\Lambda, M)$ be the discrete group of pairs $(\alpha\colon \Lambda \simeq \Lambda, f\colon M \simeq \alpha^* M)$, so that $B\,\mathrm{Aut}(\Lambda, M)$ is equivalent to a path component of $(\mathcal{X}/\mathcal{AB})^{\simeq}$. We see that $X \mapsto (X_{\leq n-1}, \Pi_n X)$ determines a map $\mathcal{M}(Y +_{\Lambda} (M, n)) \to \mathcal{M}(Y) \times_{B\,\mathrm{Aut}(\Lambda)} B\,\mathrm{Aut}(\Lambda, M)$. Following [Pst17, Theorem 2.71, Remark 3.17], we have the following.

5.4.1. **Proposition.** The above constructions describe Cartesian squares

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathcal{X}/\Lambda}(Y, B^{n+1}_{\Lambda} M) & \longrightarrow & \mathrm{Map}^{0\text{-Eq}}_{\mathcal{X}}(Y, B^{n+1}_{\Lambda} M) & \longrightarrow & \mathcal{M}(Y +_{\Lambda} (M, n)) \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \mathrm{Aut}(\Lambda) & \longrightarrow & \mathcal{M}(Y) \times_{B\,\mathrm{Aut}(\Lambda)} B\,\mathrm{Aut}(\Lambda, M) \\
 & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \mathcal{M}(Y) \times B\,\mathrm{Aut}(\Lambda, M)
\end{array}$$

of ∞-groupoids. □

We can now proceed to the realization problem. Fix $\Lambda \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{et})$. Call $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ a potential n-stage for Λ when X is n-truncated, $\pi_0 X \simeq \Lambda$, and $X_{S^1} \to X^{S^1}$ is an $(n-1)$-equivalence over X. Let $\mathcal{M}_n(\Lambda)$ be the space of potential

n-stages for Λ. Truncation defines $\mathcal{M}_n(\Lambda) \to \mathcal{M}_{n-1}(\Lambda)$, and as in [Pst17, Proposition 3.8] we identify the limit $\mathcal{M}_\infty(\Lambda)$ as the space of realizations of Λ, i.e. the space of $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ such that $\pi_0 X \simeq \Lambda$.

5.4.2. **Lemma.** Let X be a potential n-stage for Λ, and choose $\pi_0 X \simeq \Lambda$. Then

(1) The map $X_{S^k} \to X^{S^k}$ is an $(n-k)$-equivalence over X;
(2) We have $\Pi_k X \simeq \Lambda\langle k\rangle$ for $k \leq n$;
(3) The only nontrivial homotopy Λ-module of $\tau_! X$ is $\Pi_{n+2}\tau_! X \simeq \Lambda\langle n+1\rangle$.

Proof. For (1), induct using the Cartesian squares

$$\begin{array}{ccc} X_{S^{k+1}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ (X_{S^k})_{S^1} & \longrightarrow & X_{S^k} \times_X X_{S^1} \end{array}$$

for $k \geq 1$ as in Proposition 3.1.4; (2) follows. For (3), use Theorem 3.2.1. □

5.4.3. **Lemma.** Fix an n-truncated object $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ such that $\pi_0 X \simeq \Lambda$. Then $X \in \mathcal{M}_n(\Lambda)$ if and only if the map $X_{S_1} \to X^{S^1}$ induces an equivalence $(B_X X_{S^1})_{\leq n} \simeq (B_X X^{S^1})_{\leq n}$. □

5.4.4. **Proposition.** Suppose given $Y \in \mathcal{M}_{n-1}(\Lambda)$ and a Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \tau^*\Lambda \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & \tau^* B_\Lambda^{n+1}\Lambda\langle n\rangle \end{array} .$$

Then $X \in \mathcal{M}_n(\Lambda)$ if and only if f is adjoint to an equivalence $\tau_! Y \simeq B_\Lambda^{n+2}\Lambda\langle n\rangle$.

Proof. Form Cartesian squares

$$\begin{array}{ccccc} X & \longrightarrow & \tau^* Z & \longrightarrow & \tau^*\Lambda \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & \tau^*\tau_! Y & \longrightarrow & \tau^* B_\Lambda^{n+1}\Lambda\langle n\rangle \end{array} .$$

All maps here are 0-equivalences, and we wish to show that $X \in \mathcal{M}_n(\Lambda)$ if and only if $Z \simeq \Lambda$. Form the Cartesian cube

$$\begin{array}{ccccccc} B_X(X \times_Y Y_{S^1}) & & \longrightarrow & & X & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & B_Y Y_{S^1} & & \longrightarrow & & Y \\ X & & \longrightarrow & & \tau^* Z & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \\ & & Y & & \longrightarrow & & \tau^*\tau_! Y \end{array} .$$

As $X_{\leq n-1} \simeq Y_{\leq n-1}$, we have $(X_{S^1})_{\leq n-1} \simeq (Y_{S^1})_{\leq n-1} \simeq (X \times_Y Y_{S^1})_{\leq n-1}$, and thus $(B_X X_{S^1})_{\leq n} \simeq (B_X(X \times_Y Y_{S^1}))_{\leq n} \simeq (X \times_{\tau^* Z} X)_{\leq n}$. By Lemma 5.4.3, we learn that if $Z \simeq \Lambda$ then $X \in \mathcal{M}_n(\Lambda)$. Conversely, if $X \in \mathcal{M}_n(\Lambda)$ then $(X \times_{\tau^* Z} X)_{\leq n} \simeq (X \times_{\pi_0 X} X)_{\leq n}$, from which it follows that $\tau^* Z \simeq \Lambda$. □

Let $\mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n \rangle, n+1))$ be as in the beginning of this subsection, constructed with respect to $\mathrm{Psh}(\mathrm{h}\mathcal{P})$. We can identify this space using Proposition 5.4.1.

5.4.5. **Lemma.** There is an equivalence

$$\mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n \rangle, n+1)) \cong \mathrm{Map}_{\mathrm{Psh}^\Pi(h\mathcal{P})/\Lambda}(\Lambda; B^{n+2}_\Lambda \Lambda\langle n \rangle)_{\mathrm{Aut}(\Lambda, \Lambda\langle n \rangle)}.$$

Under this equivalence, $B^{n+1}_\Lambda \Lambda\langle n \rangle \in \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n \rangle, n+1))$ is sent to the zero section. □

We already have enough for a coarse obstruction theory.

5.4.6. **Proposition.** Fix $\Lambda \in \mathrm{Psh}^\Pi(\mathcal{P}, \mathcal{S}\mathrm{et})$, and let Y be an $(n-1)$-stage for Λ. Then there is an obstruction $\epsilon_n(Y) \in \pi_0 \mathrm{Map}_{h\mathcal{P}/\Lambda}(\Lambda; B^{n+2}_\Lambda \Lambda\langle n \rangle)/\mathrm{Aut}(\Lambda, \Lambda\langle n \rangle)$ which vanishes if and only if there is an n-stage X such that $X_{\leq n-1} \simeq Y$.

Proof. By Lemma 5.4.2, we have a map $\tau_!\colon \mathcal{M}_{n-1}(\Lambda) \to \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n \rangle, n+1))$. Let $\epsilon_n(Y)$ be the path component of the image of Y under this map and Lemma 5.4.5. We conclude by Proposition 5.4.4. □

The more refined statement is the following.

5.4.7. **Theorem** ([Pst17, Theorem 3.15])**.** For each $n \geq 1$, there is a Cartesian square

$$\begin{array}{ccc} \mathcal{M}_n(\Lambda) & \longrightarrow & B\,\mathrm{Aut}(\Lambda, \Lambda\langle n \rangle) \\ \downarrow & & \downarrow \\ \mathcal{M}_{n-1}(\Lambda) & \longrightarrow & \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n \rangle, n+1)) \end{array} .$$

Proof. If we choose $Y \in \mathcal{M}_{n-1}(\Lambda)$ and let $F = \{Y\} \times_{\mathcal{M}_{n-1}(\Lambda)} \mathcal{M}_n(\Lambda)$ be the space of $X \in \mathcal{M}_n(\Lambda)$ equipped with an equivalence $X_{\leq n-1} \simeq Y$, then we reduce to verifying that the bottom square in

$$\begin{array}{ccc} \mathrm{Eq}(\tau_! Y, B^{n+1}_\Lambda \Lambda\langle n \rangle) & \longrightarrow & \{\Lambda\langle n \rangle\} \\ \downarrow & & \downarrow \\ F & \longrightarrow & B\,\mathrm{Aut}(\Lambda, \Lambda\langle n \rangle) \\ \downarrow & & \downarrow \\ \{\tau_! Y\} & \longrightarrow & \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n \rangle, n+1)) \end{array}$$

is Cartesian. The outer square is Cartesian by definition, so it is sufficient to verify that the top square is Cartesian. Let F' be the space of all $X \in \mathcal{M}(Y +_\Lambda (\Lambda\langle n \rangle, n))$ equipped with an equivalence $X_{\leq n-1} \simeq Y$, so that F is a collection of path components of F', and F' fits into a Cartesian square

$$\begin{array}{ccc} F' & \longrightarrow & \mathcal{M}(Y +_\Lambda (\Lambda\langle n \rangle, n)) \\ \downarrow & & \downarrow \\ \{Y\} \times B\,\mathrm{Aut}(\Lambda, \Lambda\langle n \rangle) & \longrightarrow & \mathcal{M}(Y) \times B\,\mathrm{Aut}(\Lambda, \Lambda\langle n \rangle) \end{array} .$$

Form the diagram

$$\begin{array}{ccccc} \mathrm{Eq}(\tau_! Y, B_\Lambda^{n+1}\Lambda\langle n\rangle) & \longrightarrow & \mathrm{Map}^{0\text{-Eq}}_{\mathrm{Psh}^\Pi(\mathcal{P})}(Y, \tau^* B_\Lambda^{n+1}\Lambda\langle n\rangle) & \longrightarrow & \{\Lambda\langle n\rangle\} \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & F' & \longrightarrow & B\,\mathrm{Aut}(\Lambda, \Lambda\langle n\rangle) \end{array},$$

where the upper left horizontal map is obtained via the adjunction

$$\mathrm{Map}_{\mathrm{Psh}^\Pi(\mathrm{h}\mathcal{P})}(\tau_! Y, B_\Lambda^{n+1}\Lambda\langle n\rangle) \simeq \mathrm{Map}_{\mathrm{Psh}^\Pi(\mathcal{P})}(Y, \tau^* B_\Lambda^{n+1}\Lambda\langle n\rangle).$$

The rightmost square is Cartesian by Proposition 5.4.1, so to show the outer square is Cartesian it is sufficient to verify that the left square is Cartesian. This follows from Proposition 5.4.4. □

6. Completed algebra

If R is an $\mathbb{E}_2$-ring and $I \subset R_0$ a finitely generated ideal, then there is a good notion of I-completeness for R-modules, and one can proceed to consider various algebraic structures built from I-complete R-modules. We would like to apply our theory in this setting; in fact, this setting was our initial motivation for not restricting ourselves to finitary theories. If $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$ is the category of I-complete left R-modules, and $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}}$ is the full subcategory of I-completions of free R-modules, then to apply our theory we would like to say $\mathrm{Psh}^{\Pi,\Omega}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$, and to identify $\mathrm{Psh}^\Pi(\mathrm{h}(\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}}))$ as something recognizable in terms of $\mathrm{Psh}^\Pi(\mathrm{L}\mathcal{M}\mathrm{od}_{R_*}^{\mathrm{free}})$. The former always holds, and the latter is possible under an additional algebraic condition on the ideal I. We determine this condition in Subsection 6.2 in the more general setting of R-linear theories for a connective $\mathbb{E}_2$-ring R. Before this, in Subsection 6.1 we consider some general aspects of the interaction between localizations and theories.

6.1. **Localizations of resolution theories.** We begin with some facts about localizing monads in the 1-categorical setting.

6.1.1. **Lemma.** Let $\mathcal{C}$ be a 1-category, $L\colon \mathcal{C} \to \mathcal{C}$ a localization, and $T\colon \mathcal{C} \to \mathcal{C}$ a monad. If $LTC \to LTLC$ is an equivalence for all $C \in \mathcal{C}$, then the composite $LTLT \simeq LTT \to LT$ equips LT with the structure of a monad with the following property: L canonically lifts to a localization $L\colon \mathcal{A}\mathrm{lg}_T \to \mathcal{A}\mathrm{lg}_{LT}$ exhibiting $\mathcal{A}\mathrm{lg}_{LT}$ as the category of T-algebras whose underlying object of $\mathcal{C}$ is L-local.

Proof. This is a diagram chase; see for instance [Rez18, Proposition 11.5]. □

The monad structure on LT obtained by Lemma 6.1.1 is essentially unique, in the following sense.

6.1.2. **Lemma.** Let $\mathcal{C}$ be a 1-category, and $L\colon \mathcal{C} \to \mathcal{C}$ a localization. Let $T\colon \mathcal{C} \to \mathcal{C}$ be a functor, and suppose we have equipped $T, LT \in \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ with right-unital pairings (η, μ) and $(\widehat{\eta}, \widehat{\mu})$ in such a way that $T \to LT$ preserves this structure. Then

1. For any $C \in \mathcal{C}$, the map $LTC \to LTLC$ is an equivalence;
2. The pairing on LT is given by the composite $LT \circ LT \xleftarrow{\simeq} LT \circ T = L(T \circ T) \to LT$.

Proof. This is a diagram chase. Write I for the identity on $\mathcal{C}$, and $c\colon I \to L$ for the unit. By naturality of c, the diagram

$$\begin{CD} I @>c>> L \\ @V{\eta}VV @VV{L\eta}V \\ T @>{cT}>> LT \end{CD}$$

commutes, and the assumption that c is compatible with units implies that the composite is $\widehat{\eta}$. As a consequence, we have a commutative diagram

$$\begin{array}{ccccc} LT & \xrightarrow{LT\widehat{\eta}} & LTLT & \xrightarrow{\widehat{\mu}} & LT \\ {\scriptstyle LTc}\downarrow & \nearrow_{LTL\eta} & & & \downarrow{\scriptstyle LTc} \\ LTL & & \xrightarrow{\quad g \quad} & & LTL \end{array},$$

where g is defined so the diagram commutes. To show that LTc is an equivalence, it is sufficient to verify that g is the identity, for then an inverse is given by $\widehat{\mu} \circ LTL\eta$. Indeed, consider the diagram

$$\begin{CD} LTL @>{LTLc=LTcL}>> LTLL \\ @V{LTL\eta}VV @VV{LTL\eta L}V \\ LTLT @>{LTLTc}>> LTLTL \\ @V{\widehat{\mu}}VV @VV{\widehat{\mu}L}V \\ LT @>{LTc}>> LTL \end{CD}\ .$$

The top square commutes by naturality of η, and the bottom square commutes by naturality of $\widehat{\mu}$. The clockwise composite is the identity as $L\eta \circ c = \widehat{\eta}$ implies $LTL\eta L \circ LTcL = LT\widehat{\eta}L$, and the counterclockwise composite is g, hence g is the identity.

It remains to verify (2). Observe that the diagram

$$\begin{array}{ccccc} & & \overset{cTcT}{\frown} & & \\ TT & \xrightarrow{cTT} & LTT & \xrightarrow{LTcT} & LTLT \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle L\mu} & & \downarrow{\scriptstyle \widehat{\mu}} \\ T & \xrightarrow{cT} & LT & \xrightarrow{=} & LT \end{array}$$

commutes. Indeed, the leftmost square commutes by naturality of c, and the outermost by compatibility of c with the pairings. As L is a localization, the rightmost square commutes because the outer square commutes. Consider the diagram

$$\begin{array}{ccc} LTLT & \xrightarrow{LTL\eta T} & LTLTT \\ \downarrow{\scriptstyle \widehat{\mu}} & \nwarrow^{LTcT} & \downarrow{\scriptstyle \widehat{\mu}T} \\ LT & \xleftarrow{L\mu} & LTT \end{array}\ .$$

By our proof of (1), the clockwise composite is exactly the pairing $LTLT \simeq LTT \to LT$, so we must verify the outer square commutes. The left triangle commutes by the above, and the right triangle gives the identity on $LTLT$ by our proof of (1), so the outer square indeed commutes. □

Now let $\mathcal{P}$ be a theory, and $L\colon \operatorname{Psh}^{\Pi}(\mathcal{P}) \to \operatorname{Psh}^{\Pi}(\mathcal{P})$ a localization which preserves geometric realizations, so that the category of L-local objects is realized by the restriction $\operatorname{Psh}^{\Pi}(L\mathcal{P}) \to \operatorname{Psh}^{\Pi}(\mathcal{P})$; see Proposition 3.3.5. Let T be a monad on $\operatorname{Psh}^{\Pi}(\mathcal{P})$ which preserves geometric realizations, so that $\mathcal{A}\mathrm{lg}_T \simeq \operatorname{Psh}^{\Pi}(T\mathcal{P})$.

6.1.3. **Theorem.** Fix notation as in the preceding paragraph. Suppose that $LTh(P)$ is a T-algebra for each $P \in \mathcal{P}$ naturally in $Th(P)$. Then

(1) The map $LT \to LTL$ is a natural isomorphism;
(2) The functor LT carries the structure of a monad, informally described by $LTLT \simeq LTT \to LT$;
(3) The localization L lifts to a localization $L\colon \mathcal{A}\mathrm{lg}_T \to \mathcal{A}\mathrm{lg}_{LT}$ realizing $\mathcal{A}\mathrm{lg}_{LT}$ as the category of T-algebras whose underlying object of $\operatorname{Psh}^{\Pi}(\mathcal{P})$ is L-local.

Proof. As each $LTh(P)$ is a T-algebra, we can let $LT\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ be the full subcategory of such objects. We then have a commutative diagram

$$\begin{array}{ccc} \operatorname{Psh}^{\Pi}(LT\mathcal{P}) & \longrightarrow & \operatorname{Psh}^{\Pi}(T\mathcal{P}) \\ \downarrow & & \downarrow \\ \operatorname{Psh}^{\Pi}(L\mathcal{P}) & \longrightarrow & \operatorname{Psh}^{\Pi}(\mathcal{P}) \end{array},$$

of restriction functors which preserve geometric realizations and are the forgetful functors of monadic adjunctions. The monad associated to $\operatorname{Psh}^{\Pi}(LT\mathcal{P}) \to \operatorname{Psh}^{\Pi}(\mathcal{P})$ has underlying functor LT, and we obtain a map $T \to LT$ of monads. We obtain (1) and (2) by applying Lemma 6.1.2 to the homotopy category of $\operatorname{Psh}^{\Pi}(\mathcal{P})$. For (3), we are claiming that the above diagram is Cartesian, and that

$$\begin{array}{ccc} \operatorname{Psh}^{\Pi}(LT\mathcal{P}) & \longleftarrow & \operatorname{Psh}^{\Pi}(T\mathcal{P}) \\ \downarrow & & \downarrow \\ \operatorname{Psh}^{\Pi}(L\mathcal{P}) & \xleftarrow{\ L\ } & \operatorname{Psh}^{\Pi}(\mathcal{P}) \end{array}$$

commutes. The latter is clear, as can be checked on objects of the form $T(P)$ for $P \in \mathcal{P}$, and this readily implies the former. □

It is only a bit of extra work to include resolution theories in the story.

6.1.4. **Proposition.** Let $\mathcal{P}$ be a resolution theory, and $L\colon \operatorname{Psh}^{\Pi}(\mathcal{P}) \to \operatorname{Psh}^{\Pi}(\mathcal{P})$ be a localization which preserves geometric realizations. Suppose that $Lh(P) \in \operatorname{Psh}^{\Pi,\Omega}(\mathcal{P})$ for each $P \in \mathcal{P}$. Then

(1) $\operatorname{Psh}^{\Pi,\Omega}(L\mathcal{P}) \to \operatorname{Psh}^{\Pi,\Omega}(\mathcal{P})$ is a reflective subcategory;
(2) The diagram

$$\begin{array}{ccc} \operatorname{Psh}^{\Pi,\Omega}(L\mathcal{P}) & \longrightarrow & \operatorname{Psh}^{\Pi,\Omega}(\mathcal{P}) \\ \downarrow & & \downarrow \\ \operatorname{Psh}^{\Pi}(L\mathcal{P}) & \longrightarrow & \operatorname{Psh}^{\Pi}(\mathcal{P}) \end{array}$$

is Cartesian.

Proof. Because $Lh(P) \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ for each $P \in \mathcal{P}$, the full subcategory $L\mathcal{P} \subset \mathrm{Psh}^{\Pi}(L\mathcal{P})$ is a resolution theory, and there is a restriction $\mathrm{Psh}^{\Pi,\Omega}(L\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. This is fully faithful, being obtained from $\mathrm{Psh}^{\Pi}(L\mathcal{P}) \to \mathrm{Psh}^{\Pi}(\mathcal{P})$, and is the forgetful functor of a monadic adjunction by Proposition 3.3.4, proving (1). For (2), we are claiming that $\mathrm{Psh}^{\Pi,\Omega}(L\mathcal{P})$ consists of those objects of $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ which are L-local in $\mathrm{Psh}^{\Pi}(\mathcal{P})$, which is clear. ☐

6.1.5. **Theorem.** Let $\mathcal{P}$ be a resolution theory, and $L\colon \mathrm{Psh}^{\Pi}(\mathcal{P}) \to \mathrm{Psh}^{\Pi}(\mathcal{P})$ a localization which preserves geometric realizations such that $Lh(P) \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ for each $P \in \mathcal{P}$. Let T be a monad on $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, and suppose that

1. T satisfies the criteria of Theorem 3.3.6, so that $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P})$;
2. $LTh(P) \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ for each $P \in \mathcal{P}$;
3. $LTh(P)$ is a T-algebra for each $P \in \mathcal{P}$ naturally in TP.

Then

(a) The functor LT carries the structure of a monad, and the associated forgetful functor can be identified as $\mathrm{Psh}^{\Pi,\Omega}(LT\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$;

(b) The square

$$\begin{array}{ccc} \mathrm{Psh}^{\Pi,\Omega}(LT\mathcal{P}) & \longrightarrow & \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P}) \\ \downarrow & & \downarrow \\ \mathrm{Psh}^{\Pi,\Omega}(L\mathcal{P}) & \longrightarrow & \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \end{array}$$

is Cartesian;

(c) The square

$$\begin{array}{ccc} \mathrm{Psh}^{\Pi,\Omega}(LT\mathcal{P}) & \longleftarrow & \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P}) \\ \downarrow & & \downarrow \\ \mathrm{Psh}^{\Pi,\Omega}(L\mathcal{P}) & \xleftarrow{L} & \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \end{array}$$

commutes.

Proof. The restriction $\mathrm{Psh}^{\Pi,\Omega}(LT\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P})$ is the forgetful functor of a monadic adjunction by Proposition 3.3.4. It is obtained from $\mathrm{Psh}^{\Pi}(LT\mathcal{P}) \to \mathrm{Psh}^{\Pi}(T\mathcal{P})$, so is the inclusion of a reflective subcategory by Theorem 6.1.3. We claim that the associated localization is a lift of L; (b) follows quickly. This is the content of (c), and moreover shows that the monad associated to $\mathrm{Psh}^{\Pi,\Omega}(LT\mathcal{P}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ has underlying functor LT, proving (a). Consider the cube

$$\begin{array}{ccccc} \mathrm{Psh}^{\Pi,\Omega}(LT\mathcal{P}) & & \dashleftarrow & \mathrm{Psh}^{\Pi,\Omega}(T\mathcal{P}) & \\ \downarrow & \nwarrow & & \downarrow & \searrow \\ & \mathrm{Psh}^{\Pi}(LT\mathcal{P}) & \longleftarrow & & \mathrm{Psh}^{\Pi}(T\mathcal{P}) \\ \mathrm{Psh}^{\Pi,\Omega}(L\mathcal{P}) & & \dashleftarrow & \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) & \\ & \nwarrow \downarrow & & & \searrow \downarrow \\ & \mathrm{Psh}^{\Pi}(L\mathcal{P}) & \longleftarrow & & \mathrm{Psh}^{\Pi}(T\mathcal{P}) \end{array},$$

consisting of restrictions or left adjoints. The dashed arrows are such that the top and bottom faces commute, and the back face is the square of (c). As the rest of the diagram commutes, so does the back face, as desired. □

6.2. **R-linear theories and completions.** We now apply the theory of Subsection 6.1 to the main cases of interest; namely, those derived from I-completion. We begin by reviewing this notion of completeness; this theory is developed in [Lur18, Section 7]. Fix a connective $\mathbb{E}_2$-ring R. In [Lur18, Definition D.1.4.1], the notion of an R-linear prestable category is introduced. In short, where $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{sfgf}}$ is the category of left R-modules equivalent to $R^{\oplus n}$ for some $n < \infty$, an R-linear structure on a presentable stable category $\mathcal{M}$ is equivalent to an additive and monoidal functor $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{sfgf}} \to \mathrm{Fun}^L(\mathcal{M}, \mathcal{M})$, where the latter is the category of colimit-preserving endofunctors of $\mathcal{M}$. For convenience, we extend this definition to allow $\mathcal{M}$ to be an arbitrary presentable additive category; we will only apply the theory of [Lur18] in the case where $\mathcal{M}$ is stable.

Let $I \subset R_0$ be a finitely generated ideal, and let $\mathcal{M}$ be an R-linear stable category. An object $M \in \mathcal{M}$ is said to be *I-nilpotent* if $R[x^{-1}] \otimes_R M \simeq 0$ for all $x \in I$, is said to be *I-local* if $\mathrm{Map}_{\mathcal{M}}(N, M)$ is contractible for all I-nilpotent N, and is said to be *I-complete* if $\mathrm{Map}_{\mathcal{M}}(N, M)$ is contractible for all I-local N. Let $\mathcal{M}^{\mathrm{Cpl}(I)} \subset \mathcal{M}$ denote the full subcategory of I-complete objects. Then $\mathcal{M}^{\mathrm{Cpl}(I)}$ is a reflective subcategory of $\mathcal{M}$, with associated localization

$$\mathcal{M} \to \mathcal{M}^{\mathrm{Cpl}(I)}, \qquad M \mapsto M_I^\wedge.$$

We will need an explicit formula for I-completion. We must first fix some notation. Let $\underline{h} = \{1, \ldots, h\}$, and let $P(\underline{h})$ denote the powerset of $\underline{h}$, so that an h-cube is given by a functor from $P(\underline{h})$. Given an h-cube $V\colon P(\underline{h}) \to \mathcal{C}$ in some category $\mathcal{C}$, we will for $i \notin S \subset \underline{h}$ write $V_i\colon V(S) \to V(S \cup \{i\})$ for the resulting map. Given an h-cube V in a category $\mathcal{C}$ with finite colimits, write $\mathrm{tCof}\, V$ for the total cofiber of V.

Suppose now that I is a finitely generated ideal, and make a choice of generators $\underline{u} = (u_1, \ldots, u_h)$. If M is any object of a category with products on which $u_1, \ldots, u_h$ act, we can define the h-cube

$$K(M; \underline{u})\colon P(\underline{h}) \to \mathcal{M}, \qquad S \mapsto M[[T_1, \ldots, T_h]] = M^{\times \omega^h},$$

where

$$K(M; \underline{u})_i = (T_i - u_1)\colon M[[T_1, \ldots, T_h]] \to M[[T_1, \ldots, T_h]].$$

6.2.1. **Proposition.** Let $\mathcal{M}$ be an R-linear stable category. Then for $M \in \mathcal{M}$, we have

$$M_I^\wedge \simeq \mathrm{tCof}\, K(M; \underline{u}).$$

Proof. When $h = 1$, this is a reformulation of [Lur18, Proposition 7.3.2.1]. The general case then follows from [Lur18, Proposition 7.3.3.2]. □

Let $\mathcal{P}$ be an additive theory. Say that $\mathcal{P}$ is an R-linear theory if we have chosen an additive monoidal functor $\mathcal{M}\mathrm{od}_R^{\mathrm{sfgf}} \to \mathrm{Fun}^{\oplus}(\mathcal{P}, \mathcal{P})$, where the latter is the category of coproduct-preserving endofunctors of $\mathcal{P}$. If $\mathcal{P}$ is an additive resolution theory, say that $\mathcal{P}$ is an R-linear resolution theory if we have chosen an additive monoidal functor $\mathcal{M}\mathrm{od}_R^{\mathrm{sfgf}} \to \mathrm{Fun}^{\oplus,\Sigma}(\mathcal{P}, \mathcal{P})$, where the latter is the category of coproduct and suspension-preserving endofunctors of $\mathcal{P}$. Note that if $\mathcal{P}$ is an R-linear theory, then so is $\mathrm{h}\mathcal{P}$. If $\mathcal{P}$ is an R-linear theory, then $\mathrm{Psh}^{\Pi}(\mathcal{P})$ is an R-linear category, and

$\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$ is an R-linear stable category; if $\mathcal{P}$ is an R-linear resolution theory, then $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is an R-linear category, stable so long as $\mathcal{P}$ is a stable resolution theory.

6.2.2. **Proposition.** Let $\mathcal{P}$ be an R-linear theory. Then I-completion

$$\mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p}) \to \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p}), \qquad X \mapsto X_I^{\wedge}$$

restricts to a localization of $\mathrm{Psh}^{\Pi}(\mathcal{P})$ which preserves geometric realizations. This localization is given explicitly by

$$X_I^{\wedge} = \mathrm{tCof}\, K(X; \underline{u})$$

for $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$. □

Call $X \in \mathrm{Psh}^{\Pi}(\mathcal{P})$ *I-complete* if $X \simeq X_I^{\wedge}$. If $\mathcal{P}$ is a stable resolution theory, then $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is a stable R-linear category, so there is a possible ambiguity in speaking of I-complete objects of $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. However, this ambiguity turns out to vanish.

6.2.3. **Lemma.** Let $\mathcal{P}$ be an R-linear stable resolution theory, and fix $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. Then X is I-complete as an object of $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ if and only if it is I-complete as an object of $\mathrm{Psh}^{\Pi}(\mathcal{P})$.

Proof. If $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ is I-complete in $\mathrm{Psh}^{\Pi}(\mathcal{P})$, then $X \simeq \mathrm{tCof}(X; \underline{u})$ in $\mathrm{Psh}^{\Pi}(\mathcal{P})$. So the latter total cofiber lives in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, and thus X is I-complete in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. Suppose conversely that X is I-complete in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. By definition, X is I-complete in $\mathrm{Psh}^{\Pi}(\mathcal{P})$ if and only if for every $M \in \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p})$ which is I-local, the mapping space $\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P},\mathcal{S}\mathrm{p})}(M, X)$ is contractible. As we can identify $\mathrm{Map}_{\mathrm{Psh}^{\Pi}(\mathcal{P})}(M, X) \simeq \mathrm{Map}_{\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P},\mathcal{S}\mathrm{p})}(LM, LX)$ where $L\colon \mathrm{Psh}^{\Pi}(\mathcal{P}, \mathcal{S}\mathrm{p}) \to \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p})$ is the localization, and as LX is I-complete in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}, \mathcal{S}\mathrm{p})$, it is sufficient to verify that L preserves I-local objects. This is a consequence of [Lur18, Proposition 7.2.4.9]. □

Moreover, we can identify $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})^{\mathrm{Cpl}(I)}$ as follows.

6.2.4. **Proposition.** Let $\mathcal{P}$ be an R-linear stable resolution category. Let $\mathcal{P}_I^{\wedge} \subset \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$ be the full subcategory on the I-completions of objects of $\mathcal{P}$. Then

$$\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})^{\mathrm{Cpl}(I)} \simeq \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}_I^{\wedge}).$$

Proof. By Theorem 3.3.6, it is sufficient to verify that

$$\begin{array}{ccc} \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) & \xrightarrow{(-)_I^{\wedge}} & \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P}) \\ \downarrow & & \uparrow L \\ \mathrm{Psh}^{\Pi}(\mathcal{P}) & \xrightarrow{(-)_I^{\wedge}} & \mathrm{Psh}^{\Pi}(\mathcal{P}) \end{array}$$

commutes, which follows from Proposition 6.2.1. □

Some additional hypotheses are necessary to make use of Proposition 6.2.4. For example, $\mathrm{h}\mathcal{P}$ is an R-linear theory, so we would to identify $\mathrm{Psh}^{\Pi}(\mathrm{h}(\mathcal{P}_I^{\wedge})) \simeq \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})^{\mathrm{Cpl}(I)}$; however, this is not true in general. Determining when properties such as this hold amount to understanding when, given $X \in \mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$, the completion $X_I^{\wedge}$ as computed in $\mathrm{Psh}^{\Pi}(\mathcal{P})$ still lives in $\mathrm{Psh}^{\Pi,\Omega}(\mathcal{P})$. This turns out to be an essentially algebraic condition, which we now consider.

Call a theory $\mathcal{P}$ *pretame* if $\tau_!\colon \operatorname{Psh}^{\Pi}(\mathcal{P}) \to \operatorname{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$ preserves countable products. It follows readily from Theorem 3.2.1 that every resolution theory is pretame. The purpose of the pretameness condition is the following.

6.2.5. **Lemma.** Let $\mathcal{P}$ be a pretame R-linear theory. Then

$$\tau_!(X_I^\wedge) = (\tau_! X)_I^\wedge$$

for all $X \in \operatorname{Psh}^{\Pi}(\mathcal{P})$.

Proof. As $\mathcal{P}$ is pretame, we can identify $\tau_! K(X;\underline{u}) \simeq K(\tau_! X;\underline{u})$ as h-cubes, so the result follows from Proposition 6.2.2. □

If $\mathcal{P}$ is a discrete R-linear theory, we can describe the I-completion of discrete objects of $\operatorname{Psh}^{\Pi}(\mathcal{P})$ fairly algebraically. Given an abelian category $\mathcal{A}$, we can view an h-cube $V\colon P(\underline{h}) \to \mathcal{A}$ as an h-dimensional complex, and form the total complex C_*V. This satisfies $H_0(C_*V) = \operatorname{tCof} V$, the latter taken in $\mathcal{A}$. In general, one can show the following.

6.2.6. **Lemma.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $V\colon P(\underline{h}) \to \mathcal{A}$ be an h-cube. Then $(\mathbb{L}_n \operatorname{tCof})(V) = H_n(C_*V)$. □

In particular, if $\mathcal{A}$ is an R_0-linear abelian category with products and $M \in \mathcal{A}$, then we can form the chain complex $C_*K(M;\underline{u})$. In this case, set $K_n(M;\underline{u}) = H_n C_* K(M;\underline{u})$ and $\mathcal{A}_I M = K_0(M;\underline{u})$. As an immediate consequence of Proposition 6.2.2 and Lemma 6.2.6, we have the following.

6.2.7. **Lemma.** Let $\mathcal{P}$ be a discrete R-linear theory. For $M \in \operatorname{Psh}^{\Pi}(\mathcal{P})$ discrete, we can identify

$$\pi_n(M_I^\wedge) = K_n(M;\underline{u}).$$

In particular,

(1) $M_I^\wedge$ is h-truncated, h being the length of the sequence $\underline{u}$;
(2) $M_I^\wedge$ is discrete if and only if $K_n(M;\underline{u}) = 0$ for $n > 0$. □

We now arrive at the promised characterization.

6.2.8. **Proposition.** Let $\mathcal{P}$ be a pretame R-linear theory, and fix $X \in \operatorname{Psh}^{\Pi}(\mathcal{P})$. Suppose that $\tau_! X = \pi_0 X$. Then the following are equivalent:

(1) The object $\tau_! X_I^\wedge$ of $\operatorname{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$ is discrete;
(2) The object $K_n(\pi_0 X;\underline{u})$ of $\operatorname{Psh}^{\Pi}(\mathrm{h}\mathcal{P},\mathcal{S}\mathrm{et})$ vanishes for $n > 0$.

If $\mathcal{P}$ is a resolution theory, then $X \in \operatorname{Psh}^{\Pi,\Omega}(\mathcal{P})$, and these are equivalent to:

(3) The completion $X_I^\wedge$ computed in $\operatorname{Psh}^{\Pi}(\mathcal{P})$ lives in $\operatorname{Psh}^{\Pi,\Omega}(\mathcal{P})$.

Proof. The equivalence of (1) and (2) follows from Lemma 6.2.5 and Lemma 6.2.7, and the inclusion of (3) follows from Corollary 3.2.2. □

Say that I is *tame* on $X \in \operatorname{Psh}^{\Pi}(\mathcal{P})$ if the equivalent conditions of Proposition 6.2.8 hold for X, and say that I is tame on $\mathcal{P}$ if $h(P)$ is tame for $P \in \mathcal{P}$. From the definition, we see that I is tame on $\mathcal{P}$ if and only if it is tame on $\mathrm{h}\mathcal{P}$. The definition is chosen so that the following holds.

6.2.9. **Proposition.** Let $\mathcal{P}$ be a stable R-linear theory, and suppose that I is tame on $\mathcal{P}$. Then

(1) We can identify $\operatorname{Psh}^{\Pi}(\mathcal{P}_I^\wedge) \simeq \operatorname{Psh}^{\Pi}(\mathcal{P})^{\operatorname{Cpl}(I)}$;

(2) We can identify $\mathrm{Psh}^{\Pi}(\mathrm{h}(\mathcal{P}_I^\wedge)) \simeq \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})^{\mathrm{Cpl}(I)}$.

Proof. Because I is tame on $\mathcal{P}$, we can identify $\mathcal{P}_I^\wedge \subset \mathrm{Psh}^{\Pi}(\mathcal{P})$ as the full subcategory spanned by $h(P)_I^\wedge$ for $P \in \mathcal{P}$, where this completion is taken in $\mathrm{Psh}^{\Pi}(\mathcal{P})$, and can identify $\mathrm{h}(\mathcal{P}_I^\wedge) \subset \mathrm{Psh}^{\Pi}(\mathrm{h}\mathcal{P})$ as the full subcategory spanned by $(\pi_0 h(P))_I^\wedge$ for $P \in \mathcal{P}$. So (1) and (2) are consequences of Proposition 3.3.5. □

If S is a $\mathbb{E}_2$-ring under R and $\mathcal{P} = \mathrm{L}\mathcal{M}\mathrm{od}_S^{\mathrm{free}}$, then I is tame on $\mathcal{P}$ precisely when it is tame on $\mathrm{L}\mathcal{M}\mathrm{od}_{S_*}^{\mathrm{free}}$. Tameness in this setting coincides with the notion of tameness discussed in Greenlees-May [GM92] [GM95] and Rezk [Rez18, Section 8], and holds in a number of situations. For example, if M is an S_0-module, then I is tame on M if I is generated by a sequence which is regular on M, or if $M = N^{\oplus J}$ for some Noetherian S_0-module N and some set J [Lur18, Corollary 7.3.6.1]. We note that Proposition 6.2.9 extends by combination with Subsection 6.1 to describe unstable theories built out of stable theories in completed settings.

Appendix A. Spectral sequences of filtered objects

This appendix gives the facts we have needed about towers in a stable category with t-structure and their associated spectral sequences. We freely use material and notation from [Lur17a, Section 1.2].

A.1. **Construction and convergence.** Fix a stable category $\mathcal{C}$ with t-structure, and let $\mathcal{A}$ be the heart of $\mathcal{C}$. There is a resulting functor $\pi_0 = \tau_{\leq 0}\tau_{\geq 0}\colon \mathcal{C} \to \mathcal{A}$, and we set $\pi_p = \pi_0 \circ \Sigma^{-p}$. Fix a tower

$$X = \cdots \to X(-1) \to X(0) \to X(1) \to \cdots$$

in $\mathcal{C}$. We have for each $-\infty \leq p \leq q$ an object $X(p,q)$ in $\mathcal{C}$, where $X(-\infty, p) = X(p)$, and for $p \leq q \leq r$ we have a chosen cofiber sequence

$$X(p,q) \xrightarrow{\eta} X(p,r) \xrightarrow{\eta} X(q,r) \ .$$

In particular, $X(p,q)$ can be identified as sitting in a cofiber sequence

$$X(p) \xrightarrow{\eta} X(q) \xrightarrow{\eta} X(p,q) \ .$$

Define

$$E_{p,q}^r = \mathrm{im}\left(\pi_q X(p-r,p) \to \pi_q X(p-1,p+r-1)\right);$$

we abbreviate $E_{p,*}^r$ as E_p^r when it simplifies the notation. Using the diagrams

$$\begin{array}{ccc} X(p-r,p) & \xrightarrow{\eta} & X(p-1,p+r-1) \\ \downarrow & & \downarrow \\ \Sigma X(p-2r,p-r) & \xrightarrow{\Sigma\eta} & \Sigma X(p-r-1,p-1) \end{array} \ ,$$

we obtain maps

$$d_{p,q}^r \colon E_{p,q}^r \to E_{p-r,q-1}^r.$$

A.1.1. **Proposition** ([Lur17a, Proposition 1.2.2.7]). With notation as above,

(1) $d^r \circ d^r = 0$,
(2) There are canonical equivalences $E^{r+1} = H(E^r, d^r)$.

In particular, $\{E^r, d^r\}$ is a spectral sequence of objects of $\mathcal{A}$. □

Write $E(X)$ for this spectral sequence. We would like to identify some simple criteria for convergence. Suppose that $\mathcal{C}$ and $\mathcal{A}$ admit countable direct sums, and thus all countable colimits. For a tower X, write $X(\infty) = \operatorname{colim}_{p\to\infty} X(p)$. The following can be proved just as [Lur17a, Proposition 1.2.2.14].

A.1.2. **Proposition.** Fix a tower X, and suppose

(1) The connectivity of $X(p)$ goes to ∞ as p goes to $-\infty$;
(2) $\operatorname{colim}_{r\to\infty} \pi_* X(p, p+r) \simeq \pi_* \operatorname{colim}_{r\to\infty} X(p, p+r)$ for all $p \in \mathbb{Z} \cup \{-\infty\}$;

and moreover one of the following holds:

(3) The t-structure on $\mathcal{C}$ is compatible with filtered colimits;
(3') For all $q \in \mathbb{Z}$, the map $\pi_q X(p) \to \pi_q X(p+1)$ is an isomorphism for all but finitely many p.

Then $E(X)$ converges to $\pi_* X(\infty)$. Explicitly, if $A_q = \pi_q X(\infty)$, then

(a) For all fixed p, q and all sufficiently large r, there are canonical inclusions $E^r_{p,q} \subset E^{r+1}_{p,q}$, and in case $(3')$ these eventually stabilize;
(b) Where $F^p A_q = \operatorname{im}(\pi_q X(p) \to \pi_q X(\infty))$, we have $F^p A_q = 0$ for p sufficiently small and $A_q = \cup_p F^p A_q$, and in case $(3')$ this filtration is finite;
(c) There are canonical isomorphisms $F^p A_q / F^{p-1} A_q \cong E^\infty_{p,q}$. □

A.2. **Monoidal properties of towers.** Fix a stable category $\mathcal{C}$ with t-structure, and let $\mathcal{O}$ be a single-colored ∞-operad. Following [Lur17a, Definition 2.2.1.6], we say an $\mathcal{O}$-monoidal structure on $\mathcal{C}$ is compatible with the t-structure on $\mathcal{C}$ if it respects finite colimits, and for all $f \in \mathcal{O}(n)$, the tensor product $\otimes_f$ sends $\mathcal{C}_{\geq 0}^{\times n}$ into $\mathcal{C}_{\geq 0}$. Fix such an $\mathcal{O}$-monoidal structure on $\mathcal{C}$.

A.2.1. **Proposition.** The functor $\mathcal{C} \to \operatorname{Fun}(\mathbb{Z}, \mathcal{C})$ sending an object to its Whitehead tower is canonically lax $\mathcal{O}$-monoidal.

Proof. This functor factors as the composite of the diagonal $\mathcal{C} \to \operatorname{Fun}(\mathbb{Z}, \mathcal{C})$ and the endofunctor W of $\operatorname{Fun}(\mathbb{Z}, \mathcal{C})$ sending a tower $n \mapsto X(n)$ to the new tower $n \mapsto X(n)_{\geq -n}$. The former is lax $\mathcal{O}$-monoidal, as $\mathbb{Z} \to \{0\}$ is monoidal, hence we must verify the latter is lax $\mathcal{O}$-monoidal. This follows from [Lur17a, Proposition 2.2.1.1], for W is a colocalization of $\operatorname{Fun}(\mathbb{Z}, \mathcal{C})$, with image closed under the $\mathcal{O}$-monoidal structure by our hypotheses. □

We restrict now to the case where $\mathcal{O}$ is the nonunital $\mathbb{A}_2$-operad. In other words, we fix for ourselves a pairing $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is exact in each variable and sends $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. Writing $\mathcal{A}$ for the heart of $\mathcal{C}$, we obtain a pairing $\overline{\otimes}$ on $\mathcal{A}$ given by

$$M \mathbin{\overline{\otimes}} N = \pi_0(M \otimes N).$$

For $X', X'' \in \mathcal{C}$, there is a canonical Künneth map $\pi_0 X' \mathbin{\overline{\otimes}} \pi_0 X'' \to \pi_0(X \otimes X'')$ given by the composite

$$\pi_0 X' \mathbin{\overline{\otimes}} \pi_0 X'' = \pi_0(X'_{\geq 0} \otimes X''_{\geq 0}) \to \pi_0(X' \otimes X'').$$

There is not such a canonical map in nonzero degrees, the issue being the following. As $\otimes$ is exact in each variable, there are canonical isomorphisms $(\Sigma X') \otimes X'' \simeq \Sigma(X' \otimes X'')$ and $X' \otimes (\Sigma X'') \simeq \Sigma(X' \otimes X'')$. However, the diagram

$$\begin{array}{ccc} (\Sigma^{q'}X')\otimes(\Sigma^{q''}X'') & \longrightarrow & \Sigma^{q'}(X'\otimes(\Sigma^{q''}X'')) \\ \downarrow & & \downarrow \\ \Sigma^{q''}((\Sigma^{q'}X')\otimes X'') & \longrightarrow & \Sigma^{q'+q''}(X'\otimes X'') \end{array}$$

can only be made to commute up to a switch map $S^{q'+q''} \simeq S^{q'} \otimes S^{q''} \simeq S^{q''} \otimes S^{q'} \simeq S^{q''+q'} = S^{q'+q''}$, and so on π_0 up to a sign of $(-1)^{q'q''}$. For the rest of this section, we choose the isomorphism given by the counterclockwise composite; in other words, we choose

$$(\Sigma^{q'}X')\otimes(\Sigma^{q''}X'') = \Sigma^{q''}\Sigma^{q'}(X'\otimes X'').$$

This choice falls naturally out of the convention of pretending that $(\Sigma X')\otimes X''$ and $\Sigma(X'\otimes X'')$ are the "same", whereas $X'\otimes(\Sigma X'')$ and $\Sigma(X'\otimes X'')$ are "different". Having made a choice, we obtain a natural transformation

$$\begin{aligned}\pi_{q'}X' \mathbin{\overline{\otimes}} \pi_{q''}X'' &= \pi_0\Sigma^{-q'}X' \mathbin{\overline{\otimes}} \pi_0\Sigma^{-q''}X'' \\ &\to \pi_0(\Sigma^{-q'}X'\otimes\Sigma^{-q''}X'') \simeq \pi_{q'+q''}(X'\otimes X'').\end{aligned}$$

With this choice, the diagram

$$\begin{array}{ccc} \pi_{q'}\Sigma X' \mathbin{\overline{\otimes}} \pi_{q''}X'' & \xrightarrow{=} & \pi_{q'-1}X'\otimes\pi_{q''}X'' \\ \downarrow & & \downarrow \\ \pi_{q'+q''}\Sigma X'\otimes X'' & \xrightarrow{\simeq} & \pi_{q'+q''-1}X'\otimes X'' \end{array}$$

commutes, whereas the diagram

$$\begin{array}{ccc} \pi_{q'}X' \mathbin{\overline{\otimes}} \pi_{q''}\Sigma X'' & \xrightarrow{=} & \pi_{q'}X' \mathbin{\overline{\otimes}} \pi_{q''-1}X'' \\ \downarrow & & \downarrow \\ \pi_{q'+q''}X'\otimes\Sigma X'' & \xrightarrow{\simeq} & \pi_{q'+q''-1}X'\otimes X'' \end{array}$$

commutes up to a factor of exactly $(-1)^{q'}$. This is the origin of the signs that will appear for us.

We end this subsection by recording a concrete description of a pairing in $\mathrm{Fun}(\mathbb{Z},\mathcal{C})$.

A.2.2. **Lemma.** A pairing $X'\otimes X''\to X$ in $\mathrm{Fun}(\mathbb{Z},\mathcal{C})$ is equivalent to the choice, for all $p',p''\in\mathbb{Z}$, of a pairing $X'(p')\otimes X''(p'')\to X(p'+p'')$, together with homotopies filling in the cubes

$$\begin{array}{ccccccc} X'(p'-1)\otimes X''(p''-1) & & \longrightarrow & & X'(p')\otimes X''(p''-1) & & \\ & \searrow & & & & \searrow & \\ & & X(p'+p''-2) & & \longrightarrow & & X(p'+p''-1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'(p'-1)\otimes X''(p'') & & \longrightarrow & & X'(p')\otimes X''(p'') & & \\ & \searrow & & & & \searrow & \\ & & X(p'+p''-1) & & \longrightarrow & & X(p'+p'') \end{array}.$$

□

A.3. **Pairings of spectral sequences.** Fix conventions as in the previous subsections. Our goal in this subsection is to verify that every pairing $X' \otimes X'' \to X$ of towers gives rise to a pairing $E(X') \overline{\otimes} E(X'') \to E(X)$ of spectral sequences. Before giving the main construction, we point out the following. By a cofibering $X''(-1) \to X''(0) \to C''(0)$, we refer really to the left square in a suitable coherently commutative diagram

$$\begin{tikzcd} X''(-1) \arrow[r] \arrow[d] & X''(0) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & C''(0) \arrow[r, "\delta"] & \Sigma X''(-1) \end{tikzcd},$$

and from this we obtain the right square, and in particular the boundary map δ. As $\otimes$ is exact in both variables, for any $X' \in \mathcal{C}$ we obtain from our original cofiber sequence a cofiber sequence $X' \otimes X''(-1) \to X' \otimes X''(0) \to X' \otimes C''(0)$. Again, this refers really to the left square in

$$\begin{tikzcd} X' \otimes X''(-1) \arrow[r] \arrow[d] & X' \otimes X''(0) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X' \otimes C''(0) \arrow[r, "\delta'"] & \Sigma(X' \otimes X''(-1)) \end{tikzcd},$$

where we have implicitly identified $X' \otimes 0 \simeq 0$, and from this we obtain the right square, and in particular the boundary map δ'. This diagram is canonically equivalent to the diagram obtained by tensoring the first with X', and we find that δ' can be identified with the composite

$$X' \otimes C''(0) \xrightarrow{X' \otimes \delta} X' \otimes \Sigma X''(-1) \xrightarrow{\simeq} \Sigma(X' \otimes X''(-1)) .$$

We now proceed to the main construction. Fix the data of cofiberings

$$\begin{array}{cc} X(-2) \to X(-1) \to C(-1) & X(-1) \to X(0) \to C(0) \\ X'(-1) \to X'(0) \to C'(0) & X''(-1) \to X''(0) \to C''(0), \end{array}$$

as well as the data of a filled in cube

$$\begin{tikzcd} X'(-1) \otimes X''(-1) \arrow[rr] \arrow[dd] \arrow[dr] & & X'(0) \otimes X''(-1) \arrow[dd] \arrow[dr] & \\ & X(-2) \arrow[rr] \arrow[dd] & & X(-1) \arrow[dd] \\ X'(-1) \otimes X''(0) \arrow[rr] \arrow[dr] & & X'(0) \otimes X''(0) \arrow[dr] & \\ & X(-1) \arrow[rr] & & X(0) \end{tikzcd}.$$

Our initial cofiberings, together with the fact that $\otimes$ is exact in each variable, give rise to a canonical isomorphism from the total cofiber of the back face of this cube to $C'(0) \otimes C''(0)$. As a consequence of this, we can form the commutative diagrams

$$\begin{array}{ccc}
X'(-1)\otimes X''(0)\cup_{X'(-1)\otimes X''(-1)} X'(0)\otimes X''(-1) & \longrightarrow & X(-1) \\
\downarrow & & \downarrow \\
X'(0)\otimes X''(0) & \longrightarrow & X(0) \\
\downarrow & & \downarrow \\
C'(0)\otimes C''(0) & \dashrightarrow & C(0)
\end{array}$$

$$\begin{array}{ccc}
X'(-1)\otimes X''(-1) & \longrightarrow & X(-2) \\
\downarrow & & \downarrow \\
X'(-1)\otimes X''(0)\cup_{X'(-1)\otimes X''(-1)} X'(0)\otimes X''(-1) & \longrightarrow & X(-1) \\
\downarrow f & & \downarrow \\
X'(-1)\otimes C''(0)\oplus C'(0)\otimes X''(-1) & \dashrightarrow & C(-1)
\end{array},$$

where the columns have the structure of cofiber sequences and, the bottom squares are induced from this. By the construction of the maps involved, we have the following.

A.3.1. **Lemma.** In the diagram

$$\begin{array}{ccc}
C'(0)\otimes C''(0) & \longrightarrow & C(0) \\
\downarrow & & \downarrow \\
\Sigma(X'(-1)\otimes X''(0)\cup_{X'(-1)\otimes X''(-1)} X'(0)\otimes X''(-1)) & \longrightarrow & \Sigma X(-1) \\
\downarrow \Sigma f & & \downarrow \\
\Sigma(X'(-1)\otimes C''(0)\oplus C'(0)\otimes X''(-1)) & \longrightarrow & \Sigma C(-1)
\end{array}$$

induced from the above data, the left vertical composite is given by the sum of the maps

$$C'(0)\otimes C''(0)\to(\Sigma X'(-1))\otimes C''(0)\simeq\Sigma(X'(-1)\otimes C''(0))$$
$$C'(0)\otimes C''(0)\to C'(0)\otimes(\Sigma X''(-1))\simeq\Sigma(C'(0)\otimes X''(-1)).$$

□

We are now in a position to prove the following.

A.3.2. **Theorem.** A pairing $X'\otimes X''\to X$ of towers gives rise to a pairing $E(X')\mathbin{\overline{\otimes}}E(X'')\to E(X'')$ of spectral sequences, i.e. pairings

$$\smile\colon E^r_{p',q'}(X')\mathbin{\overline{\otimes}}E^r_{p'',q''}(X'')\to E^r_{p'+p'',q'+q''}(X)$$

satisfying the Leibniz rule

$$d^r(x'\smile x'')=d^r(x')\smile x''+(-1)^{q'}x'\smile d^r(x''),$$

where moreover the pairing on E^r is induced from naturally defined maps

$$X'(p'-r,p')\otimes X''(p''-r,p'')\to X(p'+p''-r,p'+p''),$$

and the pairing on E^{r+1} is induced by that on E^r.

Proof. From the pairing $X'\otimes X''\to X$, we obtain solid cubes

$$\begin{array}{ccccccc}
X'(p'-r)\otimes X''(p''-r) & & \longrightarrow & & X'(p')\otimes X''(p''-r) & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
& & X(p'+p''-2r) & \longrightarrow & & & X(p'+p''-r) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X'(p'-r)\otimes X''(p'') & & \longrightarrow & & X'(p')\otimes X''(p'') & & \\
& \searrow & \downarrow & & & \searrow & \downarrow \\
& & X(p'+p''-r) & \longrightarrow & & & X(p'+p'')
\end{array}.$$

Via the construction above, we obtain from this pairings

$$X'(p'-r,p')\otimes X''(p''-r,p'')\to X(p'+p''-r,p'+p'')$$

with commutative diagrams

$$\begin{array}{ccc}
X'(p'-r,p')\otimes X''(p''-r,p'') & \xrightarrow{\mu} & X(p'+p''-r,p'+p'') \\
\downarrow & & \downarrow \\
\Sigma\left(\begin{array}{c} X'(p'-2r,p'-r)\otimes X''(p''-r,p'') \\ \oplus \\ X'(p'-r,p')\otimes X''(p''-2r,p''-r)\end{array}\right) & \xrightarrow{\Sigma(\mu+\mu)} & \Sigma X(p'+p''-2r,p'+p''-r)
\end{array}.$$

As $\pi_*X(p-1,p)=E^1_p(X)$, the pairings obtained for $r=1$ give $E^1_{p'}(X')\,\overline{\otimes}\,E^1_{p''}(X'')\to E^1_{p'+p''}(X)$. The above square, together with Lemma A.3.1 identifying the left vertical composite and our conventions regarding Künneth maps, implies this satisfies the Leibniz rule, and hence passes to a pairing on E^r for all $r\geq 1$. By construction there are canonically commutative diagrams

$$\begin{array}{ccc}
X'(p'-r,p')\otimes X''(p''-r,p'') & \longrightarrow & X(p'+p''-r,p'+p'') \\
\downarrow & & \downarrow \\
X'(p'-1,p')\otimes X''(p''-1,p'') & \longrightarrow & X(p'+p''-1,p'+p'')
\end{array},$$

and these tell us that the pairing on E^r is induced from the pairing in $\mathcal{C}$. □

We end with a remark concerning convergence of this product. Fix a pairing of towers $X'\otimes X''\to X$. Suppose that $\mathcal{C}$ and $\mathcal{A}$ admit countable sums, and that these distribute across $\otimes$ and $\overline{\otimes}$. In particular, we obtain a pairing $X'(\infty)\otimes X''(\infty)\to X(\infty)$. Under the convergence conditions of Proposition A.1.2, the pairing $E(X')\,\overline{\otimes}\,E(X'')\to E(X)$ of Theorem A.3.2 passes to $E^\infty_{p'}(X')\,\overline{\otimes}\,E^\infty_{p''}(X'')\to E^\infty_{p'+p''}(X)$. As there are canonically commutative diagrams

$$\begin{array}{ccc}
X'(p')\otimes X''(p'') & \longrightarrow & X(p'+p'') \\
\downarrow & & \downarrow \\
X'(\infty)\otimes X''(\infty) & \longrightarrow & X(\infty)
\end{array},$$

this is the associated graded of the pairing $\pi_*X'(\infty)\,\overline{\otimes}\,\pi_*X''(\infty)\to\pi_*X(\infty)$.

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

Email address: balderr2@illinois.edu