STRAIGHTENING AND UNSTRAIGHTENING FOR ORDINARY CATEGORIES

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1. Intro

1.1. **Goal.** The goal of this note is to describe a reasonably simple Quillen equivalence

$$c : \mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Proj}} \rightleftarrows \mathrm{s}\mathcal{S}\mathrm{et}/A_{\mathrm{Cov}} : d$$

for A an ordinary category. We take as given a characterization of the covariant model structure which we review in Section 3.1. We refer the reader to Cisinski's beautiful book [Cis] for a pleasant proof of this characterization; this note was written when learning from there, and essentially everything here came from this.

1.2. **Conventions.** We denote by $\mathbf{\Delta}$ the simplicial indexing category, and by $\mathrm{s}\mathcal{S}\mathrm{et} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ the category of simplicial sets. The nerve construction gives a fully faithful functor $N\colon \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$, and we will tend to write $N(A) = A$ for a category A. If S is a simplicial set and $s \in S$ is an object, we will denote by s/S and S/s the slices of S under and over s respectively; in particular, $a/N(A) = N(a/A)$ and $N(A)/a = N(A/a)$ for a category A and object $a \in A$.

2. Construction

Define $c\colon \mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}) \to \mathrm{s}\mathcal{S}\mathrm{et}/A$ by the following composite:

$$\begin{aligned}\mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}) &\cong \mathrm{Fun}(A \times \mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})\\ &\xrightarrow{\int} \mathcal{C}\mathrm{at}/A \times \mathbf{\Delta}^{\mathrm{op}}\\ &\xrightarrow{N} \mathrm{s}\mathcal{S}\mathrm{et}/A \times \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\pi_1} \mathrm{s}\mathcal{S}\mathrm{et}/A.\end{aligned}$$

Here, if B is a category then

$$\int \colon \mathrm{Fun}(B, \mathcal{S}\mathrm{et}) \to \mathcal{C}\mathrm{at}/B$$

is the covariant Grothendieck construction, which sends a functor $F\colon B \to \mathcal{S}\mathrm{et}$ to the category with objects the pairs $\langle b \in B, x \in F(b)\rangle$ and morphisms $f\colon \langle b, x\rangle \to \langle b', x'\rangle$ being $f\colon b \to b'$ such that $F(f)(x) = x'$. For $F\colon A \to \mathrm{s}\mathcal{S}\mathrm{et}$, an explicit description of $c(F)$ is given by

$$c(F)_n = \coprod_{(f,g)\colon [n] \to A \times \mathbf{\Delta}^{\mathrm{op}}} F(f(0))_{g(0)}.$$

An alternate description is provided by the following observation in the case $B = A \times \mathbf{\Delta}^{\mathrm{op}}$.

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2.0.1. **Lemma.** For a category B, let $(\mathbf{\Delta}/B)^{\mathrm{op}} = \int N(B)$ denote the Grothendieck construction applied to $N(B)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Define a functor $p_0\colon (\mathbf{\Delta}/B)^{\mathrm{op}} \to B$ as follows. On objects,

$$p_0(f\colon [n] \to B) = f(0).$$

On morphisms, we send

$$\begin{array}{ccc} [m] & \xleftarrow{\ \sigma\ } & [n] \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & B & \end{array}$$

to the map $f(0) \to g(0)$ determined by the relations $0 \leq \sigma(0)$ and $f(\sigma(0)) = g(0)$. Then the diagram

$$\begin{array}{ccc} \mathrm{Fun}(B, \mathcal{S}\mathrm{et}) & \xrightarrow{\ \int\ } & \mathcal{C}\mathrm{at}/B \\ \big\downarrow{\scriptstyle p_0^*} & & \big\downarrow{\scriptstyle N} \\ \mathrm{Fun}((\mathbf{\Delta}/B)^{\mathrm{op}}, \mathcal{S}\mathrm{et}) & \xrightarrow{\ \simeq\ } & \mathrm{s}\mathcal{S}\mathrm{et}/B \end{array}$$

commutes. □

The construction of $\int N(B)$ from B can be also be described as the categorical wreath product of Δ with B along the embedding $\Delta \subset \mathcal{C}\mathrm{at}$. We can read off of Lemma 2.0.1 that the functor $c\colon \mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}) \to \mathrm{s}\mathcal{S}\mathrm{et}/A$ preserves colimits, and is thus part of an adjunction $c \dashv d$. Our goal in this note is to show that this is a Quillen equivalence.

3. Model Structures

Let us recall the relevant model structures. We will assume as known the Kan-Quillen model structure $\mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

3.1. **Covariant model structure.** For a simplicial set A, we write $(\mathrm{s}\mathcal{S}\mathrm{et}/A)_{\mathrm{Cov}}$ for the covariant model structure on simplicial sets over A. This is a model structure whose cofibrations are monomorphisms and fibrant objects are the left fibrations over A. The weak equivalences arise as follows.

For $p\colon X \to A$ and $a \in A$, define $X/a = A/a \times_A X$. If p is a left fibration, then we have weak equivalences $\mathrm{Map}_A(a/A, X) \to p^{-1}(a) \to X/a$ of Kan complexes. A map $f\colon (X, p) \to (Y, q)$ of simplicial sets over A is a covariant equivalence precisely when for each $a \in A$, the induced map $X/a \to Y/a$ is a weak equivalence. In particular, weak equivalences of left fibrations over A are exactly the fiberwise equivalences over A.

Given $p\colon X \to A$ and $S \in \mathrm{s}\mathcal{S}\mathrm{et}$, we can form $X \times S \to X \to A$. If $a \in A$, then $(X \times S)/a \cong (X/a) \times S$, so this construction preserves weak equivalences in the S variable.

3.2. **Projective model structure.** For a category A, we write $\mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Proj}}$ for the projective model structure formed with respect to the Kan-Quillen model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$. This model structure consists of levelwise weak equivalences and fibrations, and is cofibrantly generated with generating cofibrations

$$h^a \times \partial\Delta^n \subset h^a \times \Delta^n$$

and generating trivial cofibrations

$$h^a \times \Lambda^n_k \subset h^a \times \Delta^n,$$

where $h^a = A(a, -)$.

4. Key Facts

4.1. **Technical lemma.**

4.1.1. **Definition.** Call a class $\mathbf{C}$ of simplicial sets saturated by monomorphisms if the following conditions are satisfied:

(i) For any small family (X_α) of simplicial sets, if $X_\alpha \in \mathbf{C}$ for each α, then $\coprod_\alpha X_\alpha \in C$,

(ii) For any sequence

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \cdots$$

of monomorphisms of simplicial sets, if each X_i is in $\mathbf{C}$, then $\operatorname{colim}_i X_i \in \mathbf{C}$,

(iii) For any pushout

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ Y & \longrightarrow & Y' \end{array}$$

of simplicial sets in which i and j are monic and X, X', and Y are in $\mathbf{C}$, so too is Y' in $\mathbf{C}$.

4.1.2. **Lemma.** Let $\mathbf{C}$ be a class of simplicial sets which is saturated by monomorphisms, and suppose $\mathbf{C}$ contains Δ^n for $n \geq 0$. Then $\mathbf{C}$ is the class of all simplicial sets.

Proof. This follows immediately from the cellular filtration on simplicial sets. □

4.2. **Main construction.** Define a cosimplicial space $S\colon \mathbf{\Delta} \to \mathrm{s}\mathcal{S}\mathrm{et}$ by

$$S([n]) = \int \Delta^n = (\mathbf{\Delta}/[n])^{\mathrm{op}}.$$

This extends to an adjunction $S_! : \mathrm{s}\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{s}\mathcal{S}\mathrm{et} : S^*$. By Lemma 2.0.1 applied to $B = \mathbf{\Delta}^{\mathrm{op}}$, the functor

$$\int : \mathrm{s}\mathcal{S}\mathrm{et} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{s}\mathcal{S}\mathrm{et}$$

preserves colimits, so we necessarily have $S_! = \int$. Moreover, the nerves of the functors $p_0\colon \int \Delta^n \to [n]$ of Lemma 2.0.1 yield a natural transformation $S \to h$, where $h\colon \mathbf{\Delta} \to \mathrm{s}\mathcal{S}\mathrm{et}$ is the Yoneda embedding, and thus yield transpose natural transformations $S_! \to I$ and $I \to S^*$, where I denotes the identity of $\mathrm{s}\mathcal{S}\mathrm{et}$.

The following is the key point; I learned it from [Cis].

4.2.1. **Proposition.** For any simplicial set X, the map $\int X = S_!(X) \to X$ is a weak equivalence. If X is a Kan complex, then $X \to S^*(X)$ is a weak equivalence.

Proof. The functor $S_!$ visibly preserves monomorphisms. It thus suffices to verify that $S_!(X) \to X$ is always a weak equivalence. Indeed, in this case we learn that $S_! \dashv S^*$ is a Quillen adjunction and $\mathbf{L}S_! \to I$ is a natural isomorphism, where $\mathbf{L}S_!$ is the left derived functor of $S_!$. It follows that the transpose $I \to \mathbf{R}S^*$ is a natural isomorphism, where $\mathbf{R}S^*$ is the right derived functor of S^*, yielding the proposition.

Let $\mathbf{C}$ denote the class of simplicial sets X for which $\int X \to X$ is a weak equivalence. Using the fact that $\int\colon \mathcal{sSet} \to \mathcal{sSet}$ preserves monomorphisms and colimits, and that the cofibrations in $\mathcal{sSet}_{\mathrm{KQ}}$ are the monomorphisms, we find that $\mathbf{C}$ is saturated by monomorphisms. By Lemma 4.1.2, it thus suffices to verify that $\int \Delta^n \to \Delta^n$ is a weak equivalence. As $\int \Delta^n = (\mathbf{\Delta}/[n])^{\mathrm{op}}$ has an initial object, this is a map between contractible simplicial sets and is thus a weak equivalence. □

4.2.2. **Remark.** The functor $p_{\Delta^n}\colon \int \Delta^n = (\mathbf{\Delta}/[n])^{\mathrm{op}} \to [n]$ has a fully faithful left adjoint sending $j \in [n]$ to the upper inclusion $[n-j] \to [n]$. It follows that Δ^n is the localization of $\int \Delta^n$ at the maps which p_{Δ^n} inverts, which coincide with the maps which p_{Δ^n} sends to the identity. A saturation argument extends this to show in general that $p_X\colon \int X \to X$ realizes X as the localization of $\int X$ at the collection of maps which p_X sends to the identity. This yields a quick proof that every ∞-category arises as the localization of a relative category.

4.2.3. **Remark.** Proposition 4.2.1 admits the following generalization: if $T\colon \mathbf{\Delta} \to \mathcal{sSet}$ is a cosimplicial space satisfying the conditions that $T_!$ is monic—it is sufficient to check only the map $T([0]) \coprod T([0]) \to T([1])$—and $T_! T^*(\Delta^n) \to \Delta^n$ is a weak equivalence for all n, then $T_! : \mathcal{sSet}_{\mathrm{KQ}} \rightleftarrows \mathcal{sSet}_{\mathrm{KQ}} : T^*$ is a Quillen equivalence. For example, associated to any category Ω is the functor $j_\Omega\colon \Omega \to \mathcal{sSet}$ given by $j_\Omega(\omega) = N(\Omega/\omega)$. The condition that the cosimplicial space $T = j_{\Omega!} \circ j_\Omega^* \circ h$ satisfies the mentioned hypotheses is related to the condition that Ω is a weak test category.

5. Quillen Adjunction

To verify that $c : \mathrm{Fun}(A, \mathcal{sSet}_{\mathrm{KQ}})_{\mathrm{Proj}} \rightleftarrows \mathcal{sSet}/A_{\mathrm{Cov}} : d$ is a Quillen adjunction, as c visibly preserves monomorphisms it is sufficient to demonstrate that

$$c(h^a \times \Lambda^n_k) \subset c(h^a \times \Delta^n)$$

is a covariant equivalence over A for $0 \le k \le n$. For any two categories A and B,

$$\begin{array}{ccc} \mathrm{Fun}(A, \mathcal{Set}) \times \mathrm{Fun}(B, \mathcal{Set}) & \xrightarrow{\int \times \int} & \mathcal{Cat}/A \times \mathcal{Cat}/B \\ \downarrow{\scriptstyle \times} & & \downarrow{\scriptstyle \times} \\ \mathrm{Fun}(A \times B, \mathcal{Set}) & \xrightarrow{\quad \int \quad} & \mathcal{Cat}/A \times B \end{array}$$

commutes, so we find that for any simplicial set S,

$$c(h^a \times S) = (a/A) \times \int S \to A \times \mathbf{\Delta}^{\mathrm{op}} \to A.$$

It is thus sufficient to show that if $\alpha\colon S \to T$ is a weak equivalence, then

$$(a/A) \times \int S \to (a/A) \times \int T$$

is a covariant equivalence over A. Indeed, in this case we can form a diagram

$$\begin{array}{ccc} (a/A) \times \int S & \longrightarrow & (a/A) \times \int T \\ \downarrow & & \downarrow \\ (a/A) \times S & \xrightarrow{(a/A)\times\alpha} & (a/A) \times T \end{array},$$

and by assumption and Proposition 4.2.1 the bottom map and vertical maps are covariant equivalences, whence so too is the top map.

6. Quillen equivalence

6.0.1. **Lemma.** For a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ to be a Quillen equivalence, it is sufficent to suppose

(i) F both preserves and reflects all weak equivalences,
(ii) For $X \in \mathcal{N}$ fibrant, the counit $FG(X) \to X$ is a weak equivalence.

Proof. Suppose given such a Quillen adjunction $F \dashv G$. In general, this is a Quillen equivalence precisely when for all $X \in \mathcal{N}$ fibrant and $Y \in \mathcal{M}$ cofibrant, the maps

$$F(G(X)_{\mathrm{cof}}) \to FG(X) \to X, \qquad Y \to GF(Y) \to G(F(Y)_{\mathrm{fib}})$$

are weak equivalences, where we have indicated cofibrant and fibrant replacements by subscripts. Under our assumptions, the first map is immediately a weak equivalence, so we must verify that the second is as well. Consider the diagram

$$\begin{array}{ccccc} F(Y) & \xrightarrow{\eta} & FGF(Y) & \longrightarrow & FG(F(Y)_{\mathrm{fib}}) \\ & \searrow & & & \downarrow{\scriptstyle \epsilon} \\ & & & & F(Y)_{\mathrm{fib}} \end{array}.$$

As F reflects weak equivalences, it is sufficient to verify that the top row is a weak equivalence. As the counit of $F \dashv G$ is an weak equivalence, it suffices to show the diagonal is an equivalence. This can be identified as the fibrant replacement of $F(Y)$ in $\mathcal{N}$, which is indeed a weak equivalence. □

6.0.2. **Lemma.** The functor $c\colon \mathrm{Fun}(A, s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Proj}} \to s\mathcal{S}\mathrm{et}/A_{\mathrm{Cov}}$ preserves and reflects weak equivalences.

Proof. Fix $F, G\colon A \to s\mathcal{S}\mathrm{et}$ and a map $\alpha\colon F \to G$. Recall that $c(\alpha)\colon c(F) \to c(G)$ is a covariant equivalence over A if and only if for all $a \in A$, the induced map $c(F)/a \to c(G)/a$ is an equivalence. For $a \in A$, let $c(F)^a$ denote the fiber of $c(F)$ at a. Then we have a diagram

$$\begin{array}{ccccccc} c(F)/a & \longleftarrow & c(F)^a & = & \int F(a) & \xrightarrow{\simeq} & F(a) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha(a)} \\ c(G)/a & \longleftarrow & c(G)^a & = & \int G(a) & \xrightarrow{\simeq} & G(a) \end{array}$$

in which the indicated arrows are weak equivalences by Proposition 4.2.1. It is thus sufficient to demonstrate that

$$\int F(a) = c(F)^a \to c(F)/a$$

is a weak equivalence. Both of these simplicial sets are nerves of categories, $c(F)^a$ arising from the category of pairs $\langle [n] \in \mathbf{\Delta}^{\mathrm{op}}, x \in F(a)_n \rangle$ and $c(F)/a$ arising from the category of tuples $\langle [m] \in \mathbf{\Delta}^{\mathrm{op}}, b \in A, y \in F(b)_m, g\colon b \to a \rangle$. The inclusion $c(F)^a \to c(F)/a$ has a left adjoint sending

$$\langle m, b, y, g \rangle \mapsto \langle m, F(g)(y) \rangle,$$

and so is an equivalence. □

6.0.3. **Lemma.** For any left fibration $p\colon X \to A$, the counit $(c\circ d)(X,p) \to (X,p)$ is a covariant equivalence over A.

Proof. Fix a left fibration $p\colon X \to A$. As p is a left fibration, the inclusion $p^{-1}(a) \to X/a$ is a weak equivalence. Thus, as in Lemma 6.0.2, we are reduced to verifying that $(c\circ d)(X,p) \to (X,p)$ is a fiberwise equivalence over A, for which we are reduced to checking that we have a natural weak equivalence $d(X,p)(a) \simeq p^{-1}(a)$. As $c(h^a \times \Delta^n) = a/A \times \int \Delta^n$, we can identify $d(X,p) = S^*(\mathrm{Map}_A(a/A, X))$, and by Proposition 4.2.1 and our discussion of the covariant model structure,

$$S^*(\mathrm{Map}_A(a/A,X)) \simeq \mathrm{Map}_A(a/A,X) \simeq p^{-1}(a). \qquad \square$$

Putting everything together, we have proved the following.

6.0.4. **Theorem.** For a category A, the adjunction

$$c : \mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Proj}} \rightleftarrows \mathrm{s}\mathcal{S}\mathrm{et}/A_{\mathrm{Cov}} : d$$

is a Quillen equivalence. $\square$

7. Comparison with Heuts-Moerdijk model

Heuts-Moerdijk [HM15] gives a similar Quillen equivalence

$$\overline{c} : \mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Proj}} \rightleftarrows \mathrm{s}\mathcal{S}\mathrm{et}/A_{\mathrm{Cov}} : \overline{d}.$$

Their left adjoint $\overline{c}$ is given by the composite

$$\begin{aligned}\mathrm{Fun}(A, \mathrm{s}\mathcal{S}\mathrm{et}) &\cong \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Fun}(A, \mathcal{S}\mathrm{et})) \\ &\xrightarrow{p_0^*} \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Fun}((\boldsymbol{\Delta}/A)^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \\ &\cong \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}} \times (\boldsymbol{\Delta}/A)^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \\ &\xrightarrow{\delta^*} \mathrm{Fun}((\boldsymbol{\Delta}/A)^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \cong \mathrm{s}\mathcal{S}\mathrm{et}/A,\end{aligned}$$

where $\delta\colon \boldsymbol{\Delta}/A \to \boldsymbol{\Delta} \times \boldsymbol{\Delta}/A$ is given by the canonical projection and the identity. Explicitly,

$$\overline{c}(F)_n = \coprod_{f\colon [n]\to A} F(f(0))_n.$$

If S is a simplicial set, then

$$\overline{c}(h^a \times S) = (a/A) \times S \to A \times \boldsymbol{\Delta}^{\mathrm{op}} \to A,$$

so we find that $\overline{c} \dashv \overline{d}$ is a Quillen adjunction. The right adjoint can be described as

$$\overline{d}(X,p)(a) = \mathrm{Map}_A(a/A, X).$$

By Proposition 4.2.1, we obtain a natural transformation $\overline{d}(X,p) \to d(X,p)$ which is a weak equivalence if $p\colon X \to A$ is a left fibration. It follows that $\overline{c} \dashv \overline{d}$ is a Quillen equivalence, giving an alternate proof of half of [HM15, Theorem C]. Of course, this misses some of the point: we've assumed some hard facts about covariant equivalences that they're able to deduce as a consequence of their methods.

An advantage of $\overline{c} \dashv \overline{d}$ is that for $a \in A$ and $F\colon A \to \mathrm{s}\mathcal{S}\mathrm{et}$, we have a strict isomorphism $\overline{c}(F)^a \cong F(a)$. An advantage of $c \dashv d$ is that for $a \in A$, and $F\colon A \to \mathrm{s}\mathcal{S}\mathrm{et}$, the simplicial sets $c(F)/a$ and $c(F)^a$ are nerves of ordinary categories.

References

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