LOWER CATEGORY THEORY

WILLIAM BALDERRAMA

This is, roughly, a note on how to use the relation between $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$, $\mathcal{C}\mathrm{at}/\mathcal{C}$, and $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ to reason about categories.

1. Some preliminaries

1.1. **Slice categories.** The ordinary category $\mathcal{C}\mathrm{at}$ of categories has all small limits, which are computed in the most naive way. In particular, given a span

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{E},$$

the pullback $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ is the category with:

(i) Objects: pairs $\langle c \in \mathcal{C}, e \in \mathcal{E}\rangle$ such that $f(c) = g(d)$,
(ii) Morphisms: a morphism $\langle c, e\rangle \to \langle c', e'\rangle$ consists of $m\colon c \to c'$ and $n\colon e \to e'$ such that $f(m) = g(n)$.

Observe that the equalities in the above definition are strict.

Given $f\colon \mathcal{C} \to \mathcal{D}$, we define the slice categories

$$\mathcal{D}/f = \mathcal{D}^{[1]}{}_{\mathrm{ev}_1}{\times_{\mathcal{D}}}\, \mathcal{C}, \qquad f/\mathcal{D} = \mathcal{D}^{[1]}{}_{\mathrm{ev}_0}{\times_{\mathcal{D}}}\, \mathcal{C}.$$

These are equipped with projections $p\colon \mathcal{D}/f \to \mathcal{D}$ and $q\colon f/\mathcal{D} \to \mathcal{D}$, and for $d \in \mathcal{D}$ we have the slice categories

$$d/f = p^{-1}(d), \qquad f/d = q^{-1}(d).$$

Explicitly, f/d is the category with

(i) Objects: $\langle c \in \mathcal{C}, s\colon f(c) \to d\rangle$,
(ii) Morphisms: $\langle c, s\rangle \to \langle c', s'\rangle$ consists of $t\colon c \to c'$ such that $s' \circ f(t) = s$,

and d/f is described similarly.

1.2. **Adjunctions.** Given functors

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g,$$

an adjunction $f \dashv g$ is a natural isomorphism

$$\mathcal{D}(f(-), =) \cong \mathcal{C}(-, g(=))$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}\mathrm{et}$. We say that f is left adjoint to g, and g is right adjoint to f.

1.2.1. **Proposition.** Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:

(i) There is a functor $g\colon \mathcal{D} \to \mathcal{C}$ together with an adjunction $f \dashv g$,
(ii) For all $d \in \mathcal{D}$, the functor $\mathcal{D}(f(-), d)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is representable,
(iii) For all $d \in \mathcal{D}$, the slice category f/d has a terminal object.

□

Date: January 22, 2019.

1.3. **Kan extensions.** Given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and a functor $f\colon \mathcal{C} \to \mathcal{D}$, we obtain a restriction functor

$$f^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E}).$$

The adjoints to this, if they exist, are the functors $f_! \dashv f^* \dashv f_*$. Here, $f_!$ is the functor of left Kan extension and f_* is the functor of right Kan extension.

1.3.1. **Proposition.** Suppose $\mathcal{C}$ is small and $\mathcal{D}$ is locally small. Then for any functor $f\colon \mathcal{C} \to \mathcal{D}$, there are functors of left and right Kan extension

$$f_!\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{S}\mathrm{et}), \qquad f_*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{S}\mathrm{et}).$$

□

Given $f\colon \mathcal{C} \to \mathcal{D}$, we write

$$f_\circ\colon \mathrm{Fun}(\mathcal{E}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$$

for postcomposition by f.

1.4. **The Grothendieck construction and colimits of sets.** Given a functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, we want a formula for computing $\operatorname{colim} F$.

1.4.1. **Definition.** For a category $\mathcal{C}$, the Grothendieck construction is either of

$$\int\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathcal{C}\mathrm{at}/\mathcal{C}, \qquad \int^{\mathrm{op}}\colon \mathrm{Psh}(\mathcal{C}) \to \mathcal{C}\mathrm{at}/\mathcal{C}$$

defined as follows. Fix $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$. Then $\int F$ is the category with

(i) Objects: pairs $\langle c \in \mathcal{C}, x \in F(c)\rangle$,
(ii) Morphisms: $\langle c, x\rangle \to \langle c', x'\rangle$ consists of $t\colon c \to c'$ such that $F(t)(x) = x'$.

Fix $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Then $\int^{\mathrm{op}} F$ is the category with

(i) Objects: pairs $\langle c \in \mathcal{C}, x \in F(c)\rangle$,
(ii) Morphisms: $\langle c, x\rangle \to \langle c', x'\rangle$ consists of $t\colon c \to c'$ such that $F(t)(x') = x$.

◁

Observe that for $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, we can identify $\int^{\mathrm{op}} F = (\int F)^{\mathrm{op}}$. We will also need the following.

1.4.2. **Proposition.** The inclusion $\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ has a left adjoint $\pi_0\colon \mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et}$.

Construction. For a category $\mathcal{E}$, the set $\pi_0\mathcal{E}$ is defined as $\mathrm{Ob}(\mathcal{E})/\sim$, where $\sim$ is the equivalence relation generated by $e \sim e'$ whenever there exists a map $e \to e'$. □

Observe that an adjoint $f\colon \mathcal{E} \to \mathcal{D}$ induces an isomorphism $\pi_0\mathcal{E} \cong \pi_0\mathcal{D}$. The following theorem is essentially the motivation for everything that follows; the careful reader will observe that a proof is present in our proof of Theorem 2.4.2.

1.4.3. **Theorem.** For $F\colon \mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, we can identify

$$\pi_0 \int F = \operatorname{colim} F = \pi_0 \int^{\mathrm{op}} F.$$

□

1.4.4. **Remark.** There is a similar formula for limits:

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}/\mathcal{C}}(\mathcal{C}, \int F) = \lim F = \mathrm{Hom}_{\mathcal{C}\mathrm{at}/\mathcal{C}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \int^{\mathrm{op}} F).$$

◁

2. Discrete (op)fibrations

2.1. **Adjoint of the Grothendieck construction.** Recall that we have defined functors

$$\int : \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathcal{C}\mathrm{at}/\mathcal{C}, \qquad \int^{\mathrm{op}} : \mathrm{Psh}(\mathcal{C}) \to \mathcal{C}\mathrm{at}/\mathcal{C}.$$

We can go the other way as well.

2.1.1. **Definition.** The functors

$$U\colon \mathcal{C}\mathrm{at}/\mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}), \qquad U^{\mathrm{op}}\colon \mathcal{C}\mathrm{at}/\mathcal{C} \to \mathrm{Psh}(\mathcal{C})$$

are defined by

$$U(p)(c) = \pi_0(p/c), \qquad U^{\mathrm{op}}(p)(c) = \pi_0(c/p).$$

◁

Observe that these are dual in the sense that $U(p) = U^{\mathrm{op}}(p^{\mathrm{op}})$.

2.1.2. **Theorem.** There are adjunctions

$$U \dashv \int, \qquad U^{\mathrm{op}} \dashv \int^{\mathrm{op}}$$

for which the counits are natural isomorphisms. In particular, $\int$ and $\int^{\mathrm{op}}$ are fully faithful.

Proof. We will content ourselves to focus on $U \dashv \int$. Fix $p\colon \mathcal{E} \to \mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$; we would like to exhibit a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}/\mathcal{C}}(\mathcal{E}, \int F) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S}\mathrm{et})}(U(p), F).$$

Now, a functor $\alpha\colon \mathcal{E} \to \int F$ over $\mathcal{C}$ is equivalent to a collection of assignments $\alpha_c\colon p^{-1}(c) \to F(c)$ satisfying the following condition: for all $t\colon e \to e'$ in $\mathcal{E}$ with $p(t) = s\colon c \to c'$, we have $F(s)(\alpha_c(e)) = \alpha_{c'}(e')$. As a consequence, we obtain functors

$$p/c \to F(c), \qquad \langle e, s\rangle \mapsto \alpha_{p(e)}(e)$$

which are natural in c, which patch together to give a natural transformation $\alpha\colon U(p) \to F$.

Conversely, suppose given a natural transformation $\widetilde{\alpha}U(p) \to F$. This is equivalent to a collection of functors $\widetilde{\alpha}_c\colon p/c \to F(c)$ natural in c, which give rise to a functor $\alpha\colon \mathcal{E} \to \int F$ by $\alpha(e) = \widetilde{\alpha}_{p(e)}(e, 1_{p(e)})$. These constructions are mutually inverse, so that $U \dashv \int$ as claimed.

It remains to show that for $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, the counit $U(\int F) \to F$ is an isomorphism. Write $q\colon \int F \to \mathcal{C}$, so that $U(\int F)(c) = \pi_0(q/c)$. Observe that q/c is the category of tuples $\langle c' \in \mathcal{C}, x \in F(c'), t\colon c' \to c\rangle$, and the counit $U(\int F)(c) \to F(c)$ arises from the assignments

$$\langle c', x, t\rangle \mapsto F(t)(x).$$

Now, for $z \in F(c)$, let $(q/c)_z \subset q/c$ be the full subcategory on those pairs $\langle c', x, t\rangle$ which are sent to z under this map. Then $q/c = \coprod_{z\in F(c)} (q/c)_z$ and $(q/c)_z$ has terminal object $\langle c, z, 1_c\rangle$, so that $\pi_0((q/c)_z) = \{z\}$ and thus the counit is an isomorphism as claimed. □

2.2. **Discrete (op)fibrations.** As $\int$ and $\int^{\mathrm{op}}$ are fully faithful, it is a good idea to identify their essential image. Let us focus on $\int$. For $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, write $p\colon \int F \to \mathcal{C}$. Observe that for $c \in \mathcal{C}$, the fiber $p^{-1}(c) = F(c)$ is a set which depends functorially on $\mathcal{C}$. Making explicit this functoriality, we are led to the following definition.

2.2.1. **Definition.** A functor $p\colon \mathcal{E} \to \mathcal{C}$ is a discrete opfibration if for every $t\colon c \to c'$ in $\mathcal{C}$ and $e \in p^{-1}(e)$, there is a unique map $s\colon e \to e'$ such that $p(s) = t$. ◁

We picture this situation as follows:

$$\begin{array}{ccc} e & \overset{\exists! s}{\dashrightarrow} & e' \\ \downarrow & & \downarrow \\ c & \xrightarrow{t} & c' \end{array} .$$

Let $\mathcal{O}\mathrm{p}\mathcal{F}\mathrm{ib}^{\flat}(\mathcal{C}) \subset \mathcal{C}\mathrm{at}/\mathcal{C}$ be the full subcategory spanned by the discrete opfibrations. We obtain a functor

$$V\colon \mathcal{O}\mathrm{p}\mathcal{F}\mathrm{ib}^{\flat}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}), \qquad V(p)(c) = p^{-1}(c),$$

where given $t\colon c \to c'$ in $\mathcal{C}$ and $e \in p^{-1}(c)$, we set $V(p)(t)(e) = e'$, where $s\colon e \to e'$ is the unique map lifting t with source e.

2.2.2. **Theorem.** The functor $\int\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathcal{C}\mathrm{at}/\mathcal{C}$ induces an equivalence $\int\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \simeq \mathcal{O}\mathrm{p}\mathcal{F}\mathrm{ib}(\mathcal{C})$, with essential inverse V.

Proof. For any functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, we can see that $\int F$ is a discrete opfibration and $V(\int F) = F$. Conversely, if $p\colon \mathcal{E} \to \mathcal{C}$ is a discrete opfibration, then it is not difficult to check that the assignment

$$\int V(p) \to \mathcal{E}, \qquad \langle c \in \mathcal{C}, x \in V(p)(c) = p^{-1}(c)\rangle \mapsto x$$

gives a natural equivalence $\int V(p) \simeq p$. □

All of this has an analogue for presheaves.

2.2.3. **Definition.** A functor $p\colon \mathcal{E} \to \mathcal{C}$ is a discrete fibration if for every $t\colon c \to c'$ in $\mathcal{C}$ and $e' \in p^{-1}(c')$, there is a unique map $s\colon e \to e'$ such that $p(s) = t$. ◁

Observe that p is a discrete fibration if and only if p^{op} is a discrete opfibration. Setting $\mathcal{F}\mathrm{ib}^{\flat}(\mathcal{C}) \subset \mathcal{C}\mathrm{at}/\mathcal{C}$ to be the subcategory of discrete fibrations, we obtain as above an equivalence

$$\int^{\mathrm{op}} : \mathrm{Psh}(\mathcal{C}) \simeq \mathcal{F}\mathrm{ib}^{\flat}(\mathcal{C}) : V^{\mathrm{op}}.$$

2.3. **Pullbacks of discrete fibrations.** Let $\mathcal{S}\mathrm{et}_*$ denote the category of pointed sets. Let $\pi\colon \mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}$ denote the projection given by $\pi(\langle S, s \in S\rangle) = S$.

2.3.1. **Lemma.** The projection π is a discrete opfibration, and in fact $\pi = \int I_{\mathcal{S}\mathrm{et}}$. □

As one might guess from the identification $\pi = \int I_{\mathcal{S}\mathrm{et}}$, this discrete opfibration satisfies a universality property.

2.3.2. **Theorem.** For any discrete opfibration $p\colon \mathcal{E} \to \mathcal{C}$, there is an essentially unique functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ for which we have a pullback square

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{S}\mathrm{et}_* \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{S}\mathrm{et} \end{array} .$$

Here, we can identify $F = U(p)$.

Proof. For $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, the construction of $\int F$ shows that we have a pullback square

$$\begin{array}{ccc} \int F & \longrightarrow & \mathcal{S}\mathrm{et}_* \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{S}\mathrm{et} \end{array} .$$

The result follows from Theorem 2.2.2. □

For any functor $f\colon \mathcal{C} \to \mathcal{D}$, we have an adjunction

$$f_! : \mathcal{C}\mathrm{at}/\mathcal{C} \rightleftarrows \mathcal{C}\mathrm{at}/\mathcal{D} : f^*,$$

where f^* is given by pulling back along f and $f_!$ is given by postcomposition by f.

2.3.3. **Corollary.** Suppose given a discrete opfibration $p\colon \mathcal{E} \to \mathcal{D}$ and arbitrary functor $f\colon \mathcal{C} \to \mathcal{D}$. Then $f^*p\colon f^*\mathcal{E} \to \mathcal{D}$ is a discrete opfibration, and moreover $U(f^*p) = f^*U(p)$.

Proof. Consider the pullback diagrams

$$\begin{array}{ccccc} f^*\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{S}\mathrm{et}_* \\ \downarrow{\scriptstyle f^*p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} & \xrightarrow{U(p)} & \mathcal{S}\mathrm{et} \end{array} .$$

□

The analogous corollary holds for discrete fibrations and presheaves.

2.4. Computing left Kan extensions.

The following proposition can be viewed as saying that postcomposition is always well-behaved with respect to viewing categories over $\mathcal{C}$ as modeling functors $\mathcal{C} \to \mathcal{S}\mathrm{et}$.

2.4.1. **Proposition.** For any $f\colon \mathcal{C} \to \mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathcal{C}\mathrm{at}/\mathcal{C} & \xrightarrow{f_!} & \mathcal{C}\mathrm{at}/\mathcal{D} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) & \xrightarrow{f_!} & \mathrm{Fun}(\mathcal{D}, \mathcal{S}\mathrm{et}) \end{array}$$

commutes.

Proof. Each functor in this diagram is a left adjoint, and it is equivalent to show that the corresponding diagram

$$\begin{array}{ccc} \mathcal{C}\mathrm{at}/\mathcal{C} & \xleftarrow{f^*} & \mathcal{C}\mathrm{at}/\mathcal{D} \\ \uparrow{\scriptstyle \int} & & \uparrow{\scriptstyle \int} \\ \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) & \xleftarrow[f^*]{} & \mathrm{Fun}(\mathcal{D}, \mathcal{S}\mathrm{et}) \end{array}$$

of right adjoints commutes. This follows immediately from Theorem 2.2.2 and Corollary 2.3.3. □

This turns out to be essentially equivalent to the Kan formula for computing left Kan extensions as a colimit.

2.4.2. **Theorem.** Suppose given

$$F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}, \qquad f\colon \mathcal{C} \to \mathcal{D}.$$

Then for all $d \in \mathcal{D}$, we can identify

$$(f_! F)(d) = \operatorname*{colim}_{f(c)\to d} F(c).$$

Proof. Define

$$F/d\colon f/d \to \mathcal{S}\mathrm{et}, \qquad \langle c \in \mathcal{C}, s\colon f(c) \to d\rangle \mapsto F(c).$$

Then by Theorem 1.4.3, we can identify

$$\operatorname*{colim}_{f(c)\to d} F(c) = \pi_0 \int F/d.$$

On the other hand, write $p\colon \int F \to \mathcal{C}$. Then by Proposition 2.4.1, we can identify

$$(f_! F)(d) = \pi_0(fp/d).$$

In fact, $\int F/d$ and fp/d are isomorphic: the former is the category of tuples $\langle c \in \mathcal{C}, s\colon f(c) \to d, x \in F(c)\rangle$ and the latter is the category of tuples $\langle c \in \mathcal{C}, x \in F(c), s\colon f(c) \to d\rangle$. □

3. Grothendieck (op)fibrations

3.1. **Motivation.** Grothendieck (op)fibrations arise when thinking about functors into $\mathcal{C}\mathrm{at}$. Given

$$F\colon \mathcal{C} \to \mathcal{C}\mathrm{at},$$

we can define a category $\int F$ with

(i) Objects: $\langle c \in \mathcal{C}, x \in F(c)\rangle$,
(ii) Morphisms: $\langle c, x\rangle \to \langle c', x'\rangle$ consists of $s\colon c \to c'$ and $t\colon F(s)(x) \to x'$.

More generally, one can play this game for pseudofunctors $F\colon \mathcal{C} \to \mathcal{C}\mathrm{at}$, i.e. functors which are only functorial up to "coherent isomorphism". This construction gives a (non-full) embedding from $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at})$ into $\mathcal{C}\mathrm{at}/\mathcal{C}$, and the objects appearing in the essential image are the Grothendieck opfibrations. Playing the same game with $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$, one arrives at Grothendieck fibrations.

For our purposes, this identification of Grothendieck (op)fibrations is not important. Rather, Grothendieck (op)fibrations for us will be a primarily technical tool for reasoning about discrete (op)fibrations.

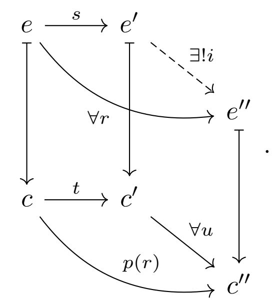
3.2. Definitions.

3.2.1. **Definition.** Fix $p\colon \mathcal{E} \to \mathcal{C}$ and $s\colon e \to e'$ in $\mathcal{E}$, and write $p(s) = t\colon c \to c'$. The map s is said to be p-coCartesian, or a p-coCartesian lift of t, if the canonical map

$$s/\mathcal{E} \to e/\mathcal{E} \times_{c/\mathcal{C}} t/\mathcal{C}$$

is an isomorphism. ◁

Tracing through this definition, keeping notation as above, we see that s is p-coCartesian precisely when the following holds: for every $r\colon e \to e''$ and $u\colon c' \to c''$ such that $p(r) = us$, there exists a unique map $i\colon e' \to e''$ such that $p(i) = u$ and $r = is$. We can picture this situation via the following diagram:



Applying this definition to $u = 1_{c'}$, we see that p-coCartesian lifts of t with source e are essentially unique.

3.2.2. **Definition.** A functor $p\colon \mathcal{E} \to \mathcal{C}$ is a Grothendieck opfibration if for every $t\colon c \to c'$ in $\mathcal{C}$ and $e \in p^{-1}(c)$, there exists a p-coCartesian lift of t with source e. ◁

3.2.3. **Example.** For any category $\mathcal{C}$, the projection $\mathrm{ev}_1\colon \mathcal{C}^{[1]} \to \mathcal{C}$ is a Grothendieck fibration. ◁

3.2.4. **Proposition.** If $p\colon \mathcal{E} \to \mathcal{C}$ is a Grothendieck fibration, then the assignment

$$V(p)\colon \mathcal{C} \to \mathcal{S}\mathrm{et} \qquad V(p)(c) = \pi_0(p^{-1}(c))$$

is canonically functorial.

Proof. Fix $t\colon c \to c'$ and $e \in p^{-1}(c)$. Choose a p-coCartesian lift $s'\colon e \to e'$ of t, and let $V(p)(t)(e)$ be the image of e' in $\pi_0(p^{-1}(c'))$. We must show that this assignment is independent of choice of s and descends to a map $\pi_0(p^{-1}(c)) \to \pi_0(p^{-1}(c'))$.

If $s''\colon e \to e''$ is another p-coCartesian lift, then by the definition of p-coCartesian with $u = 1_{c'}$, we obtain a map $i\colon e' \to e''$ such that $p(i) = 1_{c'}$, and thus e' and e'' agree in $\pi_0(p^{-1}(c'))$.

Say $m\colon e \to \overline{e}$ is a map in $p^{-1}(c)$, and let $\overline{s}'\colon \overline{e} \to \overline{e}'$ be a p-coCartesian lift of t. As s' is p-coCartesian, we obtain a map $m'\colon e' \to \overline{e}'$ lifting $1_{c'}$ such that $m's' = \overline{s}'m$, and thus e' and $\overline{e}'$ agree in $\pi_0(p^{-1}(c'))$. We can play this argument backwards to consider makes $n\colon \overline{e} \to e$, so that $V(p)(t)$ descends to a map $\pi_0(p^{-1}(c)) \to \pi_0(p^{-1}(c'))$. □

Everything in this section can be dualized. The lazy definition is that $p\colon \mathcal{E} \to \mathcal{C}$ is a Grothendieck fibration when $p^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a Grothendieck opfibration, and if p is a Grothendieck fibration then we obtain a functor $V^{\mathrm{op}}(p)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ by $V^{\mathrm{op}}(p)(c) = \pi_0(p^{-1}(c))$.

3.3. **Pullbacks of Grothendieck (op)fibrations.** Our definition of Grothendieck opfibrations via slice categories quickly gives the following.

3.3.1. **Lemma.** Suppose given a pullback

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{q'} & \mathcal{E} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathcal{C}' & \longrightarrow & \mathcal{C} \end{array}$$

of categories and a map t in $\mathcal{E}'$. If $q'(t)$ is p-coCartesian, then t is p'-coCartesian. $\square$

3.3.2. **Proposition.** Suppose given a pullback

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{q'} & \mathcal{E} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathcal{C}' & \xrightarrow{q} & \mathcal{C} \end{array}$$

of categories in which p is a Grothendieck opfibration. Then p' is a Grothendieck opfibration, and $V(p') = q^*V(p)$.

Proof. The fact that p' is a Grothendieck opfibration follows immediately from Lemma 3.3.1. For the second statement, observe that for $c' \in \mathcal{C}'$, we have $p'^{-1}(c') = p^{-1}(q(c'))$. Thus

$$V(p')(c') = \pi_0 q'^{-1}(c') = \pi_0 p^{-1}(q(c')) = V(p)(q(c')) = (q^*V(p))(c').$$

$\square$

4. Base change

4.1. **Base change of adjunctions along Grothendieck fibrations.** We will need the following easy lemma.

4.1.1. **Lemma.** Suppose $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ have terminal objects c, d, and e, and $\mathcal{C} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{E}$ is a span of categories such that $f(c) = d = g(e)$. Then $\langle c, e\rangle$ is terminal in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$. $\square$

The following theorem is one of the main technical theorems about Grothendieck (op)fibrations that we will use.

4.1.2. **Theorem.** Fix maps $p\colon \mathcal{E} \to \mathcal{D}$ and $q\colon \mathcal{K} \to \mathcal{D}$ and a functor $\alpha\colon \mathcal{E} \to \mathcal{K}$ which has a right adjoint β. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a Grothendieck fibration. Then $f^*\alpha\colon f^*\mathcal{E} \to f^*\mathcal{K}$ has a right adjoint.

Proof. By Proposition 1.2.1, our assumption is that for all $k' \in \mathcal{K}$, the slice category α/k' has some terminal object $\langle e \in \mathcal{E}, \epsilon\colon \alpha(e) \to k'\rangle$. We want to show that for all $\langle c', k'\rangle \in \mathcal{C} \times_{\mathcal{D}} \mathcal{K}$, the slice category $(\mathcal{C} \times_{\mathcal{D}} \alpha)/\langle c', k'\rangle$ has a terminal object. Write $f(c') = d' = q(k')$ and observe

$$(\mathcal{C} \times_{\mathcal{D}} \alpha)/\langle c', k'\rangle = (\mathcal{C}/c') \times_{(\mathcal{D}/d')} (\alpha/k').$$

As $\langle e, \epsilon\rangle \in \alpha/k'$ is terminal, $\alpha/k' \to \mathcal{D}/d'$ factors through $\mathcal{D}/q(\epsilon)$ by the diagram

$$\begin{array}{ccc} (\alpha/k')/\langle e, \epsilon\rangle & \longrightarrow & \mathcal{D}/q(\epsilon) \\ \downarrow{\scriptstyle =} & & \downarrow \\ \alpha/k' & \longrightarrow & \mathcal{D}/d' \end{array}.$$

We can then identify

$$(\mathcal{C}/c')\times_{(\mathcal{D}/d')}(\alpha/k') = (\mathcal{C}/c')\times_{(\mathcal{D}/d')}(\mathcal{D}/q(\epsilon))\times_{(\mathcal{D}/q(\epsilon))}(\alpha/k') = (\mathcal{C}/s)\times_{(\mathcal{D}/q(\epsilon))}(\alpha/k').$$

where $s\colon c \to c'$ is an f-Cartesian lift of $q(\epsilon)$. We conclude with an application of Lemma 4.1.1. $\square$

We will also use the dual statement, that right adjoints are stable under base change by Grothendieck opfibrations. An examination of the construction of the right adjoint in the previous theorem gives the following slight upgrade.

4.1.3. **Scholium.** Fix notation as in the previous theorem. If $g\colon \mathcal{C} \to \mathcal{A}$ is a functor and for every $s\colon d \to d'$ in $\mathcal{D}$ and $c' \in f^{-1}(d)$ we can find an f-Cartesian lift of s with target c' which is sent to an identity map under g, then $f^*\alpha$ has a right adjoint defined in $\mathcal{C}\mathrm{at}/\mathcal{A}$. □

We will draw a number corollaries from these. Let us start with the following.

4.1.4. **Corollary.** Suppose $p\colon \mathcal{E} \to \mathcal{C}$ is a left adjoint. Then for any $f\colon \mathcal{C} \to \mathcal{D}$ and $d \in \mathcal{D}$, the map $fp/d \to f/d$ induced by p is a left adjoint. In particular, $U(fp) = U(f)$.

Proof. We can identify the map $fp/\mathcal{D} \to f/\mathcal{D}$ as

$$1 \times p\colon \mathcal{D}^{[1]}\,{}_{\mathrm{ev}_0}\times_{\mathcal{D},fp} \mathcal{E} \to \mathcal{D}^{[1]}\,{}_{\mathrm{ev}_0}\times_{\mathcal{D},f} \mathcal{C}$$

which remains a left adjoint as ev_0 is a Grothendieck fibration. The projection $\mathrm{ev}_1\colon \mathcal{D}^{[1]} \to \mathcal{D}$ satisfies the hypotheses of g in Scholium 4.1.3, so that for $d \in \mathcal{D}$ the map

$$1 \times 1 \times p\colon \{d\} \times_{\mathcal{D},\mathrm{ev}_1} \mathcal{D}^{[1]}\,{}_{\mathrm{ev}_0}\times_{\mathcal{D},fp} \mathcal{E} \to \{d\} \times_{\mathcal{D},\mathrm{ev}_1} \mathcal{D}^{[1]}\,{}_{\mathrm{ev}_0}\times_{f} \mathcal{C}$$

is also a left adjoint. But this is exactly $fp/d \to f/d$. □

4.2. **Base change along arbitrary functors.** Grothendieck opfibrations, even when not discrete, are useful for thinking of $\mathcal{C}\mathrm{at}/\mathcal{C}$ as modeling functors $\mathcal{C} \to \mathcal{S}\mathrm{et}$.

4.2.1. **Lemma.** If $p\colon \mathcal{E} \to \mathcal{C}$ is a Grothendieck opfibration, then we can identify

$$U(p)(c) = V(p)(c)$$

Proof. There is a canonical inclusion $p^{-1}(c) \to p/c$, which we must show is an isomorphism on π_0. This functor is in fact a right adjoint by Theorem 4.1.2, being the base change along the Grothendieck opfibration p of the right adjoint $\{c\} \to \mathcal{C}/c$ defined over $\mathcal{C}$. □

Geometrically, we can view a Grothendieck opfibration $p\colon \mathcal{E} \to \mathcal{C}$ as a fattened version of $\int U(p)$. One benefit of this is that it is easy to approximate arbitrary maps by Grothendieck opfibrations.

4.2.2. **Proposition.** Let $p\colon \mathcal{E} \to \mathcal{C}$ be an arbitrary map, and write $q\colon p/\mathcal{C} \to \mathcal{C}$ for the canonical projection. Then q is a Grothendieck opfibration, and the inclusion $j\colon \mathcal{E} \to p/\mathcal{C}$ over $\mathcal{C}$ is a left adjoint.

Proof. Fix $t\colon c \to c'$ and $\langle e \in \mathcal{E}, s\colon p(e) \to c\rangle$ in $p/\mathcal{C}$. One can check that a q-coCartesian lift of t is given by $\langle e, ts\rangle$.

To see that j is a left adjoint, observe that for $\langle e \in \mathcal{E}, c \in \mathcal{C}, s\colon p(e) \to c\rangle$ in $p/\mathcal{C}$, the slice category $j/\langle e, c, s\rangle$ has terminal object $\langle e, 1, s\rangle$. □

As a consequence, we learn the following.

4.2.3. **Proposition.** Fix an arbitrary functor $f\colon \mathcal{C} \to \mathcal{D}$. Define $\mathbb{R}f^*\colon \mathcal{C}\mathrm{at}/\mathcal{D} \to \mathcal{C}\mathrm{at}/\mathcal{C}$ as follows: for $p\colon \mathcal{E} \to \mathcal{D}$, write $q\colon p/\mathcal{D} \to \mathcal{D}$, and set $\mathbb{R}f^*(p) = f^*q$. Then $U(\mathbb{R}f^*(p)) = f^*U(p)$ for any p.

Proof. We have

$$f^*U(p) \overset{(1)}{=} f^*U(q) \overset{(2)}{=} f^*V(q) \overset{(3)}{=} V(f^*q) \overset{(4)}{=} U(f^*q) = U(\mathbb{R}f^*(p)),$$

where (1) is by Proposition 4.2.2 and Corollary 4.1.4, (2) is by Lemma 4.2.1, (3) is by Proposition 3.3.2, and (4) combines Proposition 3.3.2 and Lemma 4.2.1. □

4.3. **Base change along Grothendieck fibrations.** We have seen that if $f\colon \mathcal{C} \to \mathcal{D}$ is arbitrary, then pulling back Grothendieck opfibrations along f is well-behaved with respect to the interpretation of maps over $\mathcal{C}$ and $\mathcal{D}$ as covariant functors into $\mathcal{S}\mathrm{et}$. Conversely, pulling back arbitrary functors along Grothendieck fibrations is well-behaved from this perspective.

4.3.1. **Proposition.** If $f\colon \mathcal{C} \to \mathcal{D}$ is a Grothendieck fibration, then

$$\begin{array}{ccc} \mathcal{C}\mathrm{at}/\mathcal{C} & \xleftarrow[f^*]{} & \mathcal{C}\mathrm{at}/\mathcal{D} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathrm{Fun}(\mathcal{C},\mathcal{S}\mathrm{et}) & \xleftarrow[f^*]{} & \mathrm{Fun}(\mathcal{D},\mathcal{S}\mathrm{et}) \end{array}$$

commutes.

Proof. By Proposition 4.2.3, we have $f^*U(p) = U(\mathbb{R}f^*p)$, where $\mathbb{R}f^*p = f^*q$, where $q\colon p/\mathcal{D} \to \mathcal{D}$. The natural map $f^*p \to f^*q$ is obtained by pulling back along f the map $\mathcal{E} \to p/\mathcal{C}$, which is a left adjoint over $\mathcal{C}$ by Proposition 4.2.2. The result follows from Theorem 4.1.2 and Corollary 4.1.4. □

4.4. **The base change formula.** We can use Proposition 4.3.1 to give a base change formula which both generalizes and provides another proof of Theorem 2.4.2.

4.4.1. **Definition.** Suppose given a commutative diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{g'} & \mathcal{C} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} \end{array}$$

of categories. The associated natural transformation

$$f'_! \circ g'^* \to g^* \circ f_!$$

of functors $\mathrm{Fun}(\mathcal{C},\mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{D}',\mathcal{S}\mathrm{et})$ is defined as the composite

$$f'_! \circ g'^* \xrightarrow{f'_! \circ g'^* \circ \eta} f'_! \circ g'^* \circ f^* \circ f_! \xrightarrow{=} f'_! \circ f'^* \circ g^* \circ p_! \xrightarrow{\epsilon \circ q^* \circ p_!} g^* \circ f_! \ .$$

◁

4.4.2. **Theorem.** Fix a pullback square

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{g'} & \mathcal{C} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} \end{array}$$

in which g is a Grothendieck fibration. Then the associated natural transformation $f'_! \circ g'^* \to g^* \circ f_!$ is a natural isomorphism.

Proof. Model $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ as a discrete opfibration $\mathcal{E} \to \mathcal{C}$. Combining Corollary 2.3.3 and Proposition 2.4.1, we see that $f'_! g'^* F$ is modeled by $f'_! g'^* \mathcal{E}$. Combining Proposition 2.4.1 and Proposition 4.3.1, we see that $g^* f_! F$ is modeled by $g^* f_! \mathcal{E}$. The map $f'_! g'^* F \to g^* f_! F$ is now modeled as the canonical map $f'_! g'^* \mathcal{E} \to g^* f_! \mathcal{E}$, which is an isomorphism just by the pasting law for pullbacks. □

We can use this to give a second proof of Theorem 2.4.2.

Proof of Theorem 2.4.2. Fix $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ and $f\colon \mathcal{C} \to \mathcal{D}$; we wish to show

$$(f_! F)(d) = \operatorname*{colim}_{f(c) \to d} F(c).$$

Consider the following pullback squares:

$$\begin{array}{ccccc} f^{-1}(d) & \longrightarrow & f/d & \xrightarrow{q} & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle f_d} & & \downarrow{\scriptstyle f} \\ \{d\} & \xrightarrow{i} & \mathcal{D}/d & \xrightarrow{p} & \mathcal{D} \end{array} .$$

Write π for the projection from a category to the terminal category. We obtain isomorphisms

$$\begin{aligned} (f_! F)(d) &= i^* f_! F \\ &= i^* p^* f_! F \\ &\overset{(1)}{\cong} i^* f_{d!} q^* F \\ &\overset{(2)}{=} \pi_! f_{d!} q^* F \\ &= \pi_! q^* F \\ &= \operatorname{colim} q^* F = \operatorname*{colim}_{f(c) \to d} F(c). \end{aligned}$$

Here, (1) is from Theorem 4.4.2, as p is a Grothendieck fibration, and (2) is from the fact that $\pi \dashv i$ so that $i^* = \pi_!$. □

5. Presheaves

5.1. **The Yoneda lemma.** For a locally small category $\mathcal{C}$, we have a functor

$$h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C}), \qquad h_{\mathcal{C}}(c)(-) = \mathcal{C}(-, c).$$

5.1.1. **Theorem.** For all $c \in \mathcal{C}$ and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, the assignment

$$\mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(h_{\mathcal{C}}(c), F) \to F(c), \qquad \alpha \mapsto \alpha_c(1_c)$$

is an isomorphism. When $F = h_{\mathcal{C}}(c')$, this is inverse to the canonical map

$$\mathcal{C}(c, c') \to \mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(h_{\mathcal{C}}(c), h_{\mathcal{C}}(c')),$$

and thus $h_{\mathcal{C}}$ is fully faithful.

Proof. Fix $c \in \mathcal{C}$ and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. We can identify the indicated map as

$$\mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(h_{\mathcal{C}}(c), F) \cong \mathrm{Hom}_{\mathcal{C}\mathrm{at}/\mathcal{C}}(\mathcal{C}/c, \int^{\mathrm{op}} F) \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}/\mathcal{C}}(\{c\}, \int^{\mathrm{op}} F),$$

and the first statement then follows from Theorem 2.2.2 as the inclusion $\{c\} \to \mathcal{C}/c$ can be identified as the unit $\{c\} \to \int^{\mathrm{op}} U^{\mathrm{op}}(\{c\})$. The second statement is easily verified. □

5.2. **Left Kan extension of representables.** We have already made use of the following.

5.2.1. **Lemma.** We can identify $\int^{\mathrm{op}} h_{\mathcal{C}}(c)$ as the canonical projection $\mathcal{C}/c \to \mathcal{C}$. $\square$

This admits the following upgrade.

5.2.2. **Proposition.** Suppose given $p\colon \mathcal{E} \to \mathcal{C}$ where $\mathcal{E}$ has terminal object e. Then $U(p) = h_{\mathcal{C}}(p(e))$.

Proof. Applying U^{op} to the zigzag $\mathcal{C}/p(e) \leftarrow \{p(e)\} = \{e\} \to \mathcal{E}$ of categories over $\mathcal{C}$ gives an isomorphism $h_{\mathcal{C}}(p(e)) \simeq U(p)$ by Corollary 4.1.4 and Lemma 5.2.1. $\square$

This has the following consequence.

5.2.3. **Proposition.** For any functor $f\colon \mathcal{C} \to \mathcal{D}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \big\downarrow{\scriptstyle h_{\mathcal{C}}} & & \big\downarrow{\scriptstyle h_{\mathcal{D}}} \\ \mathrm{Psh}(\mathcal{C}) & \xrightarrow{f_!} & \mathrm{Psh}(\mathcal{D}) \end{array} .$$

Proof. For $c \in \mathcal{C}$, we have $f_! h_{\mathcal{C}}(c) = f_! U(\mathcal{C}/c) = U(f_! \mathcal{C}/c)$, and the result follows as $\mathcal{C}/c$ has terminal object mapping to $f(c) \in \mathcal{D}$. $\square$

5.3. **Colimit of the Yoneda embedding.**

5.3.1. **Lemma.** For any category $\mathcal{D}$, the category $\mathrm{Psh}(\mathcal{D})$ admits small colimits, where for $f\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{D})$ we identify

$$(\operatorname*{colim}_{c\in\mathcal{C}} f(c))(d) = \operatorname*{colim}_{c\in\mathcal{C}}(f(c)(d)).$$

Proof. After enlarging the universe, we may suppose that $\mathcal{D}$ is small; the indicated formula then shows that this colimit is independent of this enlargement. Write π for the projection to the terminal category. Choose $d \in \mathcal{D}$ and write $j\colon \{d\} \to \mathcal{D}$. Form the diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}, \mathrm{Psh}(\mathcal{D})) & \overset{\pi_!}{\dashrightarrow} & \mathrm{Fun}([0], \mathrm{Psh}(\mathcal{D})) \\ \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} \\ \mathrm{Fun}(\mathcal{D}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S}\mathrm{et}) & \xrightarrow{(1\times\pi)_!} & \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \\ \big\downarrow{\scriptstyle (j\times 1)^*} & & \big\downarrow{\scriptstyle j^*} \\ \mathrm{Fun}(\{d\} \times \mathcal{D}, \mathcal{S}\mathrm{et}) & \xrightarrow{\pi_!} & \mathrm{Fun}(\{d\}, \mathcal{S}\mathrm{et}) \end{array} .$$

The top dashed map is indeed left adjoint to π^*, as the associated diagram of right adjoints commutes. The bottom square commutes up to isomorphism by Theorem 4.4.2, as $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ is a Grothendieck fibration.

Now, for $f \in \mathrm{Fun}(\mathcal{C}, \mathrm{Psh}(\mathcal{D}))$, the image of f around the counterclockwise composite is $\mathrm{colim}_{c\in\mathcal{C}}(f(c)(d))$, and the image of f around the clockwise composite is $(\mathrm{colim}_{c\in\mathcal{C}} f(c))(d)$; the result follows. $\square$

Let $* \in \mathrm{Psh}(\mathcal{C})$ denote the terminal presheaf.

5.3.2. **Proposition.** The colimit of $h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is $*$.

Proof. We have

$$(\operatorname{colim} h_{\mathcal{C}})(c) = \operatorname{colim}(h_{\mathcal{C}}(c)) = \pi_0 \int^{\mathrm{op}} h_{\mathcal{C}}(c) = \pi_0 \mathcal{C}/c = *.$$

□

5.4. **Presheaves as colimits of representables.** We can bootstrap the results of the previous sections to the following fundamental result.

5.4.1. **Theorem.** Let $\mathcal{C}$ be a small category and $p\colon \mathcal{E} \to \mathcal{C}$ a functor. Then

$$U(p) = \operatorname{colim} h_{\mathcal{C}} p.$$

Proof. We identify

$$\begin{aligned} \operatorname{colim} h_{\mathcal{C}} p &\overset{(1)}{=} \operatorname{colim} p_! h_{\mathcal{E}} \\ &= p_! \operatorname{colim} h_{\mathcal{E}} \\ &\overset{(2)}{=} p_! * \\ &= p_! U(I_{\mathcal{E}}) \\ &\overset{(3)}{=} U(p_! I_{\mathcal{E}}) = U(p), \end{aligned}$$

where (1) is by Proposition 5.2.3, (2) is by Proposition 5.3.2, and (3) is by Proposition 2.4.1. □

6. Equivalences of fibrations

6.1. Global equivalences, and final and cofinal functors.

6.1.1. **Definition.** Fix $p\colon \mathcal{E} \to \mathcal{C}$ and $p'\colon \mathcal{E}' \to \mathcal{C}$. A map $\alpha\colon \mathcal{E} \to \mathcal{E}'$ over $\mathcal{C}$ is a covariant, resp., contravariant, equivalence over $\mathcal{C}$ if $U(\alpha)$, resp., $U^{\mathrm{op}}(\alpha)$, is an isomorphism. ◁

6.1.2. **Definition.** A functor $\alpha\colon \mathcal{E} \to \mathcal{E}'$ is a global covariant, resp., contravariant, equivalence if for any functor $p'\colon \mathcal{E}' \to \mathcal{C}$, it induces a covariant, resp., contravariant, equivalence in $\mathcal{C}\mathrm{at}/\mathcal{C}$. ◁

The following a rephrasing of the last part of Corollary 4.1.4.

6.1.3. **Proposition.** If $\alpha\colon \mathcal{E} \to \mathcal{E}'$ is a left adjoint, then α is a global covariant equivalence. Dually, if α is a right adjoint, then α is a global contravariant equivalence. □

We can identify these global equivalences more explicitly.

6.1.4. **Definition.** The functor $p\colon \mathcal{E} \to \mathcal{C}$ is coinitial if for every $c \in \mathcal{C}$ we have $\pi_0(p/c) = *$, and is cofinal if for every $c \in \mathcal{C}$ we have $\pi_0(c/p) = *$. ◁

This is an essentially combinatorial definition: p is coinitial if every two elements of p/c can be connected by a zigzag. The naming is such that if $p\colon \mathcal{E} \to \mathcal{C}$ is an inclusion of posets, then p is cofinal precisely when it is so in the usual sense.

6.1.5. **Lemma.** If $f\colon \mathcal{C} \to \mathcal{D}$ is a discrete opfibration, then $f_!\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{S}\mathrm{et})$ is conservative.

Proof. Write π for the projection to the terminal category. Fix $d \in \mathcal{D}$, and write $q\colon p/d \to \mathcal{C}$. As f is a Grothendieck opfibration, the inclusion $j\colon p^{-1}(d) \to p/d$ has a left adjoint i by the proof of Lemma 4.2.1. As moreover $p^{-1}(d)$ is discrete, we learn from Theorem 2.4.2 that

$$(f_!F)(d) = \pi_! q^* F = \pi_! i_! q^* F = \pi_! j^* q^* F = \coprod_{c \in p^{-1}(d)} F(c).$$

The unit $F \to f^* f_! F$ is then an embedding, so that $f_!$ is conservative. □

6.1.6. **Theorem.** For $p\colon \mathcal{E} \to \mathcal{C}$, the following are equivalent:

(i) p is a global covariant equivalence,
(ii) p is coinitial,
(iii) There exists a discrete opfibration $f\colon \mathcal{C} \to \mathcal{D}$ such that p is a covariant equivalence over $\mathcal{D}$.

Proof. Observe that p is coinitial precisely when p is a covariant equivalence in $\mathcal{C}\mathrm{at}/\mathcal{C}$; thus certainly (i) implies (ii) and (ii) implies (iii). That (iii) implies (ii) follows from the previous lemma, so we have reduced to verifying that (ii) implies (i). Fix $f\colon \mathcal{C} \to \mathcal{D}$; we must show $U(fp) = U(f)$. Indeed, we have

$$U(fp) = U(f_! p) = f_! U(p) = f_! U(I_{\mathcal{C}}) = U(f_! I_{\mathcal{C}}) = U(f),$$

and the result follows. □

6.1.7. **Corollary.** For all $p\colon \mathcal{E} \to \mathcal{C}$, the unit $\mathcal{E} \to \int U(p)$ is coinitial. □

The dual statements hold.

6.2. Contravariant equivalences and colimits.

6.2.1. **Theorem.** For a functor $p\colon \mathcal{E} \to \mathcal{C}$, the following are equivalent:

(i) p is cofinal,
(ii) For any functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, we have $\operatorname{colim} F = \operatorname{colim} p^* F$.

Proof. For $c \in \mathcal{C}$, observe $\int \mathcal{C}(c, p(-)) = c/p$. As a consequence, by Theorem 1.4.3 we identify

$$\pi_0(c/p) = \operatorname{colim} \mathcal{C}(c, p(-)) = \operatorname{colim} p^* \mathcal{C}(c, -)$$

We find that p is cofinal if and only if condition (ii) holds for representable functors $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$. The result follows from Theorem 5.4.1. □

7. Colimits

We have focused on the category theory of categories of the form $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ for $\mathcal{C}$ small. Our goal in this section is to give some examples indicating how to use this theory for more general categories. We will use the following fact.

7.0.1. **Proposition.** Let $\mathcal{C}$ be a category and $p\colon \mathcal{J} \to \mathcal{C}$ a functor. Then $\operatorname{colim} p$ exists in $\mathcal{C}$ if and only if

$$c \mapsto \lim_{j \in \mathcal{J}} \mathcal{C}(p(j), c)$$

is representable, in which case $\operatorname{colim} p$ is the representing object. □

7.1. **Categories with colimits.** Fix a category $\mathcal{C}$. We define a class $\mathbb{F}$ of diagrams in $\mathcal{C}$ to be the data of a class of diagrams together with for each $\mathcal{J} \in \mathbb{F}$ a subcategory $\mathbb{F}(\mathcal{J}) \subset \mathrm{Fun}(\mathcal{J}, \mathcal{C})$. We require moreover that $[0] \in \mathbb{F}$ and $\mathbb{F}([0]) = \mathcal{C}$.

7.1.1. **Definition.** Given $\mathcal{C}$ and $\mathbb{F}$ as above, define the category $\mathrm{Psh}^{\mathbb{F}}(\mathcal{C})$ of $\mathbb{F}$-shaped presheaves on $\mathcal{C}$ as follows. Fix a universe for which $\mathcal{C}$ is locally small and all $\mathcal{J} \in \mathbb{F}$ are small. Then $\mathrm{Psh}^{\mathbb{F}}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ is the full subcategory spanned by the essential images of the composites

$$\mathbb{F}(\mathcal{J}) \xrightarrow{\subset} \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \xrightarrow{(h_{\mathcal{C}})_\circ} \mathrm{Fun}(\mathcal{J}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})) \xrightarrow{\mathrm{colim}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \ .$$

◁

Observe that, up to equivalence, $\mathrm{Psh}^{\mathbb{F}}(\mathcal{C})$ remains unchanged after enlarging the universe. As $\mathbb{F}([0]) = \mathcal{C}$, we find that $\mathrm{Psh}^{\mathbb{F}}(\mathcal{C})$ contains all representable functors and so obtain a restricted Yoneda embedding $h^{\mathbb{F}}_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Psh}^{\mathbb{F}}(\mathcal{C})$.

7.1.2. **Theorem.** The following are equivalent:

(i) For all $\mathcal{J} \in \mathbb{F}$ and $p \in \mathbb{F}(\mathcal{J})$, the colimit $\mathrm{colim}\, p$ exists in $\mathcal{C}$,
(ii) The restricted Yoneda embedding $h^{\mathbb{F}}_{\mathcal{C}}$ admits a left adjoint $L^{\mathbb{F}}_{\mathcal{C}}$.

Moreover, in this case we can identify $L^{\mathbb{F}}_{\mathcal{C}}(\mathrm{colim}\, h_{\mathcal{C}} p) = \mathrm{colim}\, p$ for $p \in \mathbb{F}(\mathcal{J})$ with $\mathcal{J} \in \mathbb{F}$. In particular, the counit $L^{\mathbb{F}}_{\mathcal{C}} h^{\mathbb{F}}_{\mathcal{C}} \to I_{\mathcal{C}}$ is an isomorphism.

Proof. The restricted Yoneda embedding $h^{\mathbb{F}}_{\mathcal{C}}$ admits a left adjoint if and only if for all $F \in \mathrm{Psh}^{\mathbb{F}}(\mathcal{C})$, the functor

$$\mathcal{C} \to \mathcal{S}\mathrm{et}, \qquad c \mapsto \mathrm{Hom}_{\mathrm{Psh}^{\mathbb{F}}(\mathcal{C})}(F, h_{\mathcal{C}}(c)) = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(F, h_{\mathcal{C}}(c))$$

is representable. Fix $F \in \mathrm{Psh}^{\mathbb{F}}(\mathcal{C})$ and choose $\mathcal{J} \in \mathbb{F}$ and $p \in \mathbb{F}(\mathcal{J})$ such that $F \cong \mathrm{colim}\, h_{\mathcal{C}} p$. Then

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(F, h_{\mathcal{C}}(c)) = \lim_{j \in \mathcal{J}} \mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(h_{\mathcal{C}}(p(j)), h_{\mathcal{C}}(c)) = \lim_{j \in \mathcal{J}} \mathcal{C}(p(j), c)$$

by Theorem 5.1.1. As a consequence, this is representable as a functor in c if and only if p has a colimit in $\mathcal{C}$. Varying F and p, we find that (i) and (ii) are equivalent, and the final statement follows easily. □

7.2. **The Kan formula revisited.** Our goal here is to generalize Theorem 2.4.2. This gives, by duality, the analogous formula for right Kan extensions.

7.2.1. **Definition.** Given $f \colon \mathcal{C} \to \mathcal{D}$ and a full subcategory $j \colon \mathcal{B} \subset \mathcal{D}$, we say that f has a partial right adjoint g along $\mathcal{B}$ if there is a functor $g \colon \mathcal{B} \to \mathcal{C}$ together with natural isomorphisms

$$\mathcal{C}(c, g(b)) \cong \mathcal{D}(f(c), j(b))$$

for all $b \in \mathcal{C}$. ◁

7.2.2. **Lemma.** Suppose $f \colon \mathcal{C} \to \mathcal{D}$ has a partial right adjoint g along $j \colon \mathcal{B} \subset \mathcal{D}$, and j has a left adjoint $k \colon \mathcal{D} \to \mathcal{B}$. Then $k \circ f \dashv g \circ j$.

Proof. For $c \in \mathcal{C}$ and $b \in \mathcal{B}$, we compute

$$\mathcal{C}(c, g(b)) \cong \mathcal{D}(f(c), j(b)) = \mathcal{B}(kf(c), b).$$

□

7.2.3. **Theorem.** Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathcal{K}$ be a category. Suppose that for all $F\colon \mathcal{C} \to \mathcal{K}$ and $d \in \mathcal{D}$, the colimit $\operatorname{colim}_{f(c)\to d} F(c)$ exists. Then restriction $f^*\colon \mathrm{Fun}(\mathcal{D},\mathcal{K}) \to \mathrm{Fun}(\mathcal{C},\mathcal{K})$ has a left adjoint $f_!$ computed by

$$(f_!F)(d) = \operatorname*{colim}_{f(c)\to d} F(c).$$

Proof. Choose a universe of sets for which everything in sight is small. Choose a class $\mathbb{F}$ of diagrams in $\mathcal{K}$ containing $F/d\colon f/d \to \mathcal{K}$ for all $F\colon \mathcal{C} \to \mathcal{K}$ and $d \in \mathcal{D}$, and for which $h_{\mathcal{K}}^{\mathbb{F}}$ has a left adjoint $L_{\mathcal{K}}^{\mathbb{F}}$; the existence of this is guaranteed by the hypotheses and Theorem 7.1.2. By Lemma 5.3.1 and Theorem 2.4.2, the composite

$$\mathrm{Fun}(\mathcal{C},\mathcal{K}) \xrightarrow{(h_{\mathcal{K}})_\circ} \mathrm{Fun}(\mathcal{C},\mathrm{Fun}(\mathcal{K}^{\mathrm{op}},\mathcal{S}\mathrm{et})) \xrightarrow{f_!} \mathrm{Fun}(\mathcal{D},\mathrm{Fun}(\mathcal{K}^{\mathrm{op}},\mathcal{S}\mathrm{et}))$$

restricts to

$$f_!^{\mathbb{F}}\colon \mathrm{Fun}(\mathcal{C},\mathcal{K}) \to \mathrm{Fun}(\mathcal{D},\mathrm{Psh}^{\mathbb{F}}(\mathcal{K}^{\mathrm{op}},\mathcal{S}\mathrm{et})),$$

which has partial right adjoint f^* along $(h_{\mathcal{K}}^{\mathbb{F}})_\circ\colon \mathrm{Fun}(\mathcal{D},\mathcal{K}) \to \mathrm{Fun}(\mathcal{D},\mathrm{Psh}^{\mathbb{F}}(\mathcal{K}^{\mathrm{op}},\mathcal{S}\mathrm{et}))$. As $L_{\mathcal{K}}^{\mathbb{F}} \dashv (h_{\mathcal{K}}^{\mathbb{F}})_\circ$, the result follows from Lemma 7.2.2. □

7.3. **Cocompletions.** Fix a category $\mathcal{C}$ and class S of diagrams containing the terminal category. Define $\mathrm{Psh}^S(\mathcal{C})$ as $\mathrm{Psh}^{\mathbb{F}}(\mathcal{C})$, where $\mathbb{F} = S$ with $\mathbb{F}(\mathcal{J}) = \mathrm{Fun}(\mathcal{J},\mathcal{C})$ for all $\mathcal{J} \in S$. Suppose $\mathrm{Psh}^S(\mathcal{C})$ admits S-shaped colimits, and that these are preserved under $\mathrm{Psh}^S(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})$. For categories $\mathcal{D}$ and $\mathcal{E}$ admitting S-shaped colimits, let $\mathrm{Fun}^{LS}(\mathcal{D},\mathcal{E}) \subset \mathrm{Fun}(\mathcal{D},\mathcal{E})$ be the full subcategory of functors preserving S-shaped colimits.

7.3.1. **Theorem.** The restricted Yoneda embedding $h_{\mathcal{C}}^S\colon \mathcal{C} \to \mathrm{Psh}^S(\mathcal{C})$ exhibits $\mathrm{Psh}^S(\mathcal{C})$ as the free cocompletion of $\mathcal{C}$ under S-shaped colimits in the following sense: for any category $\mathcal{E}$ admitting S-shaped colimits, restriction along $h_{\mathcal{C}}^S$ induces an equivalence

$$\mathrm{Fun}^{LS}(\mathrm{Psh}^S(\mathcal{C}),\mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C},\mathcal{E}).$$

Proof. Fix a category $\mathcal{E}$ admitting S-shaped colimits. Choose universes so everything in sight is small. In this case, a functor $f\colon \mathcal{C} \to \mathcal{E}$ induces $f_!\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{E}^{\mathrm{op}},\mathcal{S}\mathrm{et})$ which restricts to $f_!\colon \mathrm{Psh}^S(\mathcal{C}) \to \mathrm{Psh}^S(\mathcal{E})$ by Proposition 2.4.1. We obtain a functor

$$\mathrm{Fun}(\mathcal{C},\mathcal{E}) \to \mathrm{Fun}(\mathrm{Psh}^S(\mathcal{C}),\mathcal{E}), \qquad f \mapsto L_{\mathcal{E}}^S \circ f_!.$$

This lands in $\mathrm{Fun}^{LS}(\mathrm{Psh}^S(\mathcal{C}),\mathcal{E})$ by Theorem 7.1.2 as S-shaped colimits in $\mathrm{Psh}^S(\mathcal{C})$ are computed in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})$. We claim this provides an essential inverse to restriction along $h_{\mathcal{C}}^S$.

Fix $p\colon \mathrm{Psh}^S(\mathcal{C}) \to \mathcal{E}$ preserving S-shaped colimits. Fix $F \in \mathrm{Psh}^S(\mathcal{C})$. Choose $\mathcal{J} \in S$ and $p\colon \mathcal{J} \to \mathcal{C}$ so that $F = U(p)$. Then we identify

$$L_{\mathcal{E}}^S \circ (p \circ h_{\mathcal{C}})_!F = \operatorname{colim}\left(\mathcal{J} \xrightarrow{p} \mathcal{C} \xrightarrow{h_{\mathcal{C}}^S} \mathrm{Psh}^S(\mathcal{C}) \xrightarrow{p} \mathcal{D}\right).$$

As p preserves S-shaped colimits, this is p applied to

$$\operatorname{colim}\left(\mathcal{J} \xrightarrow{p} \mathcal{C} \xrightarrow{h_{\mathcal{C}}^S} \mathrm{Psh}^S(\mathcal{C})\right).$$

As colimits in $\mathrm{Psh}^S(\mathcal{C})$ are computed in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})$, this is just $p(F)$ by Theorem 5.4.1. We learn $L_{\mathcal{E}}^S \circ (p \circ h_{\mathcal{C}})_!F = p(F)$.

For the other direction, observe $L^S_{\mathcal{E}} \circ f_! \circ h^S_{\mathcal{C}} = L^S_{\mathcal{E}} \circ h^S_{\mathcal{E}} \circ f = F$ by Proposition 5.2.3 and Theorem 7.1.2. $\square$

7.3.2. **Remark.** One may, with a little more work, avoid the assumption that $\mathrm{Psh}^S(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ is closed under S-shaped colimits by replacing $\mathrm{Psh}^S(\mathcal{C})$ with the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ which contains the image of $h_{\mathcal{C}}$ and is closed under S-shaped colimits. $\triangleleft$

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

Email address: `balderr2@illinois.edu`