CLASSIFYING FORMAL GROUPS WITH DIEUDONNÉ THEORY

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Abstract. Dieudonné theory gives an equivalence between one-dimensional commutative formal groups over a perfect field and certain modules over a certain ring. One can then attempt to develop the classic theory of such formal groups purely within the category of these modules, as this note does. At a technical level, this replaces power series manipulations with Frobenius-semilinear algebra. The motivation for this exercise was to obtain a clean formulation and proof of Theorem 3.2.3.

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1. Basic properties of formal groups

1.1. Dieudonné modules. Fix a perfect field $\kappa$ of positive characteristic $p$. Let $W(\kappa)$ denote the ring of $p$-typical Witt vectors on $\kappa$, and let $D(\kappa)$ be the Dieudonné ring of $\kappa$. This is the noncommutative ring obtained from $W(\kappa)$ by adjoining two symbols $F$ and $V$ subject to

$$F\lambda = \lambda\sigma F, \quad V\lambda = \lambda\sigma^{-1} V, \quad FV = p = VF,$$

where $\lambda$ ranges over $W(\kappa)$ and $(-)^\sigma$ is the canonical Frobenius automorphism of $W(\kappa)$. Let $\text{LMod}_{D(\kappa)}^n$ be the category of left $D(\kappa)$-module $M$ subject to the following conditions:

1. As a $W(\kappa)$-module, $M$ is free of finite and positive rank,
2. The action of $V$ on $M$ is topologically nilpotent, and $M = \lim_k M/V^k M$,
3. The quotient $M/VM$ is a simple $W(\kappa)$-module, i.e. $M/VM \simeq \kappa$.

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Observe that the condition $FV = p = VF$ tells us that for $M \in \text{LMod}^p_D$ the operators $F$ and $V$ determine each other. In fact, we need not have introduced $F$ at all; see Corollary 1.3.4. The main Dieudonné theorem relevant to us is the following.

1.1.1. Theorem. The category $\text{LMod}^p_D$ is equivalent to the category of one-dimensional commutative formal groups over $\text{Spec} \kappa$.

We will not need the details of the construction of this equivalence as we will be working entirely within $\text{LMod}^p_D$, but let us say something to establish where our conventions sit. Let $\mathfrak{Hopf}^c_\kappa$ denote the category of Hopf algebras $H$ over $\kappa$ such that $\text{Spf} H^\vee$ is a connected formal group. There is an equivalence $\text{DM}(\text{H}) \cong \text{H}(\kappa)(p^\infty)^{\oplus h}$, where $h$ is the height and $d$ is the dimension of $\text{Spf} H$. Let $\text{DM}(\kappa)[p^n] \subset \text{DM}(\kappa)$ be the $p^n$-torsion subgroup. Multiplication by $p$ gives maps $\text{DM}(\kappa)[p^n] \to \text{DM}(\kappa)[p^{n-1}]$, and we define $\text{DM}(\kappa) = \lim_n \text{DM}(\kappa)[p^n]$. Under the above noncanonical isomorphism, we identify $\text{DM}(\kappa)[p^n] \to \text{DM}(\kappa)[p^{n-1}]$ as the quotient map $\text{W}(\kappa)/(p^n)^{\oplus h} \to \text{W}(\kappa)/(p^{n-1})^{\oplus h}$, and can then identify

$$\text{DM}(\kappa) \cong \text{W}(\kappa)^{\oplus h}, \quad \text{DM}(\kappa)/\text{VDM}(\kappa) \cong \kappa^{\oplus d}.$$ 

Specializing to the case where $d = 1$, the functor $\text{Spf} H^\vee \to \hat{\text{DM}}(H)$ is our equivalence.

1.1.2. Remark. The Serre dual of $\text{LMod}^p_D$ is the category $\text{LMod}^f_D$, defined in the same way except with $F$ in place of $V$. For $M \in \text{LMod}^f_D$, we have $\text{Hom}_D(M, \text{W}(\kappa)) \in \text{LMod}^f_D$, where we give $\text{Hom}_D(M, \text{W}(\kappa))$ the $D(\kappa)$-module structure

$$(Vf)(a) = f(Fa), \quad (Ff)(a) = f(Va)$$

for $f : M \to \text{W}(\kappa)$. This gives a duality between $\text{LMod}^p_D$ and $\text{LMod}^f_D$.

1.2. The Honda module. Fix a positive integer $h$ and let $\text{LMod}^{V,h}_D \subset \text{LMod}^p_D$ be the full subcategory on those objects free of rank $h$ over $\text{W}(\kappa)$. There is a distinguished object $H \in \text{LMod}^{V,h}_D$ given by

$$H = \text{W}(\kappa)\{x, Vx, \ldots, V^{h-1}x\}, \quad V^h x = px, \quad Fx = V^{h-1}x.$$ 

This is the free $D(\kappa)$-module on a generator $x$ subject to the condition $V^h x = px$. We obtain for any $M \in \text{LMod}^{V,h}_D$ an identification

$$\text{Hom}(H, M) \cong \{a \in M : V^h a = pa\}, \quad f \mapsto f(x).$$ 

1.2.1. Proposition. Suppose that $\kappa$ contains all $(p^h - 1)$th roots of unity, and let $\mathbb{F}_{p^h} \subset \kappa$ be the subfield generated by these. Then there is a canonical identification

$$\text{End}(H) = W(\mathbb{F}_{p^h})\{V\}/(V^h = p, Va = a^{p-1}V),$$

where $a$ ranges through $W(\mathbb{F}_{p^h})$. 
Proof. As above, identify \( \text{End}_{D(\kappa)}(H) = \{ a \in H : F^h a = pa \} \). Write an arbitrary element \( a \in H \) as \( a = \sum_{i=0}^{h-1} a_i V^i \). Then
\[
F^h a = \sum_{i=0}^{h-1} a_i^{\sigma^{-h}} V^{h+i} x = \sum_{i=0}^{h-1} a_i^{\sigma^{-h}} p V^i x.
\]
We find that \( F^h a = pa \) precisely when \( a_i^{\sigma^{-h}} = a_i \) for each \( i \), which holds precisely when each \( a_i \in W(F_{p^h}) \). \( \square \)

1.2. Remark. The object more commonly seen in the study of formal groups is \( E = W(F_{p^h})/(F^h = p, F a = a^p) \). There is an isomorphism
\[
\text{End}_{D(\kappa)}(H) \to E^{\text{op}}, \quad \sum_{i=0}^{h-1} a_i V^i \mapsto \sum_{i=0}^{h-1} F^i a_i.
\]
This corresponds to Remark 1.1.2.

Let \( \mathcal{S}_h = \text{Aut}_{D(\kappa)}(H) \). As underlying \( H \) is a free module over \( W(\kappa) \) of rank \( h \) and with distinguished basis, we have canonical inclusions
\[
\text{End}_{D(\kappa)}(H) \subset \text{End}_{W(\kappa)}(W(\kappa)^{\oplus h}), \quad \mathcal{S}_h \subset \text{GL}_h(W(\kappa)).
\]
We can give \( \text{End}_{D(\kappa)}(H) \), and thus \( \mathcal{S}_h \), the topology arising from these inclusions and the \( p \)-adic topology on \( W(\kappa) \). As \( h \) is finite, this is equivalent to the topology induced from the filtration on \( \text{End}_{D(\kappa)} \) by powers of \( V \), and makes \( \text{End}_{D(\kappa)}(H) \) into a profinite ring and \( \mathcal{S}_h \) into a profinite group.

We will prove the following theorem in the next section.

1.2.3. Theorem. Suppose \( \kappa \) is algebraically closed. Then every \( M \in \text{LMod}_{D(\kappa)}^{\nu,h} \) is isomorphic to \( H \).

We can rephrase this as follows. For a category \( \mathcal{C} \), let \( \pi_0 \mathcal{C} \) denote the set of isomorphism classes of objects of \( \mathcal{C} \). Then the previous theorem states
\[
\pi_0 \text{LMod}_{D(\kappa)}^{\nu,h} = \{ H \},
\]
so long as \( \kappa \) is algebraically closed. We will later leverage this into an identification of \( \pi_0 \text{LMod}_{D(\kappa)}^{\nu,h} \) for a general perfect field \( \kappa \).

1.3. Semilinear algebra. Let \( R \) be a ring, let \( \phi : R \to R \) an endomorphism, and let \( M \) and \( N \) be \( R \)-modules. For \( r \in R \), we may write \( \phi(r) = \phi_r \). A \( \phi \)-semilinear map \( f : M \to N \) is defined as a function satisfying
\[
f(m + n) = f(m) + f(n), \quad f(rm) = \phi_r f(m).
\]
Given such \( f \), define \( N^{(\phi)} = R \phi \otimes_R N \). Then
\[
M \to N^{(\phi)}, \quad m \mapsto 1 \otimes f(m)
\]
is an \( R \)-linear map. Of particular interest is the case where \( R = \kappa \) is a perfect field, \( \phi = \sigma^{\pm 1} \), and \( M = N \) is a finite-dimensional vector space. It turns out that in this case \( M \) is the sum of an \( f \)-nilpotent subspace and a subspace generated by \( f \)-fixed elements; we will not need this exact fact, but mention it as it provides some motivation for the following results. We first deal with the case of certain nilpotent operators.
1.3.1. Lemma. Let $k$ be a field, and $\phi$ an endomorphism of $k$. Let $U$ be a rank $h$ vector space over $k$, and $V: U \to U$ a $\phi$-semilinear operator. Suppose that $V$ is nilpotent, and $U/VU \cong k$. Then there is an $x \in U$ such that $x, Vx, \ldots, V^{h-1}x$ give a basis for $U$.

Proof. Set $U_0 = U$, and inductively define $U_n$ to be the image of $U_{n-1} \to U^{(\phi^n)}$. Our assumptions give us a sequence

$$U_0 \to U_1 \to \cdots \to U_{n-1} \to U_n = 0$$

of surjective $k$-linear maps, each with one-dimensional kernel. In particular, the $\phi$-semilinear operator $V$ satisfies $V^{h-1} \neq 0$ and $V^h = 0$. Let $x \in U$ be such that $V^{h-1}x \neq 0$. It is sufficient to show $x, Vx, \ldots, V^{h-1}x$ are linearly independent. Indeed, a linear relation of the form $\sum_{i=0}^{h-1} \lambda_i V^i x = 0$ yields, upon application of $V^{h-1}$, the identity $\phi^{h-i} \lambda_i V^{h-1} x = 0$, so that $\lambda_i = 0$ as $\phi$ is necessarily injective.

1.3.2. Proposition. Let $k$ be a field, $\phi$ an endomorphism of $k$, and $M$ a free module over $W(k)$ of finite rank $h$ equipped with a $\phi$-semilinear topologically nilpotent endomorphism $V$ satisfying $M/VM \cong k$. Then there exists $y \in M$ such that $V^{h-1}y \neq 0$ in $M/(p)$. Moreover, for any such $y$, we have

$$p^n M \cong W(k)\{V^{nh}y, V^{nh+1}y, \ldots, V^{nh+h-1}y\}$$

for each $n$. In particular, $V^h y = \sum_{i=0}^{h-1} pa_i V^i y$ with $a_0 \in W(k)^\times$, and $p^{-1} V^h$ is a $\phi$-semilinear automorphism of $M$.

Proof. By Lemma 1.3.1, we find $\overline{y} \in M/(p)$ such that $\overline{y}, V\overline{y}, \ldots, V^{h-1}\overline{y}$ form a basis for $M/(p)$. Choose a lift $y$ over $\overline{y}$ to $M$. Then $y, Vy, \ldots, V^{h-1}y$ form a basis for $M$, and $V^h y, \ldots, V^{2h-1}y$ lie in $pM$. By repeating our arguments with each $p^n M$, it is sufficient to verify that these form a basis for $pM$. Out of $y, \ldots, V^{h-1}y$, only $y$ is not in the image of $V$; thus when we expand $V^h y = \sum_{i=0}^{h-1} pa_i V^i y$, the element $a_0 \in W(k)$ must be unit. We learn that for $0 \leq i \leq h-1$, the expansion of $V^{h+i} y$ in our basis mod $p^2$ is of the form $p b_i V^i y + \cdots + p b_{h-1} V^{h-1} y$ with $b_i$ a unit. It is easily seen from this that $V^h y, \ldots, V^{2h-1} y$ project to a basis for $pM/p^2 M$, and thus form a basis of $pM$.

1.3.3. Corollary. Let $k$ and $M$ be as above, and moreover let $N$ be of the same type as $M$. Suppose given a $W(k)$-linear map $f: M \to N$ which commutes with $V$. Then for each $r \geq 0$, there are unique automorphisms $\mu: M \to M^{(\phi^r)}$ and $\eta: N \to N^{(\phi^r)}$ fitting into a diagram

$$
\begin{array}{ccc}
M & \xrightarrow{p^r} & M \\
\downarrow{f} & & \downarrow{f} \\
N & \xrightarrow{p^r} & N \\
\end{array}
\quad \begin{array}{ccc}
M^{(\phi^r)} & \xrightarrow{\mu} & M^{(\phi^r)} \\
\downarrow{f^{(\phi^r)}} & & \downarrow{f^{(\phi^r)}} \\
N^{(\phi^r)} & \xrightarrow{\eta} & N^{(\phi^r)} \\
\end{array}
$$

of $W(k)$-linear maps that commute with $V$. Here, $f^{(\phi^r)}$ indicates functoriality of $(-)^{(\phi^r)}$ applied to $f$. □
1.3.4. Corollary. Let $\kappa$ be a perfect field, and $M$ a free module over $W(\kappa)$ of finite rank $h$ equipped with a $\sigma^{-1}$-semilinear topologically endomorphism $V$ satisfying $M/VM \cong \kappa$. Then $M$ can be uniquely upgraded to an object of $L\text{Mod}^v_{D(\kappa)}$.

Proof. As $VF = p = FV$ and $M$ is torsion-free, we see that there is at most one choice of $F$. Choose $y \in M$ such that $M \cong W(\kappa)\{y, Vy, \ldots, V^{h-1}y\}$. We are forced to define $FV^iy = pV^{i-1}y$ for $1 \leq i \leq h-1$. As $M/VM \cong \kappa$, necessarily $py = Va$ for some $a \in M$, and we are forced to define $Fy = a$. This description on the basis $y, Vy, \ldots, V^{h-1}y$ extends by semilinearity to a description on all of $M$. \hfill $\Box$

1.3.5. Proposition. Let $\kappa$ be an algebraically closed field, and $U$ a finite-dimensional vector space over $\kappa$. Let $T: U \to U$ be a $\sigma^{-1}$-semilinear automorphism. Then there exists some nonzero $x \in U$ which is fixed by $T$.

Proof. By replacing $T$ with $T^{-1}$, we may instead deal with a $\sigma$-linear automorphism. Let us work for now with a general $\phi$-semilinear automorphism $T: U \to U$, where $\phi$ is some endomorphism of $k$.

Pick $y \in U$. As $U$ is finite-dimensional, there is some minimal $n$ such that $T^{n+1}y$ is in the span of $y, \ldots, T^{n}y$; write $T^{n+1}y = \sum_{i=0}^{n} a_i T^i y$. As $T$ is an automorphism, not all $a_i$ are zero. After possibly replacing $y$ with some $T^i y$, we may suppose $a_0 \neq 0$, so that $y, Ty, \ldots, T^ny$ are linearly independent. For indeterminates $\lambda_0, \ldots, \lambda_n$, write

$$x = \sum_{i=0}^{n} \lambda_i T^i y.$$

Then solving $Tx = x$ amounts to solving

$$(\lambda_0 - \phi \lambda_n a_0) y + (\lambda_1 - \phi \lambda_0 - \phi \lambda_0 a_1) Ty + \cdots + (\lambda_n - \phi \lambda_{n-1} - \phi \lambda_n a_n) T^ny = 0.$$ 

As $y, Ty, \ldots, T^ny$ are linearly independent, this is equivalent to asking each coefficient in parentheses to vanish. Starting with $\lambda_0 = \lambda_n^a a_0$, we can substitute each equation into the next, allowing us to reduce to solving a single equation. Writing $\lambda = \lambda_n$ and $b_i = \phi^i a_{n-i}$, this comes out to solving

$$\lambda = \phi^{n+1} \lambda b_n + \phi^n \lambda b_{n-1} + \cdots + \phi \lambda b_0.$$

In our case, $\phi = \sigma$. Recall also $b_0, \ldots, b_n$ are not all zero. So we are reduced to finding a nonzero root of the nontrivial separable equation

$$t = t^{p^n+1} b_n + t^{p^n} b_{n-1} + \cdots + t^p b_0,$$

which can be done under our assumption that $k$ is algebraically closed. \hfill $\Box$

We would like to lift the previous proposition from $\kappa$-vector spaces to suitable $W(\kappa)$-modules.

1.3.6. Lemma. Let $M$ be a $W(\kappa)$-module which is locally $p$-power torsion, and let $T: M \to M$ be a $\sigma^{-1}$-semilinear map. Let $M^T = \{a \in M : Ta = a\}$, which is a subgroup of $M$. Then the resulting map $W(\kappa) \otimes M^T \to M$ is injective.

Proof. Let $M_0^T \subset M^T$ be a finitely generated subgroup; it will suffice to show that $W(\kappa) \otimes M_0^T \to M$ is injective. Write

$$M_0^T = \bigoplus_{1 \leq i \leq k} \mathbb{Z}/(p^i),$$

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so that \( M_0^T \subset M \) is given by the choice of \( a_1, \ldots, a_k \in M \) satisfying \( Ta_i = a_i \) and \( p^k a_i = 0 \). Suppose towards a contradiction we had

\[
\lambda_1 a_1 + \cdots + \lambda_k a_k = 0
\]

for some \( \lambda_i \in W(\kappa)/(p^k) \) with not all \( \lambda_i \) zero. We can suppose this relation is chosen so that each \( \lambda_i \) is nonzero, and \( k \) is minimal so that such a relation holds. Moreover, we may suppose \( \lambda_1 = p^c \) for some integer \( c \). Applying \( T \) to the above gives

\[
p^c a_1 + \cdots + \lambda_k^{-1} a_k = 0.
\]

Subtracting these equations yields

\[
(\lambda_2 - \lambda_2^{-1}) a_2 + \cdots + (\lambda_k - \lambda_k^{-1}) a_k = 0,
\]

so that \( \lambda_i = \lambda_i^{-1} \) for each \( i \) by minimality of \( k \). But then \( \lambda_1 a_1 + \cdots + \lambda_k a_k \in \mathbb{Z}_p \otimes M^T = M^T \), and certainly \( M^T \subset M \) is injective.

1.3.7. Lemma. Suppose \( \kappa \) is algebraically closed. Let \( M \) be an Artinian \( W(\kappa) \)-module, and \( T : M \to M \) a \( \sigma^{-1} \)-semilinear injection. Then \( M \) is generated by \( M^T = \{ x \in M : Tx = x \} \) over \( W(\kappa) \).

Proof. By considering the induced \( W(\kappa) \)-linear injection \( M \to M^{(\sigma^{-1})} \) between Artinian \( W(\kappa) \)-modules of equal length, we find that \( T \) is an isomorphism. Let \( M' \subset M \) be the \( W(\kappa) \)-submodule generated by \( M^T \), and let \( M'' = M/M' \). Consider the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
& & \downarrow{T'-1} & & \downarrow{T-1} & & \downarrow{T''-1} & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}
\]

As \( \kappa \) is algebraically closed, \( \sigma^{-1} - \text{id} : \kappa \to \kappa \) is surjective. By induction on composition series, we learn that \( \sigma^{-1} - 1 : W(\kappa) \otimes A \to W(\kappa) \otimes A \) is a surjection for any finite abelian group \( A \). In case \( A = M^T \), under the isomorphism \( W(\kappa) \otimes M^T = M' \) given by Lemma 1.3.6 we find that \( T-1 : M' \to M' \) is surjective. As \( \ker(T'-1) \cong \ker(T - 1) \) by construction, the long exact sequence associated to the above diagram tells us that \( T''-1 \) is an injective endomorphism of \( M'' \). So \( T'' \) is a \( \sigma^{-1} \)-semilinear automorphism of \( M'' \) with no fixed points. The same then must be true of \( T'' \) acting on the \( p \)-torsion submodule of \( M'' \), contradicting Proposition 1.3.5.

Putting the previous two lemmata together, we learn that under the assumptions and notation of the previous lemma, the map \( W(\kappa) \otimes M^T \to M \) is an isomorphism.

1.3.8. Lemma. Let

\[
I_1 \xleftarrow{j_1} I_2 \xleftarrow{j_2} I_3 \xleftarrow{j_3} \cdots
\]

be a tower of \( p \)-complete Hausdorff abelian groups, and suppose \( j_n(I_{n+1}) \subset pI_n \) for each \( n \). Then

\[
\lim_n I_n = 0 = \lim_n^1 I_n.
\]
Proof. The assertion that \( \lim_n I_n = 0 \) is immediate, so let us verify \( \lim^1_n I_n = 0 \). Recall that \( \lim^1_n I_n \) can be identified as the cokernel of
\[
\prod_n I_n \to \prod_n I_n, \quad (a_n)_n \mapsto (a_n - j_n(a_{n+1}))_n,
\]
so we must show this map is surjective. Indeed, fix an element \((b_n)_n \in \prod_n I_n\). Then \((b_n)_n = (a_n - j_n(a_{n+1}))_n\) has the solution
\[
a_n = \sum_{k \geq n} (j_n \circ \cdots \circ j_{k-1})(b_k),
\]
which converges under our assumptions. \(\square\)

1.3.9. **Lemma.** Suppose given \( W(\kappa) \)-modules \( N_1, N_2, \ldots \) fitting into exact sequences
\[
N_{n+1} \xrightarrow{p^n} N_{n+1} \to N_n \to 0,
\]
and suppose \( N_1 \) is a finite dimensional vector space over \( \kappa \). Let \( N = \lim_n N_n \). Then \( N \) is a finitely generated \( W(\kappa) \)-module, and \( N/(p^n) = N_n \).

**Proof.** The assumptions give for all \( n \) an exact sequence of towers whose \( m \)’th level is
\[
N_{n+m} \xrightarrow{p^n} N_{n+m} \to N_n \to 0.
\]
The leftmost towers consist of surjections, so by the Mittag-Leffler condition for vanishing \( \lim^1 \) we obtain in the limit an exact sequence
\[
N \xrightarrow{p^n} N \to N_n \to 0
\]
telling us that \( N/(p^n) = N_n \). We learn that \( N = \lim_n N_n/(p^n) \) is a \( p \)-complete Hausdorff \( W(\kappa) \)-module such that \( N/(p) \) is a finite-dimensional \( \kappa \)-vector space. It follows by the Nakayama lemma for complete Hausdorff modules that \( N \) is finitely generated over \( W(\kappa) \). \(\square\)

1.3.10. **Proposition.** Let \( \kappa \) be an algebraically closed field. Let \( M \) be an \( W(\kappa) \)-module free of finite rank \( h \) and \( T : M \to M \) a \( \sigma^{-1} \)-semilinear automorphism. Let \( M^T = \{ a \in M : Ta = a \} \). Then \( W(\kappa) \otimes M^T \cong M \), where \( \otimes \) is taken with respect to the \( p \)-adic topology.

**Proof.** For each \( n \geq 1 \), let \( M^T_n = \{ a \in M : Ta = a \ (\text{mod} \ p^n) \} \). We obtain a tower \( M^T_1 \hookrightarrow M^T_2 \hookrightarrow \cdots \) of inclusions, and \( M^T = \lim_n M^T_n \). Let \( I_n = p^nM \cap M^T_n \), and observe \( M^T_n/I_n = M/(p^n)^T \). Combining this with Lemma 1.3.6 and Lemma 1.3.7 we learn
\[
W(\kappa) \otimes M^T_n/I_n \cong M/(p^n).
\]
In particular, this gives us exact sequences
\[
W(\kappa) \otimes M^T_{n+1}/I_{n+1} \xrightarrow{p^n} W(\kappa) \otimes M^T_{n+1}/I_{n+1} \to W(\kappa) \otimes M^T_n/I_n \to 0,
\]
and as \( \mathbb{Z}_p \subset W(\kappa) \) is faithfully flat this passes to exact sequences
\[
M^T_{n+1}/I_{n+1} \xrightarrow{p^n} M^T_{n+1}/I_{n+1} \to M^T_n/I_n \to 0.
\]
By Lemma 1.3.8, we know \( \lim_n I_n = \lim_n I_n^1 = 0 \), so that \( M^T = \lim_n M_n^T = \lim_n T_n^I / I_n \). By Lemma 1.3.9, we learn \( M^T / (p^n) = M_n^T / I_n \). So we can identify \( W(\kappa) \otimes M^T \to M \) as

\[
W(\kappa) \otimes M^T = \lim_{m,n} W(\kappa) / (p^m) \otimes M^T / (p^n) \\
= \lim_{m,n} W(\kappa) / (p^m) \otimes M_n^T / I_n \\
= \lim_{n} W(\kappa) \otimes M_n^T / I_n \\
= \lim_{n} M / (p^n) = M.
\]

Finally, Theorem 1.2.3 is an immediate consequence of the following.

1.3.11. Proposition. Let \( \kappa \) be an algebraically closed field, and \( M \) a free module over \( W(\kappa) \) of finite rank \( h \) equipped with a \( \sigma^{-1} \)-linear topologically nilpotent endomorphism \( V : M \to M \) such that \( M/V \to \kappa \). Then there exists \( x \in M \) such that \( M \cong W(\kappa) \{ x, Vx, \ldots, V^{h-1}x \} \) and \( V^h x = px \).

Proof. By Proposition 1.3.2, we know that \( p^{-1}V^h : M \to M \) is a \( \sigma^{-1} \)-semilinear automorphism, and moreover there is \( y \in M \) such that \( y, V y, \ldots, V^{h-1}y \) forms a basis for \( M \). On the other hand, Proposition 1.3.10 tells us that \( M \) admits a basis \( x_0, \ldots, x_{h-1} \) with \( (p^{-1}V^h)x_i = x_i \) for each \( i \). If we write \( y = \sum_{i=0}^{h-1} a_i x_i \) and choose \( n \) such that \( a_n \) is nonzero in \( M / (p) \), then Proposition 1.3.2 tells us that \( x_n, V x_n, \ldots, V^{h-1}x_n \) is a basis for \( M \). So \( x = x_n \) has the stated properties. \( \square \)

2. Classification of formal groups

2.1. Galois cohomology. For our purposes, a topological group will be a group \( G \) equipped with topology arising from an inverse limit construction \( G = \lim_i G / U_i \), where \( U_i \subset G \) is a normal subgroup not necessarily of finite index. Let \( \Gamma \) and \( S \) be topological groups, and suppose we have an action of \( \Gamma \) on \( S \). We can define the semidirect product group \( S \rtimes \Gamma \) with multiplication \( (s, \gamma) \cdot (t, \delta) = (s \cdot t, \gamma \cdot \delta) \). This is equipped with a projection \( \pi : S \rtimes \Gamma \to \Gamma \), and we define the set of \( S \)-valued cocycles on \( \Gamma \) by

\[
Z^1(\Gamma, S) = \{ \alpha : \Gamma \to S : \pi \circ \alpha = \text{id}_\Gamma \}.
\]

Equivalently,

\[
Z^1(\Gamma, S) = \{ \alpha : \Gamma \to S : \alpha(\gamma \cdot \delta) = \alpha(\gamma) \cdot \gamma \alpha(\delta) \} \subset \text{Hom}(\Gamma, S).
\]

This set is equipped with an \( S \)-action by conjugation: for \( s \in S \) and \( \alpha \in Z^1(\Gamma, S) \), we define

\[
\alpha^s(\gamma) = s^{-1} \cdot \alpha(\gamma) \cdot \gamma s.
\]

Let \( H^1(\Gamma, S) = Z^1(\Gamma, S)_S \) be the set of orbits of this action. If \( S \) is abelian, then \( H^1(\Gamma, S) \) agrees with the usual continuous group cohomology of \( \Gamma \) with coefficients in \( S \), as can be seen directly from a suitable bar resolution. In general, \( H^1(\Gamma, S) \) is a set pointed at the zero map \( \Gamma \to S \). Observe that \( H^1(\Gamma, S) = 0 \) precisely when for every \( \alpha \in Z^1(\Gamma, S) \) there exists some \( s \in S \) such that \( \alpha(\gamma) = s^{-1} \cdot \gamma s \) for all \( \gamma \in \Gamma \).
2.1.1. Lemma. Suppose $S = \lim_{n \to 1} S/U_n$ is a complete Hausdorff topological group, and set $U_0 = S$. Suppose $H^1(\Gamma, U_n/U_{n+1}) = 0$ for each $n \geq 0$. Then $H^1(\Gamma, S) = 0$.

Proof. Let $\alpha \in Z^1(\Gamma, S)$. We find that $\alpha(\gamma) = s_1^{-1} \cdot s_1 \mod U_1$ for some fixed $s_1 \in S$. Replacing $\alpha$ with $\alpha^{s_1^{-1}}$, we may suppose $\alpha$ lands in $U_1 \subset S$. Repeating this argument, we find elements $s_1, s_2, \ldots \in S$ such that $s_n \in U_n$ and $f(s_n, \ldots, s_1)^{-1}$ lies in $U_n$ for each $n$. As $S$ is complete Hausdorff with respect to the filtration by the subgroups $U_n$, we can take $s_\infty = \lim_{n \to \infty} s_n \cdots s_1$ and find $f s_\infty^{-1}$ lies in $\bigcap U_n = 0$. So $f$ is zero in $H^1(\Gamma, S)$. ∎

Fix now a field $k$ of positive characteristic $p$, and let $k \subset K$ be a Galois extension with Galois group $\Gamma$. Write $W(K)$ for the ring of $p$-typical Witt vectors on $K$, and consider this as equipped with the $p$-adic topology. We obtain a continuous action of $\Gamma$ on the Witt components of $W(K)$. The quotients $W(K) \to W(K)/(p^n)$ pass to surjections $GL_n(K) \to GL_n(W(K)/(p^n))$, and these make $GL_n(W(K))$ into a topological group. The action of $\Gamma$ on the coordinates of $GL_n(W(K))$ is continuous with respect to this topology. The following can be seen as an analogue of Hilbert’s Satz 90.

2.1.2. Theorem. $H^1(\Gamma, GL_n(W(K))) = 0$.

Proof. Let $U_n$ be the kernel of the quotient map $GL_n(W(K)) \to GL_n(W(K)/(p^n))$. One can show that

$$U_0/U_1 = (K^\times)^n, \quad U_{n+1}/U_n = K^{\oplus n}.$$ 

So by Lemma 2.1.1, we are reduced to verifying $H^1(\Gamma, K^\times) = 0 = H^1(\Gamma, K)$. The first is exactly Hilbert’s Satz 90. For the second, let $\alpha \in Z^1(\Gamma, K)$, realized as a map $\alpha: \Gamma \to K$. As $K$ is discrete and $\Gamma$ is profinite, this factors through $\alpha': G \to L$, where $k \subset L \subset K$ is some finite Galois extension and $G = Aut_k(L)$. It is then sufficient to show $\alpha' = 0$ in $H^1(G, L)$. This follows from the normal basis theorem, telling us that $L \cong k[G]$ as $k[G]$-modules, and the identification of $H^1$ with ordinary group cohomology in this case. ∎

2.2. The classification theorem. Fix a perfect field $\kappa$ of positive characteristic, and fix an algebraic closure $\kappa \subset \overline{\kappa}$. Let $\Gamma = Aut_k(\overline{\kappa})$ be the absolute Galois group of $\kappa$. For $M \in LMod_M^{D(\kappa)}$, let $M_{\overline{\kappa}} = W(\overline{\kappa}) \otimes W(\kappa) M \in LMod_M^{D(\overline{\kappa})}$. The continuous action of $\Gamma$ on $W(\overline{\kappa})$ gives rise to a continuous action of $\Gamma$ on $M_{\overline{\kappa}}$ by left $D(\kappa)$-module maps, and we can identify $M_{\overline{\kappa}}^D = M$.

Fix a positive integer $h$, and write $H = W(\kappa)\{x, v x, \ldots, v^{h-1} x\}$ for the object of $D(\kappa)$, considered in Section 1.2. We have an action of $\Gamma$ on $Hom_D(H, \kappa)$. Recall that we can identify a map $f: H_{\overline{\kappa}} \to M_{\overline{\kappa}}$, subject to $f a = pa$ for $\gamma \in \Gamma$. For $\gamma \in \Gamma$, we have $\gamma a \in M_{\overline{\kappa}}$, and we claim this is associated to some other $\gamma f: H_{\overline{\kappa}} \to M_{\overline{\kappa}}$. Indeed, as $\Gamma$ acts by $D(\kappa)$-linear maps, we have

$$V^h \gamma(a) = \gamma(V^h a) = \gamma(pa) = p \gamma(a).$$

This action restricts to an action of $\Gamma$ on $\kappa$ on $\overline{\kappa}$. In case $M = H$, this gives an action of $\Gamma$ on $S_h$. Our goal for the rest of this section is to demonstrate the following:

$$\pi_0 LMod_{D(\kappa)}^{h,h} \cong H^1(\Gamma, S_h).$$
Fix $M \in \text{LMod}_{D(\kappa)}^{v,h}$. Choose an isomorphism $f : H_{\kappa} \to M_{\kappa}$; the existence of such is guaranteed by Theorem 1.2.3. Define then

$$\alpha_{M,f} : \Gamma \to S_h, \quad \alpha_{M,f}(\gamma) = f^{-1} \cdot \gamma f.$$  

2.2.1. **Claim.** We have $\alpha_{M,f} \in Z^1(\Gamma, S_h)$.

**Proof.** We must verify the cocycle condition and continuity. For the cocycle condition, we directly calculate for $\gamma, \delta \in \Gamma$ that

$$\alpha_{M,f}(\gamma \delta) = f^{-1} \cdot \gamma \delta f = f^{-1} \cdot \gamma f \cdot \gamma f^{-1} \cdot \gamma \delta f = f^{-1} \cdot f^\gamma \cdot f^\delta (f^{-1} \cdot \gamma f) = \alpha_{M,f}(\gamma) \cdot \alpha_{M,f}(\delta).$$

For continuity, write $f(x) = \sum_i \lambda_i \otimes a_i$ with $\lambda_i \in W(\overline{p})$ and $a_i \in M$ and let $\kappa \subset K$ be the field extension obtained by adjoining the first $n$ Witt components of the coefficients $\lambda_i$. If $\gamma|K = \delta|K$, then $\gamma f \equiv \delta f \pmod{p^{n+1}}$, and thus $\alpha_{M,f}(\gamma) \equiv \alpha_{M,f}(\delta) \pmod{p^{n+1}}$. So we learn that $\alpha_{M,f}$ is continuous. \hfill \Box

We have now assigned to every $M \in \text{LMod}_{D(\kappa)}^{v,h}$ and choice of isomorphism $f : H_{\kappa} \to M_{\kappa}$ an element $\alpha_{M,f} \in Z^1(\Gamma, S_h)$.

2.2.2. **Claim.** Upon passing to $H^1(\Gamma, S_h)$, the above assignment factors through $\pi_0 \text{LMod}_{D(\kappa)}^{v,h}$.

**Proof.** Suppose given $M' \in \text{LMod}_{D(\kappa)}^{v,h}$ with chosen isomorphisms $f_\epsilon : H_{\kappa} \to M'_{\kappa}$ for $\epsilon \in \{1, 2\}$. Let $g : M^1 \to M^2$ be an isomorphism of left $D(\kappa)$-modules, and define $h \in S_h$ by the diagram

$$
\begin{array}{ccc}
H_{\kappa} & \xrightarrow{f_1} & M^1_{\kappa} \\
\downarrow h & & \downarrow g \\
H_{\kappa} & \xrightarrow{f_2} & M^2_{\kappa}
\end{array}
$$

As $g$ is defined over $W(\kappa)$, we can calculate for $\gamma \in \Gamma$ that

$$f_2^{-1} \cdot \gamma f_2 = (g \cdot f_1 \cdot h)^{-1} \cdot \gamma (g \cdot f_1 \cdot h) = h^{-1} \cdot f_1^{-1} \cdot g^{-1} \cdot \gamma (g \cdot f_1) \cdot \gamma h = h^{-1} \cdot f_1^{-1} \cdot \gamma f_1 \cdot \gamma h.$$  

Thus $\alpha^{h}_{M^1,f_1} = \alpha^{h}_{M^2,f_2}$, so that $\alpha_{M^1,f_1}$ and $\alpha_{M^2,f_2}$ agree in $H^1(\Gamma, S_h)$. \hfill \Box

We have now constructed

$$\alpha : \pi_0 \text{LMod}_{D(\kappa)}^{v,h} \to H^1(\Gamma, S_h).$$

2.2.3. **Theorem.** The map $\alpha$ is an isomorphism.

**Proof.** Let us first check that $\alpha$ is injective. Suppose given $M' \in \text{LMod}_{D(\kappa)}^{v,h}$ and choose isomorphisms $f_\epsilon : H_{\kappa} \to M'_{\kappa}$ for $\epsilon \in \{1, 2\}$. Suppose that for some $h \in S_h$ we have $\alpha^{h}_{M^1,f_1} = \alpha^{h}_{M^2,f_2}$. Then for all $\gamma \in \Gamma$, the diagram

$$
\begin{array}{ccc}
H_{\kappa} & \xrightarrow{f_1} & M^1_{\kappa} \\
\downarrow h & & \downarrow g \\
H_{\kappa} & \xrightarrow{f_2} & M^2_{\kappa}
\end{array}
$$

with $g$ defined over $W(\kappa)$, we can calculate for $\gamma \in \Gamma$ that

$$f_2^{-1} \cdot \gamma f_2 = (g \cdot f_1 \cdot h)^{-1} \cdot \gamma (g \cdot f_1 \cdot h) = h^{-1} \cdot f_1^{-1} \cdot g^{-1} \cdot \gamma (g \cdot f_1) \cdot \gamma h = h^{-1} \cdot f_1^{-1} \cdot \gamma f_1 \cdot \gamma h.$$  

Thus $\alpha^{h}_{M^1,f_1} = \alpha^{h}_{M^2,f_2}$, so that $\alpha_{M^1,f_1}$ and $\alpha_{M^2,f_2}$ agree in $H^1(\Gamma, S_h)$. \hfill \Box

We have now constructed

$$\alpha : \pi_0 \text{LMod}_{D(\kappa)}^{v,h} \to H^1(\Gamma, S_h).$$
2.3. Case of a finite field. We maintain notation from the previous section. Suppose now \( \kappa = \mathbb{F}_p^r \) is a finite field. Then \( \Gamma = \mathbb{Z} \) is cyclically generated by \( \sigma^{-r} \). We learn that evaluation on \( \sigma^{-r} \) gives an isomorphism
\[
\text{classifying formal groups with Dieudonné theory 11}
\]
\[
\begin{array}{ccc}
H_{\pi} & \rightarrow & H_{\pi} \\
\downarrow f_2 & & \downarrow f_3 \\
M_{\pi}^2 & \rightarrow & M_{\pi}^1 \\
\uparrow f_2 & & \uparrow f_1 \\
H_{\pi} & \rightarrow & H_{\pi}
\end{array}
\]
commutes, telling us
\[
f_2 \cdot h \cdot f_1^{-1} = \gamma \left( f_2 \cdot h \cdot f_1^{-1} \right) : M_{\pi}^1 \cong M_{\pi}^2
\]
for all \( \gamma \in \Gamma \). So we may take fixed points with respect to \( \Gamma \) to obtain an isomorphism \( M_{\pi}^1 \cong M_{\pi}^2 \).

Let us now check \( \alpha \) is surjective. Fix \( \eta \in Z^1(\Gamma, S_h) \), which we can push forward to \( Z^1(\Gamma, \text{GL}_n(W(\pi))) \). By Theorem 2.1.2, we know that \( \eta(\gamma) = T^{-1} \cdot \gamma T \) in \( \text{GL}_n(W(\pi)) \) for all \( \gamma \in \Gamma \) and some matrix \( T \in \text{GL}_n(W(\pi)) \). Let \( N = W(\pi) \otimes_{W(\kappa)} W(\kappa) \{ y_0, \ldots, y_{k-1} \} \), and using the distinguished bases treat \( T \) as an isomorphism \( T : H_{\pi} \cong N \) of \( W(\pi) \)-modules. We can give \( N \) the structure of a left \( D(\pi) \)-module such that this is an isomorphism of left \( D(\pi) \)-modules; explicitly, if \( V = V_H : H_{\pi} \rightarrow H_{\pi} \), then \( V_N = T \circ V_H \circ T^{-1} \). Observe now that for all \( \gamma \in \Gamma \) we have
\[
\gamma V_N = \gamma (T \cdot V_H \cdot T^{-1})
\]
\[
= \gamma T \cdot \gamma V_H \cdot \gamma T^{-1}
\]
\[
= T \cdot \eta(\gamma) \cdot V_H \cdot \eta(\gamma)^{-1} \cdot T^{-1}
\]
\[
= T \cdot V_H \cdot T^{-1} = V_N,
\]
the last equality as \( \eta(\gamma) \in S_h \) commutes with \( F_H \). As \( V_N \) is \( \Gamma \)-equivariant, we can take fixed points with respect to \( \Gamma \) to obtain the left \( D(\kappa) \)-module \( M = N^\Gamma \). Evidently \( \alpha_{M,T} = \eta \), so this concludes the proof. \( \square \)

2.3. Case of a finite field. We maintain notation from the previous section. Suppose now \( \kappa = \mathbb{F}_{p^r} \) is a finite field. Then \( \Gamma = \mathbb{Z} \) is cyclically generated by \( \sigma^{-r} \). We learn that evaluation on \( \sigma^{-r} \) gives an isomorphism
\[
Z^1(\Gamma, S_h) \cong S_h.
\]
Thus \( H^1(\Gamma, S_h) \) is a quotient of \( S_h \), where two elements \( g_1, g_2 \in S_h \) are equivalent precisely when there is some \( h \in S_h \) such that
\[
g_2 = h^{-1} \cdot g_1 \cdot \sigma^{-r} h.
\]
If \( \kappa = \mathbb{F}_{p^h} \), so that \( \mathbb{F}_{p^h} \subset \kappa \), then \( \Gamma \) acts trivially on \( S_h \) and this description yields an isomorphism
\[
\alpha : \pi_0 \text{LMod}_{D(\kappa)}^{\pi,h} \cong S_h/(\text{conj}).
\]
Let \( M \in \text{LMod}_{D(\kappa)}^{\pi,h} \) with \( \kappa = \mathbb{F}_{p^r} \). Observe that over \( \mathbb{F}_{p^r} \), the \( \sigma^{-hr} \)-semilinear map \( V^{hr} \) is in fact linear. It thus induces a \( D(\pi) \)-linear map \( M_{\pi} \rightarrow M_{\pi} \), but this is not the map \( V^{hr} \) associated \( M_{\pi} \), when treating \( V^{hr} \) in this way, we will write \( V^{hr} = B_M \). Choose an isomorphism \( f : H_{\pi} \rightarrow M_{\pi} \). We can give now another description of the conjugacy class \( \alpha(M) \) associated to \( M \). By Corollary 1.3.3, we have a commutative diagram
This tells us that $f^{-1} \sigma^{-hr} f = f^{-1} \mu \cdot f$. So $f^{-1} \mu \cdot f \in \mathbb{S}_h$ is a representative of the conjugacy class associated to $M$. Moreover, $\mu$ is defined in $W(\mathbb{F}_{p^r})$: in particular, $V^{hr} = \mu^a p^r$, so that the conjugacy class associated to $M_{\mu,hr}^a$ is $f^{-1} \mu^a : f$.

As the conjugacy class of the identity of $\mathbb{S}_h$ is a singleton, we learn that $M \in \text{LMod}_{D(\mathbb{F}_{p^r})}^{\mu,hr}$ is isomorphic to $H$ if and only if $\mu = 1$ in the above, i.e. $V^{hr} = p^r$ on $M$.

2.3.1. Example. Consider the case $h = 1$. In this case, the action of $\Gamma$ on $\mathbb{S}_1$ is trivial and $\mathbb{S}_1 = \mathbb{Z}_p^\times$ is abelian, so an object $M \in \text{LMod}_{D(\mathbb{F}_{p^r})}^{\mu,1}$ is classified by an element of $\mathbb{Z}_p^\times$. Such an object $M$ can be written as $M = W(\mathbb{F}_{p^r}) \{y\}$ with $V y = a p^i$ for some $a \in W(\mathbb{F}_{p^r})^\times$. The above description tells us that the invariant of $M$ is the norm $N(a) = \prod_{i=0}^{r-1} a^\sigma$.

3. Determinants

3.1. Exterior powers. For $M \in \text{LMod}_{D(\kappa)}^{\mu}$, consider $\Lambda^n M$, the exterior power taken over $W(\kappa)$.

3.1.1. Theorem. The $W(\kappa)$-module $\Lambda^n M$ admits a natural left $D(\kappa)$-module structure determined by

$$V(a_1 \wedge \cdots \wedge a_n) = V a_1 \wedge \cdots \wedge V a_n,$$

$$F(a_1 \wedge \cdots \wedge a_n) = p^{−(n−1)} Fa_1 \wedge \cdots \wedge Fa_n.$$

Proof. This is evidently natural; we must only verify that $F$ can be defined this way. Choose $y \in M$ so that $M \cong W(\kappa) \{y, Vy, \ldots, V^{h-1}y\}$. Then a basis for $\Lambda^n M$ is given by the elements $V_i y = V^{i_1}y \wedge \cdots \wedge V^{i_n} y$ with $I = \{i_1 < \cdots < i_n\} \subset \{0, \ldots, h-1\}$. As $F V = p$ in $M$, we find that with the above definition $F(V_i y)$ lives in $\Lambda^n M \subset \mathbb{Q} \otimes \Lambda^n M$. \hfill \Box

3.1.2. Example. Consider the case $M = H$. Then $\Lambda^n H$ has a basis consisting of the elements $V_i x$ as above, and we can identify

$$V(V^{i_1} x \wedge \cdots \wedge V^{i_n} x) = \begin{cases} V^{i_1+1} x \wedge \cdots \wedge V^{i_n+1} x, & i_n \neq h - 1, \\ (-1)^{n-1} p x \wedge V^{i_1+1} x \wedge \cdots \wedge V^{i_n+1} x, & i_n = h - 1; \end{cases}$$

$$F(V^{i_1} y \wedge \cdots \wedge V^{i_n} y) = \begin{cases} pV^{i_1-1} y \wedge \cdots \wedge V^{i_n-1} y, & i_1 \neq 0, \\ (-1)^{n-1} V^{i_2-1} x \wedge \cdots \wedge V^{i_n-1} x \wedge V^{h-1} x, & i_1 = 0. \end{cases}$$

Fix $M \in \text{LMod}_{D(\kappa)}^{\mu,h}$. For general $n$, we cannot say that $\Lambda^n M$ lives in $\text{LMod}_{D(\kappa)}^{\mu,h}$; rather, it turns out that $\Lambda^n M$ is associated to a connected $p$-divisible group of
height \( \binom{h}{n} \) and dimension \( \binom{h-1}{n-1} \). In the case \( h = n \), one can check directly that \( \Lambda^h M \in \text{LMod}^{v,1}_{D(\kappa)} \). We obtain a functor
\[
\Lambda^h : \text{LMod}^{v,h}_{D(\kappa)} \to \text{LMod}^{v,1}_{D(\kappa)},
\]
and by Theorem 2.2.3 this passes to a map
\[
\lambda : H^1(\Gamma, \mathcal{S}_h) \to H^1(\Gamma, \mathcal{S}_1),
\]
where \( \Gamma \) is the absolute Galois group of \( \kappa \). By Proposition 1.2.1, we can identify \( \mathcal{S}_1 = \mathbb{Z}_p^n \), and this has trivial action by \( \Gamma \). We can thus identify \( H^1(\Gamma, \mathcal{S}_1) = \text{Hom}(\Gamma, \mathbb{Z}_p^n) \), so that \( \lambda \) gives \( \text{Hom}_\Gamma(\Gamma, S \times \Gamma) \to \text{Hom}(\Gamma, \mathbb{Z}_p^n) \).

Say now \( \kappa = \mathbb{F}_p^r \) is a finite field, and fix \( M \in \text{LMod}^{h}_{D(\mathbb{F}_p^r)} \). We have now \( H^1(\Gamma, \mathcal{S}_1) = \mathbb{Z}_p^n \). Let us describe the resulting element \( \lambda(M) \in \mathbb{Z}_p^n \). As in Example 2.3.1, we can write \( V^r = \Lambda^h V^r : \Lambda^h M \to \Lambda^h M \) as \( V^r = p^r a \) for some \( a \in W(\mathbb{F}_p^r)^\times \), and then \( \lambda(M) = N(a) = \prod_{i=0}^{r-1} a_i \). Choose now \( y \in M \) such that \( M \cong W(\mathbb{F}_p^r)\{y, V y, \ldots, V^{h-1} y\} \), and write \( V^h y = \sum_{i=0}^{h-1} p a_i V^i y \) with \( a_0 \in W(\mathbb{F}_p^r)^\times \).

3.1.3. Proposition. With notation as above, we have
\[
a = \lambda(M) = (-1)^{r(h-1)} N(a_0) = (-1)^{r(h-1)} \prod_{i=0}^{r-1} a_i.
\]

Proof. Continuing the previous discussion, we have \( \Lambda^h M \cong W(\mathbb{F}_p^r)\{y \wedge \cdots \wedge V^{h-1} y\} \) and \( V^h y = \sum_{i=0}^{h-1} a_i V^i y \). Observe that
\[
V (y \wedge V y \wedge \cdots \wedge V^{h-1} y) = V y \wedge V^2 \wedge \cdots \wedge V^{h-1} y \wedge V^h y
= \sum_{i=0}^{h-1} V y \wedge \cdots \wedge V^{h-1} y \wedge p a_i V^i y
= V y \wedge \cdots \wedge V^{h-1} y \wedge a_0 y
= (-1)^{h-1} p a_0 y \wedge V y \wedge \cdots \wedge V^{h-1} y.
\]
Continuing in this manner, we learn
\[
V^r (y \wedge V y \wedge \cdots \wedge V^{h-1} y) = p^r (-1)^{r(h-1)} a_0^{-r-1} \cdots a_0^{-1} a_0 y \wedge V y \wedge \cdots \wedge V^{h-1} y
= p^r (-1)^{r(h-1)} N(a_0) y \wedge V y \wedge \cdots \wedge V^{h-1} y,
\]
so that \( a = (-1)^{r(h-1)} N(a_0) \) as stated.

3.2. Classical determinant. Fix a perfect field \( \kappa \) containing all \( (p^h-1) \) th roots of unity, and write \( \mathbb{F}_p^r \subset \kappa \) for the subfield generated by these elements. Write still \( H = W(\kappa)\{x, V x, \ldots, V^{h-1} x\} \) with \( V^{h-1} x = px \). Then \( \text{End}_{D(\kappa)}(H) \) was identified in Proposition 1.2.1, and is independent of the choice of such \( \kappa \). Moreover, we have an injection \( \text{End}_{D(\kappa)}(H) \to \text{End}_{W(\kappa)}(W(\kappa)^\oplus h) \) which lands in the image of the induction map \( \text{End}_{W(\mathbb{F}_p^r)}(W(\mathbb{F}_p^r)^\oplus h) \subset \text{End}_{W(\kappa)}(W(\kappa)^\oplus h) \). Explicitly, for \( a = \sum_{i=0}^{h-1} a_i V^i \) in \( \text{End}_{D(\kappa)}(H) \), we have \( a_i \in W(\mathbb{F}_p^r) \) for each \( a_i \) and the matrix
associated to the image of \( a \) in \( \text{End}_{W(F_{p^h})}(W(F_{p^h})^{\oplus h}) \) is

\[
\begin{pmatrix}
a_0 & pa_{h-1}^{-1} & pa_{h-2}^{-1} & \cdots & pa_{1}^{-1} \\
a_1 & a_{h-1}^{-1} & pa_{h-2}^{-1} & \cdots & pa_{2}^{-1} \\
a_2 & a_{h-1}^{-1} & a_{0}^{-1} & \cdots & pa_{3}^{-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{h-2} & a_{h-3}^{-1} & a_{h-2}^{-1} & \cdots & pa_{h-1}^{-1} \\
a_{h-1} & a_{h-2}^{-1} & a_{h-3}^{-1} & \cdots & a_{0}^{-1}
\end{pmatrix}^{(h-1)}
\]

Define now

\[
d : \text{End}_{D(\kappa)}(H) \to \text{End}_{W(F_{p^h})}(W(F_{p^h})^{\oplus h}) \xrightarrow{\det} W(F_{p^h}).
\]

This is a homomorphism, with \( W(F_{p^h}) \) given its multiplicative monoid structure.

3.2.1. Proposition. The image of \( d \) lies in \( W(F_{p^h}) \subset W(F_{p^h}) \).

Proof. One can check that \( p^{-1} \text{End}_{D(\kappa)}(H) \) is a well defined algebra, and we have a commutative diagram

\[
\begin{CD}
\text{End}_{D(\kappa)}(H) @>>> p^{-1} \text{End}_{D(\kappa)}(H) \\
@VVdV @VVd^{-1}V \\
\text{End}_{W(F_{p^h})}(W(F_{p^h})^{\oplus h}) @>>> \text{End}_{\mathbb{Q} \otimes W(F_{p^h})}(\mathbb{Q} \otimes W(F_{p^h})^{\oplus h})
\end{CD}
\]

of injections. Fix \( a \in \text{End}_{D(\kappa)}(H) \). Observe that inverting \( p \) also inverts \( V \), and we can identify \( V a V^{-1} = a^{-1} \). Thus \( d(a) = d(V a V^{-1}) = d(a^{-1}) = d(a) a^{-1} \), so that \( d(a) \in W(F_{p^h}) \).

3.2.2. Remark. Let \( T \in \text{End}_{W(\kappa)}(W(\kappa)^{\oplus h}) \) be associated to some \( a \in \text{End}_{D(\kappa)}(H) \). Then the coefficients of the characteristic polynomial \( q(t) \) of \( T \) all lie in \( W(F_{p^h}) \).

Indeed, we can write

\[
q(t) = \sum_{i=0}^{h} (-1)^i \text{Tr}(\Lambda_i T) t^{h-i},
\]

and as trace is invariant under cyclic permutations we can argue as above.

More generally, fix \( M \in \text{LMod}_{D(\kappa)}^{v,h} \) and a \( D(\kappa) \)-linear endomorphism \( g : M \to M \). Let \( q(t) \) be the characteristic polynomial of \( g \) considered as a \( W(\kappa) \)-linear endomorphism. Then the coefficients of \( q(t) \) lie in \( W(F_{p^h}) \subset W(\kappa) \). Indeed, \( g \) induces a \( D(\varpi) \)-linear endomorphism \( M_{\varpi} \to M_{\varpi} \), and we know \( M_{\varpi} \cong H_{\varpi} \); so we can appeal to the fact that the characteristic polynomial of a linear map is independent of choice of basis.

We obtain the determinant mapping \( d : S_h \to W(F_{p^h})^\times = \mathbb{Z}_p^\times \).

3.2.3. Theorem. Fix \( M \in \text{LMod}_{D(\kappa)}^{v,h} \), and let \( \alpha(M) \in S_h \) represent the invariant in \( H^1(\Gamma, S_h) \) associated to \( M \). Let \( \lambda(M) \in S_1 = \mathbb{Z}_p^\times \) be the invariant associated to \( \Lambda^h M \). Then

\[
\lambda(M) = (-1)^{r(h-1)} d(\alpha(M)).
\]
Proof. Write $\pi = F_p$ and choose an isomorphism $f : H_\pi \to M_\pi$ so that we can say $\alpha(M)$ is represented by $f^{-1} \cdot \pi^{-r}$. By the definition of ordinary determinants, we can identify $d(f^{-1} \cdot \pi^{-r}) = \Lambda^h(f^{-1}) \cdot \Lambda^h(\pi^{-r})$ as automorphisms of $\Lambda^h H_\pi \cong W(\pi)$. By Corollary 1.3.3 and Proposition 3.1.3, the diagram
\[
\begin{array}{ccc}
\Lambda^h H_\pi & \stackrel{\rho^r}{\longrightarrow} & \Lambda^h H_\pi ^{(1-r)} \\
\downarrow f & & \downarrow \Lambda^h (f^{-1} \cdot \pi^{-r}) \\
\Lambda^h M_\pi & \stackrel{\rho^r}{\longrightarrow} & \Lambda^h M_\pi ^{(1-r)}
\end{array}
\]
commutes. This yields the result. □

3.2.4. Corollary. Fix notation as above. Choose $y \in M$ giving an isomorphism $M \cong W(\pi) \{y, Vy, \ldots, V^{h-1}y\}$, and write $V^h y = \sum \pi^{-1} y$. Then $d(\alpha(M)) = N(a_0) = \prod \pi^{-1} a_0^r$.

Proof. Combine Theorem 3.2.3 and Proposition 3.1.3. □

3.2.5. Remark. The proof of Theorem 3.2.3 did not rely on Proposition 3.2.1, and so gives a second proof of the fact that $d : S_h \to W(\pi) \times$ lands in $\mathbb{Z}_p^\times$. □

3.2.6. Example. Suppose $h$ is even, and write $H = W(\pi) \{x, Vx, \ldots, V^{h-1}x\}$ with $V^h x = px$. Let $M = \Lambda^h H$. Then we can identify $M = W(\pi) \{y\}$ with $V y = -py$. Let $H^1 = W(\pi) \{x\}$ with $V x = px$. Then an isomorphism $f : H^1 \to M$ is given by $a \in W(\pi) \times$ satisfying
\[
a^\sigma + a = 0.
\]

We have a Galois extension $W(\pi) \subset W(\pi^2)$, and the resulting trace map is
\[
W(\pi^2) \to W(\pi), \quad a \mapsto a^\sigma + a.
\]

The kernel is a free one-dimensional $W(\pi)$-module, and any element therein not divisible by $p$ thus gives an isomorphism $H^1_{\pi^2} \to M_{\pi^2}$. We learn that $M_{\pi^2} \cong H^1_{\pi^2}$ if and only if $r$ is even. □