CLASSIFYING FORMAL GROUPS WITH DIEUDONNÉ THEORY

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Abstract. Dieudonné theory gives an equivalence between one-dimensional commutative formal groups over a perfect field and certain modules over a certain ring. One can then attempt to develop the classic theory of such formal groups purely within the category of these modules, as this note does. At a technical level, this replaces power series manipulations with Frobenius-semilinear algebra. The motivation for this exercise was to obtain a clean formulation and proof of Theorem 3.2.3.

Contents

1. Basic properties of formal groups

1.1. **Dieudonné modules.** Fix a perfect field κ of positive characteristic p. Let $W(\kappa)$ denote the ring of p-typical Witt vectors on κ, and let $D(\kappa)$ be the Dieudonne ring of κ. This is the noncommutative ring obtained from $W(\kappa)$ by adjoining two symbols F and V subject to

$$F\lambda = \lambda^\sigma F, \qquad V\lambda = \lambda^{\sigma^{-1}} V, \qquad FV = p = VF,$$

where λ ranges over $W(\kappa)$ and $(-)^\sigma$ is the canonical Frobenius automorphism of $W(\kappa)$. Let $\mathrm{L}\mathcal{M}\mathrm{od}^v_{D(\kappa)}$ be the category of left $D(\kappa)$-module M subject to the following conditions:

1. As a $W(\kappa)$-module, M is free of finite and positive rank,
2. The action of V on M is topologically nilpotent, and $M = \lim_k M/V^k M$,
3. The quotient M/VM is a simple $W(\kappa)$-module, i.e. $M/VM \simeq \kappa$.

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Observe that the condition $FV = p = VF$ tells us that for $M \in \mathrm{L\mathcal{M}od}^v_{D(\kappa)}$, the operators F and V determine each other. In fact, we need not have introduced F at all; see Corollary 1.3.4. The main Dieudonné theorem relevant to us is the following.

1.1.1. **Theorem.** The category $\mathrm{L\mathcal{M}od}^v_{D(\kappa)}$ is equivalent to the category of one-dimensional commutative formal groups over $\operatorname{Spec} \kappa$. □

We will not need the details of the construction of this equivalence as we will be working entirely within $\mathrm{L\mathcal{M}od}^v_{D(\kappa)}$, but let us say something to establish where our conventions sit. Let $\mathcal{H}\mathrm{opf}^c_\kappa$ denote the category of Hopf algebras H over κ such that $\operatorname{Spf} H^\vee$ is a connected formal group. There is an equivalence DM from $\mathcal{H}\mathrm{opf}^c_\kappa$ to the category of left $D(\kappa)$-modules M such that for any $a \in M$, there is some n such that $V^n a = 0$. Suppose now $\operatorname{Spf} H^\vee$ is a connected p-divisible group. Then $H \in \mathcal{H}\mathrm{opf}^c_\kappa$, so we can look at $\mathrm{DM}(H) \in \mathrm{L\mathcal{M}od}_{D(\kappa)}$. As H is a p-divisible Hopf algebra, $\mathrm{DM}(H)$ is p-divisible, and we can noncanonically identify

$$\mathrm{DM}(H) \cong W(\kappa)/(p^\infty)^{\oplus h}, \qquad \ker[V\colon \mathrm{DM}(H) \to \mathrm{DM}(H)] \cong \kappa^{\oplus d},$$

where h is the height and d is the dimension of $\operatorname{Spf} H$. Let $\mathrm{DM}(H)[p^n] \subset \mathrm{DM}(H)$ be the p^n-torsion subgroup. Multiplication by p gives maps $\mathrm{DM}(H)[p^n] \to \mathrm{DM}(H)[p^{n-1}]$, and we define $\widehat{\mathrm{DM}}(H) = \lim_n \mathrm{DM}(H)[p^n]$. Under the above noncanonical isomorphism, we identify $\mathrm{DM}(H)[p^n] \to \mathrm{DM}(H)[p^{n-1}]$ as the quotient map $W(\kappa)/(p^n)^{\oplus h} \to W(\kappa)/(p^{n-1})^{\oplus h}$, and can then identify

$$\widehat{\mathrm{DM}}(H) \cong W(\kappa)^{\oplus h}, \qquad \widehat{\mathrm{DM}}(H)/V\widehat{\mathrm{DM}}(H) \cong \kappa^{\oplus d}.$$

Specializing to the case where $d = 1$, the functor $\operatorname{Spf} H^\vee \mapsto \widehat{\mathrm{DM}}(H)$ is our equivalence.

1.1.2. **Remark.** The Serre dual of $\mathrm{L\mathcal{M}od}^v_{D(\kappa)}$ is the category $\mathrm{L\mathcal{M}od}^f_{D(\kappa)}$, defined in the same way except with F in place of V. For $M \in \mathrm{L\mathcal{M}od}^v_{D(\kappa)}$, we have $\mathrm{Hom}_{W(\kappa)}(M, W(\kappa)) \in \mathrm{L\mathcal{M}od}^f_{D(\kappa)}$, where we give $\mathrm{Hom}_{W(\kappa)}(M, W(\kappa))$ the $D(\kappa)$-module structure

$$(Vf)(a) = f(Fa), \qquad (Ff)(a) = f(Va)$$

for $f\colon M \to W(\kappa)$. This gives a duality between $\mathrm{L\mathcal{M}od}^v_{D(\kappa)}$ and $\mathrm{L\mathcal{M}od}^f_{D(\kappa)}$. ◁

1.2. **The Honda module.** Fix a positive integer h and let $\mathrm{L\mathcal{M}od}^{v,h}_{D(\kappa)} \subset \mathrm{L\mathcal{M}od}^v_{D(\kappa)}$ be the full subcategory on those objects free of rank h over $W(\kappa)$. There is a distinguished object $H \in \mathrm{L\mathcal{M}od}^{h,v}_{D(\kappa)}$ given by

$$H = W(\kappa)\{x, Vx, \ldots, V^{h-1}x\}, \qquad V^h x = px, \qquad Fx = V^{h-1}x.$$

This is the free $D(\kappa)$-module on a generator x subject to the condition $V^h x = px$. We obtain for any $M \in \mathrm{L\mathcal{M}od}^v_{D(\kappa)}$ an identification

$$\mathrm{Hom}_{D(\kappa)}(H, M) \cong \{a \in M : V^h a = pa\}, \qquad f \mapsto f(x).$$

1.2.1. **Proposition.** Suppose that κ contains all $(p^h - 1)$'th roots of unity, and let $\mathbb{F}_{p^h} \subset \kappa$ be the subfield generated by these. Then there is a canonical identification

$$\mathrm{End}_{D(\kappa)}(H) = W(\mathbb{F}_{p^h})\langle V\rangle/(V^h = p,\, Va = a^{\sigma^{-1}}V\},$$

where a ranges through $W(\mathbb{F}_{p^h})$.

Proof. As above, identify $\mathrm{End}_{D(\kappa)}(H) = \{a \in H : F^h a = pa\}$. Write an arbitrary element $a \in H$ as $a = \sum_{i=0}^{h-1} a_i V^i$. Then

$$F^h a = \sum_{i=0}^{h-1} a_i^{\sigma^{-h}} V^{h+i} x = \sum_{i=0}^{h-1} a_i^{\sigma^{-h}} p V^i x.$$

We find that $F^h a = pa$ precisely when $a_i^{\sigma^{-h}} = a_i$ for each i, which holds precisely when each $a_i \in W(\mathbb{F}_{p^h})$. □

1.2.2. **Remark.** The object more commonly seen in the study of formal groups is $E = W(\mathbb{F}_{p^h})\langle F\rangle/(F^h = p,\, Fa = a^\sigma F)$. There is an isomorphism

$$\mathrm{End}_{D(\kappa)}(H) \to E^{\mathrm{op}}, \qquad \sum_{i=0}^{h-1} a_i V^i \mapsto \sum_{i=0}^{h-1} F^i a_i.$$

This corresponds to Remark 1.1.2. ◁

Let $\mathbb{S}_h = \mathrm{Aut}_{D(\kappa)}(H)$. As underlying H is a free module over $W(\kappa)$ of rank h and with distinguished basis, we have canonical inclusions

$$\mathrm{End}_{D(\kappa)}(H) \subset \mathrm{End}_{W(\kappa)}(W(\kappa)^{\oplus h}), \qquad \mathbb{S}_h \subset \mathrm{GL}_h(W(\kappa)).$$

We can give $\mathrm{End}_{D(\kappa)}(H)$, and thus $\mathbb{S}_h$, the topology arising from these inclusions and the p-adic topology on $W(\kappa)$. As h is finite, this is equivalent to the topology induced from the filtration on $\mathrm{End}_{D(\kappa)}$ by powers of V, and makes $\mathrm{End}_{D(\kappa)}(H)$ into a profinite ring and $\mathbb{S}_h$ into a profinite group.

We will prove the following theorem in the next section.

1.2.3. **Theorem.** Suppose κ is algebraically closed. Then every $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{D(\kappa)}^{v,h}$ is isomorphic to H.

We can rephrase this as follows. For a category $\mathcal{C}$, let $\pi_0\mathcal{C}$ denote the set of isomorphism classes of objects of $\mathcal{C}$. Then the previous theorem states

$$\pi_0 \mathrm{L}\mathcal{M}\mathrm{od}_{D(\kappa)}^{v,h} = \{H\},$$

so long as κ is algebraically closed. We will later leverage this into an identification of $\pi_0 \mathrm{L}\mathcal{M}\mathrm{od}_{D(\kappa)}^{v,h}$ for a general perfect field κ.

1.3. **Semilinear algebra.** Let R be a ring, let $\phi\colon R \to R$ an endomorphism, and let M and N be R-modules. For $r \in R$, we may write $\phi(r) = {}^\phi r$. A ϕ-semilinear map $f\colon M \to N$ is defined as a function satisfying

$$f(m+n) = f(m) + f(n), \qquad f(rm) = {}^{\phi)} f(m).$$

Given such f, define $N^{(\phi)} = R_\phi \otimes_R N$. Then

$$M \to N^{(\phi)}, \qquad m \mapsto 1 \otimes f(m)$$

is an R-linear map. Of particular interest is the case where $R = \kappa$ is a perfect field, $\phi = \sigma^{\pm 1}$, and $M = N$ is a finite-dimensional vector space. It turns out that in this case M is the sum of an f-nilpotent subspace and a subspace generated by f-fixed elements; we will not need this exact fact, but mention it as it provides some motivation for the following results. We first deal with the case of certain nilpotent operators.

1.3.1. **Lemma.** Let k be a field, and ϕ an endomorphism of k. Let U be a rank h vector space over k, and $V: U \to U$ a ϕ-semilinear operator. Suppose that V is nilpotent, and $U/VU \cong k$. Then there is $x \in U$ such that $x, Vx, \ldots, V^{h-1}x$ give a basis for U.

Proof. Set $U_0 = U$, and inductively define U_n to be the image of $U_{n-1} \to U^{(\phi^n)}$. Our assumptions give us a sequence

$$U_0 \to U_1 \to \cdots \to U_{h-1} \to U_h = 0$$

of surjective k-linear maps, each with one-dimensional kernel. In particular, the ϕ-semilinear operator V satisfies $V^{h-1} \neq 0$ and $V^h = 0$. Let $x \in U$ be such that $V^{h-1}x \neq 0$. It is sufficient to show $x, Vx, \ldots, V^{h-1}x$ are linearly independent. Indeed, a linear relation of the form $\sum_{i=l}^{h-1} \lambda_i V^i x = 0$ yields, upon application of V^{h-1-l}, the identity ${}^{\phi^{h-1-l}}\lambda_l V^{h-1}x = 0$, so that $\lambda_l = 0$ as ϕ is necessarily injective. □

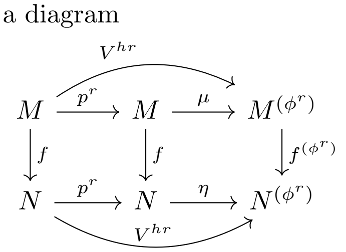
1.3.2. **Proposition.** Let k be a field, ϕ an endomorphism of k, and M a free module over $W(k)$ of finite rank h equipped with a ϕ-semilinear topologically nilpotent endomorphism V satisfying $M/VM \cong k$. Then there exists $y \in M$ such that $V^{h-1}y \neq 0$ in $M/(p)$. Moreover, for any such y, we have

$$p^n M \cong W(k)\{V^{nh}y, V^{nh+1}y, \ldots, V^{nh+h-1}y\}$$

for each n. In particular, $V^h y = \sum_{i=0}^{h-1} pa_i V^i y$ with $a_0 \in W(k)^\times$, and $p^{-1}V^h$ is a ϕ-semilinear automorphism of M.

Proof. By Lemma 1.3.1, we find $\overline{y} \in M/(p)$ such that $\overline{y}, V\overline{y}, \ldots, V^{h-1}\overline{y}$ form a basis for $M/(p)$. Choose a lift y over $\overline{y}$ to M. Then $y, Vy, \ldots, V^{h-1}y$ form a basis for M, and $V^h y, \ldots, V^{2h-1}y$ lie in pM. By repeating our arguments with each $p^n M$, it is sufficient to verify that these form a basis for pM. Out of $y, \ldots, V^{h-1}y$, only y is not in the image of V; thus when we expand $V^h y = \sum_{i=0}^{h-1} pa_i V^i y$, the element $a_0 \in W(k)$ must be unit. We learn that for $0 \leq i \leq h-1$, the expansion of $V^{h+i}y$ in our basis mod p^2 is of the form $pb_i V^i y + \cdots + pb_{h-1}V^{h-1}y$ with b_i a unit. It is easily seen from this that $V^h y, \ldots, V^{2h-1}y$ project to a basis for pM/p^2M, and thus form a basis of pM. □

1.3.3. **Corollary.** Let k and M be as above, and moreover let N be of the same type as M. Suppose given a $W(k)$-linear map $f: M \to N$ which commutes with V. Then for each $r \geq 0$, there are unique automorphisms $\mu: M \to M^{(\phi^r)}$ and $\eta: N \to N^{(\phi^r)}$ fitting into a diagram



of $W(\kappa)$-linear maps that commute with V. Here, $f^{(\phi^r)}$ indicates functoriality of $(-)^{(\phi^r)}$ applied to f. □

1.3.4. **Corollary.** Let κ be a perfect field, and M a free module over $W(\kappa)$ of finite rank h equipped with a σ^{-1}-semilinear topologically endomorphism V satisfying $M/VM \cong \kappa$. Then M can be uniquely upgraded to an object of $\mathrm{L}\mathcal{M}\mathrm{od}^{v}_{D(\kappa)}$.

Proof. As $VF = p = FV$ and M is torsion-free, we see that there is at most one choice of F. Choose $y \in M$ such that $M \cong W(\kappa)\{y, Vy, \ldots, V^{h-1}y\}$. We are forced to define $FV^i y = pV^{i-1}y$ for $1 \leq i \leq h-1$. As $M/VM \cong \kappa$, necessarily $py = Va$ for some $a \in M$, and we are forced to define $Fy = a$. This description on the basis $y, Vy, \ldots, V^{h-1}y$ extends by semilinearity to a description on all of M. □

1.3.5. **Proposition.** Let κ be an algebraically closed field, and U a finite-dimensional vector space over κ. Let $T\colon U \to U$ be a σ^{-1}-semilinear automorphism. Then there exists some nonzero $x \in U$ which is fixed by T.

Proof. By replacing T with T^{-1}, we may instead deal with a σ-linear automorphism. Let us work for now with a general ϕ-semilinear automorphism $T\colon U \to U$, where ϕ is some endomorphism of k.

Pick $y \in U$. As U is finite-dimensional, there is some minimal n such that $T^{n+1}y$ is in the span of $y, \ldots, T^n y$; write $T^{n+1}y = \sum_{i=0}^{n} a_i T^i y$. As T is an automorphism, not all a_i are zero. After possibly replacing y with some $T^i y$, we may suppose $a_0 \neq 0$, so that $y, Ty, \ldots, T^n y$ are linearly independent. For indeterminates $\lambda_0, \ldots, \lambda_n$, write

$$x = \sum_{i=0}^{n} \lambda_i T^i y.$$

Then solving $Tx = x$ amounts to solving

$$(\lambda_0 - {}^{\phi}\lambda_n a_0)y + (\lambda_1 - {}^{\phi}\lambda_0 - {}^{\phi}\lambda_n a_1)Ty + \cdots + (\lambda_n - {}^{\phi}\lambda_{n-1} - {}^{\phi}\lambda_n a_n)T^n y = 0.$$

As $y, Ty, \ldots, T^n y$ are linearly independent, this is equivalent to asking each coefficient in parentheses to vanish. Starting with $\lambda_0 = \lambda_n^{\phi} a_0$, we can substitute each equation into the next, allowing us to reduce to solving a single equation. Writing $\lambda = \lambda_n$ and $b_i = {}^{\phi^i} a_{n-i}$, this comes out to solving

$$\lambda = {}^{\phi^{n+1}}\lambda b_n + {}^{\phi^n}\lambda b_{n-1} + \cdots + {}^{\phi}\lambda b_0.$$

In our case, $\phi = \sigma$. Recall also $b_0, \ldots, b_n$ are not all zero. So we are reduced to finding a nonzero root of the nontrivial separable equation

$$t = t^{p^{n+1}} b_n + t^{p^n} b_{n-1} + \cdots + t^p b_0,$$

which can be done under our assumption that k is algebraically closed. □

We would like to lift the previous proposition from κ-vector spaces to suitable $W(\kappa)$-modules.

1.3.6. **Lemma.** Let M be a $W(\kappa)$-module which is locally p-power torsion, and let $T\colon M \to M$ be a σ^{-1}-semilinear map. Let $M^T = \{a \in M : Ta = a\}$, which is a subgroup of M. Then the resulting map $W(\kappa) \otimes M^T \to M$ is injective.

Proof. Let $M_0^T \subset M^T$ be a finitely generated subgroup; it will suffice to show that $W(\kappa) \otimes M_0^T \to M$ is injective. Write

$$M_0^T = \bigoplus_{1 \leq i \leq k} \mathbb{Z}/(p^{t_i}),$$

so that $M_0^T \subset M$ is given by the choice of $a_1, \ldots, a_k \in M$ satisfying $Ta_i = a_i$ and $p^{t_i} a_i = 0$. Suppose towards a contradiction we had

$$\lambda_1 a_1 + \cdots + \lambda_k a_k = 0$$

for some $\lambda_i \in W(\kappa)/(p^{t_i})$ with not all λ_i zero. We can suppose this relation is chosen so that each λ_i is nonzero, and k is minimal so that such a relation holds. Moreover, we may suppose $\lambda_1 = p^c$ for some integer c. Applying T to the above gives

$$p^c a_1 + \cdots + \lambda_k^{\sigma^{-1}} a_k = 0.$$

Subtracting these equations yields

$$(\lambda_2 - \lambda_2^{\sigma^{-1}})a_2 + \cdots + (\lambda_k - \lambda_k^{\sigma^{-1}})a_k = 0,$$

so that $\lambda_i = \lambda_i^{\sigma^{-1}}$ for each i by minimality of k. But then $\lambda_1 a_1 + \cdots + \lambda_k a_k \in \mathbb{Z}_p \otimes M^T = M^T$, and certainly $M^T \subset M$ is injective. □

1.3.7. **Lemma.** Suppose κ is algebraically closed. Let M be an Artinian $W(\kappa)$-module, and $T\colon M \to M$ a σ^{-1}-semilinear injection. Then M is generated by $M^T = \{x \in M : Tx = x\}$ over $W(\kappa)$.

Proof. By considering the induced $W(\kappa)$-linear injection $M \to M^{(\sigma^{-1})}$ between Artinian $W(\kappa)$-modules of equal length, we find that T is an isomorphism. Let $M' \subset M$ be the $W(\kappa)$-submodule generated by M^T, and let $M'' = M/M'$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle T'-1} & & \big\downarrow{\scriptstyle T-1} & & \big\downarrow{\scriptstyle T''-1} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}.$$

As κ is algebraically closed, $\sigma^{-1} - \mathrm{id}\colon \kappa \to \kappa$ is surjective. By induction on composition series, we learn that $\sigma^{-1} - 1\colon W(\kappa) \otimes A \to W(\kappa) \otimes A$ is a surjection for any finite abelian group A. In case $A = M^T$, under the isomorphism $W(\kappa) \otimes M^T \cong M'$ given by Lemma 1.3.6 we find that $T - 1\colon M' \to M'$ is surjective. As $\ker(T' - 1) \cong \ker(T - 1)$ by construction, the long exact sequence associated to the above diagram tells us that $T'' - 1$ is an injective endomorphism of M''. So T'' is a σ^{-1}-semilinear automorphism of M'' with no fixed points. The same then must be true of T'' acting on the p-torsion submodule of M'', contradicting Proposition 1.3.5. □

Putting the previous two lemmata together, we learn that under the assumptions and notation of the previous lemma, the map $W(\kappa) \otimes M^T \to M$ is an isomorphism.

1.3.8. **Lemma.** Let

$$I_1 \xleftarrow{j_1} I_2 \xleftarrow{j_2} I_3 \xleftarrow{j_3} \cdots$$

be a tower of p-complete Hausdorff abelian groups, and suppose $j_n(I_{n+1}) \subset pI_n$ for each n. Then

$$\lim_n I_n = 0 = \lim_n{}^1 I_n.$$

Proof. The assertion that $\lim_n I_n = 0$ is immediate, so let us verify $\lim^1_n I_n = 0$. Recall that $\lim^1_n I_n$ can be identified as the cokernel of

$$\prod_n I_n \to \prod_n I_n, \qquad (a_n)_n \mapsto (a_n - j_n(a_{n+1}))_n,$$

so we must show this map is surjective. Indeed, fix an element $(b_n)_n \in \prod_n I_n$. Then $(b_n)_n = (a_n - j_n(a_{n+1}))_n$ has the solution

$$a_n = \sum_{k \geq n} (j_n \circ \cdots \circ j_{k-1})(b_k),$$

which converges under our assumptions. □

1.3.9. **Lemma.** Suppose given $W(\kappa)$-modules $N_1, N_2, \ldots$ fitting into exact sequences

$$N_{n+1} \xrightarrow{p^n} N_{n+1} \longrightarrow N_n \longrightarrow 0\ ,$$

and suppose N_1 is a finite dimensional vector space over κ. Let $N = \lim_n N_n$. Then N is a finitely generated $W(\kappa)$-module, and $N/(p^n) = N_n$.

Proof. The assumptions give for all n an exact sequence of towers whose m'th level is

$$N_{n+m} \xrightarrow{p^n} N_{n+m} \longrightarrow N_n \longrightarrow 0\ .$$

The leftmost towers consist of surjections, so by the Mittag-Leffler condition for vanishing $\lim^1$ we obtain in the limit an exact sequence

$$N \xrightarrow{p^n} N \longrightarrow N_n \longrightarrow 0$$

telling us that $N/(p^n) = N_n$. We learn that $N = \lim_n N/(p^n)$ is a p-complete Hausdorff $W(\kappa)$-module such that $N/(p)$ is a finite-dimensional κ-vector space. It follows by the Nakayama lemma for complete Hausdorff modules that N is finitely generated over $W(\kappa)$. □

1.3.10. **Proposition.** Let κ be an algebraically closed field. Let M be an $W(\kappa)$-module free of finite rank h and $T\colon M \to M$ a σ^{-1}-semilinear automorphism. Let $M^T = \{a \in M : Ta = a\}$. Then $W(\kappa) \widehat{\otimes} M^T \cong M$, where $\widehat{\otimes}$ is taken with respect to the p-adic topology.

Proof. For each $n \geq 1$, let $M^T_n = \{a \in M : Ta \equiv a \pmod{p^n}\}$. We obtain a tower $M^T_1 \leftarrow M^T_2 \leftarrow \cdots$ of inclusions, and $M^T = \lim_n M^T_n$. Let $I_n = p^n M \cap M^T_n$, and observe $M^T_n / I_n = M/(p^n)^T$. Combining this with Lemma 1.3.6 and Lemma 1.3.7 we learn

$$W(\kappa) \otimes M^T_n / I_n \cong M/(p^n).$$

In particular, this gives us exact sequences

$$W(\kappa) \otimes M^T_{n+1}/I_{n+1} \xrightarrow{p^n} W(\kappa) \otimes M^T_{n+1}/I_{n+1} \longrightarrow W(\kappa) \otimes M^T_n/I_n \longrightarrow 0,$$

and as $\mathbb{Z}_p \subset W(\kappa)$ is faithfully flat this passes to exact sequences

$$M^T_{n+1}/I_{n+1} \xrightarrow{p^n} M^T_{n+1}/I_{n+1} \longrightarrow M^T_n/I_n \longrightarrow 0.$$

By Lemma 1.3.8, we know $\lim_n I_n = \lim^1_n I_n = 0$, so that $M^T = \lim_n M_n^T = \lim_n M_n^T/I_n$. By Lemma 1.3.9, we learn $M^T/(p^n) = M_n^T/I_n$. So we can identify $W(\kappa)\,\widehat{\otimes}\,M^T \to M$ as

$$\begin{aligned} W(\kappa)\,\widehat{\otimes}\,M^T &= \lim_{m,n} W(\kappa)/(p^m) \otimes M^T/(p^n) \\ &= \lim_{m,n} W(\kappa)/(p^m) \otimes M_n^T/I_n \\ &= \lim_{n} W(\kappa) \otimes M_n^T/I_n \\ &= \lim_{n} M/(p^n) = M. \end{aligned}$$

□

Finally, Theorem 1.2.3 is an immediate consequence of the following.

1.3.11. **Proposition.** Let κ be an algebraically closed field, and M a free module over $W(\kappa)$ of finite rank h equipped with a σ^{-1}-linear topologically nilpotent endomorphism $V\colon M \to M$ such that $M/VM \cong \kappa$. Then there exists $x \in M$ such that $M \cong W(\kappa)\{x, Vx, \ldots, V^{h-1}x\}$ and $V^h x = px$.

Proof. By Proposition 1.3.2, we know that $p^{-1}V^h\colon M \to M$ is a σ^{-1}-semilinear automorphism, and moreover there is $y \in M$ such that $y, Vy, \ldots, V^{h-1}y$ forms a basis for M. On the other hand, Proposition 1.3.10 tells us that M admits a basis $x_0, \ldots, x_{h-1}$ with $p^{-1}V^h x_i = x_i$ for each i. If we write $y = \sum_{i=0}^{h-1} a_i x_i$ and choose n such that a_n is nonzero in $M/(p)$, then Proposition 1.3.2 tells us that $x_n, Vx_n, \ldots, V^{h-1}x_n$ is a basis for M. So $x = x_n$ has the stated properties. □

2. Classification of formal groups

2.1. **Galois cohomology.** For our purposes, a topological group will be a group G equipped with topology arising from an inverse limit construction $G = \lim_i G/U_i$, where $U_i \subset G$ is a normal subgroup not necessarily of finite index. Let Γ and S be topological groups, and suppose we have an action of Γ on S. We can define the semidirect product group $S \rtimes \Gamma$ with multiplication $(s,\gamma)\cdot(t,\delta) = (s \cdot {}^{\gamma}t, \gamma\cdot\delta)$. This is equipped with a projection $\pi\colon S \rtimes \Gamma \to \Gamma$, and we define the set of S-valued cocycles on Γ by

$$Z^1(\Gamma, S) = \{\alpha\colon \Gamma \to S \rtimes \Gamma : \pi \circ \alpha = \mathrm{id}_\Gamma\}.$$

Equivalently,

$$Z^1(\Gamma, S) = \{\alpha\colon \Gamma \to S : \alpha(\gamma\cdot\delta) = \alpha(\gamma)\cdot {}^{\gamma}\alpha(\delta)\} \subset \mathrm{Hom}(\Gamma, S).$$

This set is equipped with an S-action by conjugation: for $s \in S$ and $\alpha \in Z^1(\Gamma, S)$, we define

$$\alpha^s(\gamma) = s^{-1}\cdot\alpha(\gamma)\cdot {}^{\gamma}s.$$

Let $H^1(\Gamma, S) = Z^1(\Gamma, S)_S$ be the set of orbits of this action. If S is abelian, then $H^1(\Gamma, S)$ agrees with the usual continuous group cohomology of Γ with coefficients in S, as can be seen directly from a suitable bar resolution. In general, $H^1(\Gamma, S)$ is a set pointed at the zero map $\Gamma \to S$. Observe that $H^1(\Gamma, S) = 0$ precisely when for every $\alpha \in Z^1(\Gamma, S)$ there exists some $s \in S$ such that $\alpha(\gamma) = s^{-1}\cdot {}^{\gamma}s$ for all $\gamma \in \Gamma$.

2.1.1. **Lemma.** Suppose $S = \lim_{n\geq 1} S/U_n$ is a complete Hausdorff topological group, and set $U_0 = S$. Suppose $H^1(\Gamma, U_n/U_{n+1}) = 0$ for each $n \geq 0$. Then $H^1(\Gamma, S) = 0$.

Proof. Let $\alpha \in Z^1(\Gamma, S)$. We find that $\alpha(\gamma) \equiv s_1^{-1} \cdot {}^{\gamma}s_1 \pmod{U_1}$ for some fixed $s_1 \in S$. Replacing α with $\alpha^{s_1^{-1}}$, we may suppose α lands in $U_1 \subset S$. Repeating this argument, we find elements $s_1, s_2, \ldots \in S$ such that $s_n \in U_{n-1}$ and $f^{(s_n\cdots s_1)^{-1}}$ lies in U_n for each n. As S is complete Hausdorff with respect to the filtration by the subgroups U_i, we can take $s_\infty = \lim_{n\to\infty} s_n \cdots s_1$ and find $f^{s_\infty^{-1}}$ lies in $\bigcap U_n = 0$. So f is zero in $H^1(\Gamma, S)$. □

Fix now a field k of positive characteristic p, and let $k \subset K$ be a Galois extension with Galois group Γ. Write $W(K)$ for the ring of p-typical Witt vectors on K, and consider this as equipped with the p-adic topology. We obtain a continuous action of Γ on the Witt components of $W(K)$. The quotients $W(K) \to W(K)/(p^n)$ pass to surjections $\mathrm{GL}_n(W(K)) \to \mathrm{GL}_n(W(K)/(p^n))$, and these make $\mathrm{GL}_n(W(K))$ into a topological group. The action of Γ on the coordinates of $\mathrm{GL}_n(W(K))$ is continuous with respect to this topology. The following can be seen as an analogue of Hilbert's Satz 90.

2.1.2. **Theorem.** $H^1(\Gamma, \mathrm{GL}_n(W(K))) = 0$.

Proof. Let U_n be the kernel of the quotient map $\mathrm{GL}_n(W(K)) \to \mathrm{GL}_n(W(K)/(p^n))$. One can show that

$$U_0/U_1 = (K^\times)^n, \qquad U_{n+1}/U_n = K^{\oplus n}.$$

So by Lemma 2.1.1, we are reduced to verifying $H^1(\Gamma, K^\times) = 0 = H^1(\Gamma, K)$. The first is exactly Hilbert's Satz 90. For the second, let $\alpha \in Z^1(\Gamma, K)$, realized as a map $\alpha\colon \Gamma \to K$. As K is discrete and Γ is profinite, this factors through $\alpha'\colon G \to L$, where $k \subset L \subset K$ is some finite Galois extension and $G = \mathrm{Aut}_k(L)$. It is then sufficient to show $\alpha' = 0$ in $H^1(G, L)$. This follows from the normal basis theorem, telling us that $L \cong k[G]$ as $k[G]$-modules, and the identification of H^1 with ordinary group cohomology in this case. □

2.2. **The classification theorem.** Fix a perfect field κ of positive characteristic, and fix an algebraic closure $\kappa \subset \overline{\kappa}$. Let $\Gamma = \mathrm{Aut}_k(\overline{\kappa})$ be the absolute Galois group of κ. For $M \in \mathrm{L}\mathcal{M}\mathrm{od}^v_{D(\kappa)}$, let $M_{\overline{\kappa}} = W(\overline{\kappa}) \mathbin{\widehat{\otimes}}_{W(\kappa)} M \in \mathrm{L}\mathcal{M}\mathrm{od}^v_{D(\overline{\kappa})}$. The continuous action of Γ on $W(\overline{\kappa})$ gives rise to a continuous action of Γ on $M_{\overline{\kappa}}$ by left $D(\kappa)$-module maps, and we can identify $M_{\overline{\kappa}}^\Gamma = M$.

Fix a positive integer h, and write $H = W(\kappa)\{x, Vx, \ldots, V^{h-1}x\}$ for the object of $\mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)}$ considered in Section 1.2. We have an action of Γ on $\mathrm{Hom}_{D(\overline{\kappa})}(H_{\overline{\kappa}}, M_{\overline{\kappa}})$ defined as follows. Recall that we can identify a map $f\colon H_{\overline{\kappa}} \to M_{\overline{\kappa}}$ as the element $a = f(x) \in M_{\overline{\kappa}}$ subject to $V^h a = pa$. For $\gamma \in \Gamma$ we have $\gamma(a) \in M_{\overline{\kappa}}$, and we claim this is associated to some other ${}^{\gamma}f\colon H_{\overline{\kappa}} \to M_{\overline{\kappa}}$. Indeed, as Γ acts by $D(\kappa)$-linear maps, we have

$$V^h\gamma(a) = \gamma(V^h a) = \gamma(pa) = p\gamma(a).$$

This action restricts to an action of Γ on $\mathrm{Iso}_{D(\overline{\kappa})}(H_{\overline{\kappa}}, M_{\overline{\kappa}})$. In case $M = H$, this gives an action of Γ on $\mathbb{S}_h$. Our goal for the rest of this section is to demonstrate the following:

$$\pi_0\mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)} \cong H^1(\Gamma, \mathbb{S}_h).$$

Fix $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)}$. Choose an isomorphism $f\colon H_{\overline{\kappa}} \to M_{\overline{\kappa}}$; the existence of such is guaranteed by Theorem 1.2.3. Define then

$$\alpha_{M,f}\colon \Gamma \to \mathbb{S}_h, \qquad \alpha_{M,f}(\gamma) = f^{-1} \cdot {}^{\gamma} f.$$

2.2.1. **Claim.** We have $\alpha_{M,f} \in Z^1(\Gamma, \mathbb{S}_h)$.

Proof. We must verify the cocycle condition and continuity. For the cocycle condition, we directly calculate for $\gamma, \delta \in \Gamma$ that

$$\begin{aligned} \alpha_{M,f}(\gamma\delta) &= f^{-1} \cdot {}^{\gamma\delta} f \\ &= f^{-1} \cdot {}^{\gamma} f \cdot {}^{\gamma} f^{-1} \cdot {}^{\gamma\delta} f \\ &= f^{-1} \cdot f^{\gamma} \cdot {}^{\gamma}\left(f^{-1} \cdot {}^{\delta} f\right) = \alpha_{M,f}(\gamma) \cdot {}^{\gamma}\alpha_{M,f}(\delta). \end{aligned}$$

For continuity, write $f(x) = \sum_i \lambda_i \otimes a_i$ with $\lambda_i \in W(\overline{\kappa})$ and $a_i \in M$ and let $\kappa \subset K$ be the field extension obtained by adjoining the first n Witt components of the coefficients λ_i. If $\gamma|_K = \delta|_K$, then ${}^{\gamma} f \equiv {}^{\delta} f \pmod{p^{n+1}}$, and thus $\alpha_{M,f}(\gamma) \equiv \alpha_{M,f}(\delta) \pmod{p^{n+1}}$. So we learn that $\alpha_{M,f}$ is continuous. □

We have now assigned to every $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)}$ and choice of isomorphism $f\colon H_{\overline{\kappa}} \to M_{\overline{\kappa}}$ an element $\alpha_{M,f} \in Z^1(\Gamma, \mathbb{S}_h)$.

2.2.2. **Claim.** Upon passing to $H^1(\Gamma, \mathbb{S}_h)$, the above assignment factors through $\pi_0\mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)}$.

Proof. Suppose given $M^\epsilon \in \mathrm{L}\mathcal{M}\mathrm{od}_D(\kappa)$ with chosen isomorphisms $f_\epsilon\colon H_{\overline{\kappa}} \to M^\epsilon_{\overline{\kappa}}$ for $\epsilon \in \{1, 2\}$. Let $g\colon M^1 \to M^2$ be an isomorphism of left $D(\kappa)$-modules, and define $h \in \mathbb{S}_h$ by the diagram

$$\begin{array}{ccc} H_{\overline{\kappa}} & \xrightarrow{f^1} & M^1_{\overline{\kappa}} \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle g} \\ H_{\overline{\kappa}} & \xrightarrow{f^2} & M^2_{\overline{\kappa}} \end{array} .$$

As g is defined over $W(\kappa)$, we can calculate for $\gamma \in \Gamma$ that

$$\begin{aligned} f_2^{-1} \cdot {}^{\gamma} f_2 &= (g \cdot f_1 \cdot h)^{-1} \cdot {}^{\gamma}\left(g \cdot f_1 \cdot h\right) \\ &= h^{-1} \cdot f_1^{-1} \cdot g^{-1} \cdot {}^{\gamma}\left(g \cdot f_1\right) \cdot {}^{\gamma} h = h^{-1} \cdot f_1^{-1} \cdot {}^{\gamma} f_1 \cdot {}^{\gamma} h. \end{aligned}$$

Thus $\alpha^h_{M^1,f_1} = \alpha_{M^2,f_2}$, so that α_{M^1,f_1} and α_{M^2,f_2} agree in $H^1(\Gamma, \mathbb{S}_h)$. □

We have now constructed

$$\alpha\colon \pi_0\mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)} \to H^1(\Gamma, \mathbb{S}_h).$$

2.2.3. **Theorem.** The map α is an isomorphism.

Proof. Let us first check that α is injective. Suppose given $M^\epsilon \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)}$ and choose isomorphisms $f_\epsilon\colon H_{\overline{\kappa}} \to M^\epsilon_{\overline{\kappa}}$ for $\epsilon \in \{1, 2\}$. Suppose that for some $h \in \mathbb{S}_h$ we have $\alpha^h_{M^1,f_1} = \alpha^h_{M^2,f_2}$. Then for all $\gamma \in \Gamma$, the diagram

$$\begin{array}{ccc}
H_{\overline{\kappa}} & \xrightarrow{h} & H_{\overline{\kappa}} \\
\downarrow {}^{\gamma}f_2 & & \downarrow {}^{\gamma}f_1 \\
M^2_{\overline{\kappa}} & & M^1_{\overline{\kappa}} \\
\uparrow f_2 & & \uparrow f_1 \\
H_{\overline{\kappa}} & \xrightarrow{h} & H_{\overline{\kappa}}
\end{array}$$

commutes, telling us

$$f_2 \cdot h \cdot f_1^{-1} = {}^{\gamma}\left(f_2 \cdot h \cdot f_1^{-1}\right) : M^1_{\overline{\kappa}} \cong M^2_{\overline{\kappa}}$$

for all $\gamma \in \Gamma$. So we may take fixed points with respect to Γ to obtain an isomorphism $M^1 \cong M^2$.

Let us now check α is surjective. Fix $\eta \in Z^1(\Gamma, \mathbb{S}_h)$, which we can push forward to $Z^1(\Gamma, \mathrm{GL}_n(W(\overline{\kappa})))$. By Theorem 2.1.2, we know that $\eta(\gamma) = T^{-1} \cdot {}^{\gamma}T$ in $\mathrm{GL}_n(W(\overline{\kappa}))$ for all $\gamma \in \Gamma$ and some matrix $T \in \mathrm{GL}_n(W(\overline{\kappa}))$. Let $N = W(\overline{\kappa}) \otimes_{W(\kappa)} W(\kappa)\{y_0, \ldots, y_{h-1}\}$, and using the distinguished bases treat T as an isomorphism $T\colon H_{\overline{\kappa}} \cong N$ of $W(\overline{\kappa})$-modules. We can give N the structure of a left $D_{\overline{\kappa}}$-module such that this is an isomorphism of left $D(\overline{\kappa})$-modules; explicitly, if $V = V_H\colon H_{\overline{\kappa}} \to H_{\overline{\kappa}}$, then $V_N = T \circ V_H \circ T^{-1}$. Observe now that for all $\gamma \in \Gamma$ we have

$$\begin{aligned}
{}^{\gamma}V_N &= {}^{\gamma}\left(T \cdot V_H \cdot T^{-1}\right) \\
&= {}^{\gamma}T \cdot {}^{\gamma}V_H \cdot {}^{\gamma}T^{-1} \\
&= T \cdot \eta(\gamma) \cdot V_H \cdot \eta(\gamma)^{-1} \cdot T^{-1} \\
&= T \cdot V_H \cdot T^{-1} = V_N,
\end{aligned}$$

the last equality as $\eta(\gamma) \in \mathbb{S}_h$ commutes with F_H. As V_N is Γ-equivariant, we can take fixed points with respect to Γ to obtain the left $D(\kappa)$-module $M = N^{\Gamma}$. Evidently $\alpha_{M,T} = \eta$, so this concludes the proof. □

2.3. **Case of a finite field.** We maintain notation from the previous section. Suppose now $\kappa = \mathbb{F}_{p^r}$ is a finite field. Then $\Gamma = \widehat{\mathbb{Z}}$ is cyclically generated by σ^{-r}. We learn that evaluation on σ^{-r} gives an isomorphism

$$Z^1(\Gamma, \mathbb{S}_h) \cong \mathbb{S}_h.$$

Thus $H^1(\Gamma, \mathbb{S}_h)$ is a quotient of $\mathbb{S}_h$, where two elements $g_1, g_2 \in \mathbb{S}_h$ are equivalent precisely when there is some $h \in \mathbb{S}_h$ such that

$$g_2 = h^{-1} \cdot g_1 \cdot {}^{\sigma^{-r}}h.$$

If $\kappa = \mathbb{F}_{p^{hr}}$, so that $\mathbb{F}_{p^h} \subset \kappa$, then Γ acts trivially on $\mathbb{S}_h$ and this description yields an isomorphism

$$\alpha\colon \pi_0\mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)} \cong \mathbb{S}_h/(\mathrm{conj}).$$

Let $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)}$ with $\kappa = \mathbb{F}_{p^{hr}}$. Observe that over $\mathbb{F}_{p^{hr}}$, the σ^{-hr}-semilinear map V^{hr} is in fact linear. It thus induces a $D(\overline{\kappa})$-linear map $M_{\overline{\kappa}} \to M_{\overline{\kappa}}$, but this is not the map V^{hr} associated $M_{\overline{\kappa}}$; when treating V^{hr} in this way, we will write $V^{hr} = B_M$. Choose an isomorphism $f\colon H_{\overline{\kappa}} \to M_{\overline{\kappa}}$. We can give now another description of the conjugacy class $\alpha(M)$ associated to M. By Corollary 1.3.3, we have a commutative diagram

$$
\begin{array}{ccccc}
 & & \overset{B_H}{\frown} & & \\
H_{\overline{\kappa}} & \xrightarrow{p^r} & H_{\overline{\kappa}} & \xrightarrow{=} & H_{\overline{\kappa}} \\
\big\downarrow f & & \big\downarrow f & & \big\downarrow {}^{\sigma^{-hr}}f \\
M_{\overline{\kappa}} & \xrightarrow{p^r} & M_{\overline{\kappa}} & \xrightarrow{\mu} & M_{\overline{\kappa}} \\
 & & \underset{B_M}{\smile} & &
\end{array}
.
$$

This tells us that $f^{-1} \cdot {}^{\sigma^{-hr}} f = f^{-1} \cdot \mu \cdot f$. So $f^{-1} \cdot \mu \cdot f \in \mathbb{S}_h$ is a representative of the conjugacy class associated to M. Moreover, μ is defined in $W(\mathbb{F}_{p^{hr}})$; in particular, $V^{hrs} = \mu^s p^{rs}$, so that the conjugacy class associated to $M_{\mathbb{F}_{p^{hrs}}}$ is $f^{-1} \cdot \mu^s \cdot f$.

As the conjugacy class of the identity of $\mathbb{S}_h$ is a singleton, we learn that $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\mathbb{F}_{p^{hr}})}$ is isomorphic to H if and only if $\mu = 1$ in the above, i.e. $V^{hr} = p^r$ on M.

2.3.1. **Example.** Consider the case $h = 1$. In this case, the action of Γ on $\mathbb{S}_1$ is trivial and $\mathbb{S}_1 = \mathbb{Z}_p^\times$ is abelian, so an object $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,1}_{D(\mathbb{F}_{p^r})}$ is classified by an element of $\mathbb{Z}_p^\times$. Such an object M can be written as $M = W(\mathbb{F}_{p^r})\{y\}$ with $Vy = apy$ for some $a \in W(\mathbb{F}_{p^r})^\times$. The above description tells us that the invariant of M is the norm $N(a) = \prod_{i=0}^{r-1} a^{\sigma^i}$. ◁

3. Determinants

3.1. **Exterior powers.** For $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v}_{D(\kappa)}$, consider $\Lambda^n M$, the exterior power taken over $W(\kappa)$.

3.1.1. **Theorem.** The $W(\kappa)$-module $\Lambda^n M$ admits a natural left $D(\kappa)$-module structure determined by

$$
\begin{aligned}
V(a_1 \wedge \cdots \wedge a_n) &= Va_1 \wedge \cdots \wedge Va_n, \\
F(a_1 \wedge \cdots \wedge a_n) &= p^{-(n-1)} Fa_1 \wedge \cdots \wedge Fa_n.
\end{aligned}
$$

Proof. This is evidently natural; we must only verify that F can be defined this way. Choose $y \in M$ so that $M \cong W(\kappa)\{y, Vy, \ldots, V^{h-1}y\}$. Then a basis for $\Lambda^n M$ is given by the elements $V^I y = V^{i_1} y \wedge \cdots \wedge V^{i_n} y$ with $I = \{i_1 < \cdots < i_n\} \subset \{0, \ldots, h-1\}$. As $FV = p$ in M, we find that with the above definition $F(V^I y)$ lives in $\Lambda^n M \subset \mathbb{Q} \otimes \Lambda^n M$. □

3.1.2. **Example.** Consider the case $M = H$. Then $\Lambda^n H$ has a basis consisting of the elements $V^I x$ as above, and we can identify

$$
V(V^{i_1}x \wedge \cdots \wedge V^{i_n}x) = \begin{cases} V^{i_1+1}x \wedge \cdots \wedge V^{i_n+1}x, & i_n \neq h-1, \\ (-1)^{n-1} px \wedge V^{i_1+1}x \wedge \cdots \wedge V^{i_{n-1}+1}x, & i_n = h-1; \end{cases}
$$

$$
F(V^{i_1}y \wedge \cdots \wedge V^{i_n}y) = \begin{cases} pV^{i_1-1}y \wedge \cdots \wedge V^{i_n-1}y, & i_1 \neq 0, \\ (-1)^{n-1} V^{i_2-1}x \wedge \cdots \wedge V^{i_n-1}x \wedge V^{h-1}x, & i_1 = 0. \end{cases}
$$

◁

Fix $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)}$. For general n, we cannot say that $\Lambda^n M$ lives in $\mathrm{L}\mathcal{M}\mathrm{od}^{v,\binom{h}{n}}_{D(\kappa)}$; rather, it turns out that $\Lambda^n M$ is associated to a connected p-divisible group of height $\binom{h}{n}$ and dimension $\binom{h-1}{n-1}$. In the case $h = n$, one can check directly that $\Lambda^h M \in \mathrm{L}\mathcal{M}\mathrm{od}^{v,1}_{D(\kappa)}$. We obtain a functor

$$\Lambda^h \colon \mathrm{L}\mathcal{M}\mathrm{od}^{v,h}_{D(\kappa)} \to \mathrm{L}\mathcal{M}\mathrm{od}^{v,1}_{D(\kappa)},$$

and by Theorem 2.2.3 this passes to a map

$$\lambda \colon H^1(\Gamma, \mathbb{S}_h) \to H^1(\Gamma, \mathbb{S}_1),$$

where Γ is the absolute Galois group of κ. By Proposition 1.2.1, we can identify $\mathbb{S}_1 = \mathbb{Z}_p^\times$, and this has trivial action by Γ. We can thus identify $H^1(\Gamma, \mathbb{S}_1) = \mathrm{Hom}(\Gamma, \mathbb{Z}_p^\times)$, so that λ gives $\mathrm{Hom}_\Gamma(\Gamma, \mathbb{S} \rtimes \Gamma) \to \mathrm{Hom}(\Gamma, \mathbb{Z}_p^\times)$.

Say now $\kappa = \mathbb{F}_{p^r}$ is a finite field, and fix $M \in \mathrm{L}\mathcal{M}\mathrm{od}^h_{D(\mathbb{F}_{p^r})}$. We have now $H^1(\Gamma, \mathbb{S}_1) = \mathbb{Z}_p^\times$. Let us describe the resulting element $\lambda(M) \in \mathbb{Z}_p^\times$. As in Example 2.3.1, we can write $V^r = \Lambda^h V^r \colon \Lambda^h M \to \Lambda^h M$ as $V^r = p^r a$ for some $a \in W(\mathbb{F}_{p^r})^\times$, and then $\lambda(M) = N(a) = \prod_{i=0}^{r-1} a^{\sigma^i}$. Choose now $y \in M$ such that $M \cong W(\mathbb{F}_{p^r})\{y, Vy, \ldots, V^{h-1}y\}$, and write $V^h y = \sum_{i=0}^{h-1} p a_i V^i y$ with $a_0 \in W(\mathbb{F}_{p^r})^\times$.

3.1.3. **Proposition.** With notation as above, we have

$$a = \lambda(M) = (-1)^{r(h-1)} N(a_0) = (-1)^{r(h-1)} \prod_{i=0}^{r-1} a_0^{\sigma^i}.$$

Proof. Continuing the previous discussion, we have $\Lambda^h M \cong W(\mathbb{F}_{p^r})\{y \wedge \cdots \wedge V^{h-1} y\}$ and $V^h y = \sum_{i=0}^{h-1} a_i V^i y$. Observe that

$$\begin{aligned} V\left(y \wedge Vy \wedge \cdots \wedge V^{h-1}y\right) &= Vy \wedge V^2 \wedge \cdots \wedge V^{h-1}y \wedge V^h y \\ &= \sum_{i=0}^{h-1} Vy \wedge \cdots \wedge V^{h-1} y \wedge p a_i V^i y \\ &= Vy \wedge \cdots \wedge V^{h-1} y \wedge a_0 y \\ &= (-1)^{h-1} p a_0 y \wedge Vy \wedge \cdots \wedge V^{h-1} y. \end{aligned}$$

Continuing in this manner, we learn

$$\begin{aligned} V^r(y \wedge Vy \wedge \cdots \wedge V^{h-1}y) &= p^r (-1)^{r(h-1)} a_0^{\sigma^{-(r-1)}} \cdots a_0^{\sigma^{-1}} a_0 y \wedge Vy \wedge \cdots \wedge V^{h-1} y \\ &= p^r (-1)^{r(h-1)} N(a_0) y \wedge Vy \wedge \cdots \wedge V^{h-1} y, \end{aligned}$$

so that $a = (-1)^{r(h-1)} N(a_0)$ as stated. □

3.2. **Classical determinant.** Fix a perfect field κ containing all $(p^h - 1)$'th roots of unity, and write $\mathbb{F}_{p^h} \subset \kappa$ for the subfield generated by these elements. Write still $H = W(\kappa)\{x, Vx, \ldots, V^{h-1}x\}$ with $V^h x = px$. Then $\mathrm{End}_{D(\kappa)}(H)$ was identified in Proposition 1.2.1, and is independent of the choice of such κ. Moreover, we have an injection $\mathrm{End}_{D(\kappa)}(H) \to \mathrm{End}_{W(\kappa)}(W(\kappa)^{\oplus h})$ which lands in the image of the induction map $\mathrm{End}_{W(\mathbb{F}_{p^h})}(W(\mathbb{F}_{p^h})^{\oplus h}) \subset \mathrm{End}_{W(\kappa)}(W(\kappa)^{\oplus h})$. Explicitly, for $a = \sum_{i=0}^{h-1} a_i V^i$ in $\mathrm{End}_{D(\kappa)}(H)$, we have $a_i \in W(\mathbb{F}_{p^h})$ for each a_i and the matrix associated to the image of a in $\mathrm{End}_{W(\mathbb{F}_{p^h})}(W(\mathbb{F}_{p^h})^{\oplus h})$ is

$$\begin{pmatrix} a_0 & pa_{h-1}^{\sigma^{-1}} & pa_{h-2}^{\sigma^{-2}} & \cdots & pa_1^{\sigma^{-(h-1)}} \\ a_1 & a_0^{\sigma^{-1}} & pa_{h-1}^{\sigma^{-2}} & \cdots & pa_2^{\sigma^{-(h-1)}} \\ a_2 & a_1^{\sigma^{-1}} & a_0^{\sigma^{-2}} & \cdots & pa_3^{\sigma^{-(h-1)}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{h-2} & a_{h-3}^{\sigma^{-1}} & a_{h-4}^{\sigma^{-2}} & \cdots & pa_{h-1}^{\sigma^{-(h-1)}} \\ a_{h-1} & a_{h-2}^{\sigma^{-1}} & a_{h-3}^{\sigma^{-2}} & \cdots & a_0^{\sigma^{-(h-1)}} \end{pmatrix}$$

Define now

$$d\colon \mathrm{End}_{D(\kappa)}(H) \to \mathrm{End}_{W(\mathbb{F}_{p^h})}(W(\mathbb{F}_{p^h})^{\oplus h}) \xrightarrow{\det} W(\mathbb{F}_{p^h}).$$

This is a homomorphism, with $W(\mathbb{F}_{p^h})$ given its multiplicative monoid structure.

3.2.1. **Proposition.** The image of d lies in $W(\mathbb{F}_p) \subset W(\mathbb{F}_{p^h})$.

Proof. One can check that $p^{-1}\,\mathrm{End}_{D(\kappa)}(H)$ is a well defined algebra, and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_{D(\kappa)}(H) & \longrightarrow & p^{-1}\,\mathrm{End}_{D(\kappa)}(H) \\ \big\downarrow d & & \big\downarrow \\ \mathrm{End}_{W(\mathbb{F}_{p^h})}(W(\mathbb{F}_{p^h})^{\oplus h}) & \longrightarrow & \mathrm{End}_{\mathbb{Q}\otimes W(\mathbb{F}_{p^h})}(\mathbb{Q}\otimes W(\mathbb{F}_{p^h})^{\oplus h}) \end{array}$$

of injections. Fix $a \in \mathrm{End}_{D(\kappa)}(H)$. Observe that inverting p also inverts V, and we can identify $VaV^{-1} = a^{\sigma^{-1}}$. Thus $d(a) = d(VaV^{-1}) = d(a^{\sigma^{-1}}) = d(a)^{\sigma^{-1}}$, so that $d(a) \in W(\mathbb{F}_p)$. □

3.2.2. **Remark.** Let $T \in \mathrm{End}_{W(\kappa)}(W(\kappa)^{\oplus h})$ be associated to some $a \in \mathrm{End}_{D(\kappa)}(H)$. Then the coefficients of the characteristic polynomial $q(t)$ of T all lie in $W(\mathbb{F}_p)$. Indeed, we can write

$$q(t) = \sum_{i=0}^{h} (-1)^i \,\mathrm{Tr}(\Lambda^i T) t^{h-i},$$

and as trace is invariant under cyclic permutations we can argue as above.

More generally, fix $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{D(\kappa)}^{v,h}$ and a $D(\kappa)$-linear endomorphism $g\colon M \to M$. Let $q(t)$ be the characteristic polynomial of g considered as a $W(\kappa)$-linear endomorphism. Then the coefficients of $q(t)$ lie in $W(\mathbb{F}_p) \subset W(\kappa)$. Indeed, g induces a $D(\overline{\kappa})$-linear endomorphism $M_{\overline{\kappa}} \to M_{\overline{\kappa}}$, and we know $M_{\overline{\kappa}} \cong H_{\overline{\kappa}}$; so we can appeal to the fact that the characteristic polynomial of a linear map is independent of choice of basis. ◁

We obtain the determinant mapping $d\colon \mathbb{S}_h \to W(\mathbb{F}_p)^\times = \mathbb{Z}_p^\times$.

3.2.3. **Theorem.** Fix $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{D(\mathbb{F}_{p^r})}^{v,h}$, and let $\alpha(M) \in \mathbb{S}_h$ represent the invariant in $H^1(\Gamma, \mathbb{S}_h)$ associated to M. Let $\lambda(M) \in \mathbb{S}_1 = \mathbb{Z}_p^\times$ be the invariant associated to $\Lambda^h M$. Then

$$\lambda(M) = (-1)^{r(h-1)} d(\alpha(M)).$$

Proof. Write $\overline{\kappa} = \overline{\mathbb{F}}_p$ and choose an isomorphism $f\colon H_{\overline{\kappa}} \to M_{\overline{\kappa}}$ so that we can say $\alpha(M)$ is represented by $f^{-1} \cdot {}^{\sigma^{-r}} f$. By the definition of ordinary determinants, we can identify $d(f^{-1} \cdot {}^{\sigma^{-r}} f) = \Lambda^h(f^{-1}) \cdot \Lambda^h({}^{\sigma^{-r}} f)$ as automorphisms of $\Lambda^h H_{\overline{\kappa}} \cong W(\overline{\kappa})$. By Corollary 1.3.3 and Proposition 3.1.3, the diagram

$$\begin{array}{ccccc} \Lambda^h H_{\overline{\kappa}} & \xrightarrow{p^r} & \Lambda^h H_{\overline{\kappa}} & \xrightarrow{(-1)^{r(h-1)}} & \Lambda^h H_{\overline{\kappa}} \\ \downarrow{\scriptstyle f} & & {\scriptstyle \Lambda^h(f)}\downarrow & & {\scriptstyle \Lambda^h({}^{\sigma^{-r}} f)}\downarrow{\scriptstyle = {}^{\sigma^{-r}}\Lambda^h(f)} \\ \Lambda^h M_{\overline{\kappa}} & \xrightarrow{p^r} & \Lambda^h M_{\overline{\kappa}} & \xrightarrow{\lambda(M)} & \Lambda^h M_{\overline{\kappa}} \end{array}$$

commutes. This yields the result. □

3.2.4. **Corollary.** Fix notation as above. Choose $y \in M$ giving an isomorphism $M \cong W(\mathbb{F}_{p^r})\{y, Vy, \ldots, V^{h-1}y\}$, and write $V^h y = \sum_{i=0}^{h-1} p a_0 y$. Then $d(\alpha(M)) = N(a_0) = \prod_{i=0}^{r-1} a_0^{\sigma^i}$.

Proof. Combine Theorem 3.2.3 and Proposition 3.1.3. □

3.2.5. **Remark.** The proof of Theorem 3.2.3 did not rely on Proposition 3.2.1, and so gives a second proof of the fact that $d\colon \mathbb{S}_h \to W(\mathbb{F}_{p^h})^\times$ lands in $\mathbb{Z}_p^\times$. ◁

3.2.6. **Example.** Suppose h is even, and write $H = W(\mathbb{F}_p)\{x, Vx, \ldots, V^{h-1}x\}$ with $V^h x = px$. Let $M = \Lambda^h H$. Then we can identify $M = W(\mathbb{F}_p)\{y\}$ with $Vy = -py$. Let $H^1 = W(\mathbb{F}_p)\{x\}$ with $Vx = px$. Then an isomorphism $f\colon H^1_{\overline{\kappa}} \to M_{\overline{\kappa}}$ is given by $a \in W(\overline{\kappa})^\times$ satisfying

$$a^\sigma + a = 0.$$

We have a Galois extension $W(\mathbb{F}_p) \subset W(\mathbb{F}_{p^2})$, and the resulting trace map is

$$W(\mathbb{F}_{p^2}) \to W(\mathbb{F}_p), \qquad a \mapsto a^\sigma + a.$$

The kernel is a free one-dimensional $W(\mathbb{F}_p)$-module, and any element therein not divisible by p thus gives an isomorphism $H^1_{\mathbb{F}_{p^2}} \to M_{\mathbb{F}_{p^2}}$. We learn that $M_{\mathbb{F}_{p^r}} \cong H^1_{\mathbb{F}_{p^r}}$ if and only if r is even. ◁

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