COMBINATORIAL RULES FOR THREE BASES OF POLYNOMIALS

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ABSTRACT. We present combinatorial rules (one theorem and two conjectures) concerning three bases of \( \mathbb{Z}[x_1, x_2, ...] \). First, we prove a "splitting" rule for the basis of Key polynomials [Demazure '74], thereby establishing a new positivity theorem about them. Second, we introduce an extension of [Kohnert '90]'s "moves" to conjecture the first combinatorial rule for a certain deformation [Lascoux '01] of the Key polynomials. Third, we use the same extension to conjecture a new rule for the Grothendieck polynomials [Lascoux-Schützenberger '82].

In memory of Alain Lascoux, who inspired this paper one night in Osaka

1. INTRODUCTION

1.1. Overview. This paper contributes to the study of certain bases of the ring of polynomials \( \text{Pol} = \mathbb{Z}[x_1, x_2, ...] \) that are defined by symmetrizing operators. Our two main sources on this subject, and the specific perspective we pursue, are A. Lascoux’s books [14, 11].

The Schur basis of the ring \( \Lambda \) of symmetric polynomials is central to algebraic combinatorics in at least two ways. These polynomials have fundamental applications outside of the theory of symmetric functions, specifically to representation theory of the symmetric group and of the general linear group, and to Schubert calculus, see, e.g., [20]. Moreover, understanding combinatorial descriptions of the Schur polynomials has led to a rich theory of Young tableaux. In particular, the problem of how to multiply two Schur polynomials, and expand back into the Schur basis, is important in the aforementioned applications. This problem is solved by the Littlewood-Richardson rule.

Now, since the ring of symmetric polynomials is a subring of \( \text{Pol} \), one considers the following basic question [11]:

How does one lift properties of \( \Lambda \) (and its Schur basis) to the entirety of \( \text{Pol} \)?

A number of bases, that may be considered natural lifts of the Schur basis, are considered in [14, 11]. These include the Schubert, Grothendieck, Macdonald and Key polynomials; we will also consider a deformation of the Key polynomials defined by A. Lascoux [13]. These are lifts of the Schur basis in the sense that a certain subset in each of these families is precisely the Schur basis, or otherwise deforms the elements of the Schur basis. In fact, like the Schur polynomials, each of these families have applications to representation theory and geometry.

Now, one would like to find combinatorial descriptions for the polynomials in each of these families. Indeed, such descriptions exist. Yet at present, there is no analogue of the Littlewood-Richardson rule. That is, for each basis, one desires a combinatorial
description of how to multiply and expand in the basis so that one recovers a Littlewood-Richardson rule in the special case of Schur polynomials. For instance, for the case of Schubert polynomials, and more generally, the case of Grothendieck polynomials, this is a longstanding open problem in combinatorial Schubert calculus, cf. [20].

There is a close tie between Littlewood-Richardson rules and the Young tableau description of Schur polynomials. Therefore, by analogy, one would like to find alternative combinatorial descriptions of the aforementioned bases of $\Pol$. The hope is that such alternatives might shed light on finding corresponding generalizations of the Littlewood-Richardson rule.

Our work consists of a new combinatorial description for three of the aforementioned bases of polynomials. We give one theorem and two conjectures, which we summarize as follows:

First, we prove a “splitting” rule for the basis of Key polynomials $\{\kappa_\alpha|\alpha \in \mathbb{Z}^\infty_{\geq 0}\}$, thereby establishing a new positivity theorem about these polynomials. This family was introduced by [5] and first studied combinatorially in [16, 15]. Combinatorial rules for their monomial expansion are known, see, e.g., [16, 15, 21, 8]. Our rule refines the rule of [21, Theorem 5(1)]. Our rule is also analogous to the splitting rule [4, Corollary 3] for the basis of Schubert polynomials $\{S_w|w \in S_\infty\}$.

Second, we investigate the aforementioned basis of polynomials $\{\Omega_\alpha|\alpha \in \mathbb{Z}^\infty_{\geq 0}\}$ defined by A. Lascoux [13] that deforms the Key basis. By extending the Kohnert moves of [10] we conjecturally give the first combinatorial rule for the $\Omega$-polynomials.

Third, in [10], the Kohnert moves were used to conjecture the first combinatorial rule for Schubert polynomials (a proof was later presented in [25]). Similarly, we use the extended Kohnert moves to give a conjecture for the basis of Grothendieck polynomials $\{G_w|w \in S_\infty\}$ [17]. This rule appears significantly different than earlier (proved) rules, such as those in [7, 13, 3, 19].

1.2. Splitting Key polynomials. Let $S_\infty$ be the group of permutations of $\mathbb{N}$ with finitely many non-fixed points. This group acts on $\Pol$ by permuting the variables. Let $s_i$ be the simple transposition interchanging $x_i$ and $x_{i+1}$. The divided difference operator acts on $\Pol$ by

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}.$$ 

Define the Demazure operator by setting

$$\pi_i(f) = \partial_i(x_i \cdot f), \quad \text{for } f \in \Pol.$$ 

Let $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}^\infty_{\geq 0}$ and assume throughout that $|\alpha| = \sum_i \alpha_i < \infty$. Define the Key polynomial $\kappa_\alpha$ to be

$$x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\cdots, \quad \text{if } \alpha \text{ is weakly decreasing.}$$

Otherwise, set

$$\kappa_\alpha = \pi_i(\kappa_{\hat{\alpha}}) \text{ where } \hat{\alpha} = (\ldots, \alpha_{i+1}, \alpha_i, \ldots) \text{ and } \alpha_{i+1} > \alpha_i.$$ 

(The $\pi_i$’s (and $\partial_i$’s also) are well-known to satisfy the braid relations for $S_n$ and so the $\kappa_\alpha$’s are independent of the order in which the $\pi_i$’s are applied.) Since the leading term (under the pure reverse lexicographic order) of $\kappa_\alpha$ is $x^\alpha$, the Key polynomials form a $\mathbb{Z}$-basis of $\Pol$. 

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The Key polynomials lift the Schur polynomials: when
\begin{equation}
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, 0, 0, \ldots), \text{ where } \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_t, \text{ then}
\end{equation}

\begin{equation}
\kappa_{\alpha} = s_{(\alpha_1, \ldots, \alpha_2, \alpha_1)}(x_1, \ldots, x_t).
\end{equation}

A descent of $\alpha$ is an index $i$ such that $\alpha_i \geq \alpha_{i+1}$; a strict descent is an index $i$ such that $\alpha_i > \alpha_{i+1}$. Fix descents $d_1 < d_2 < \ldots < d_k$ of $\alpha$ containing all strict descents of $\alpha$. Since $\pi_i$ is a symmetrizing operator, $\kappa_{\alpha}$ is separately symmetric in each collection:

\[X_1 = \{x_1, x_2, \ldots, x_{d_1}\}, \quad X_2 = \{x_{d_1+1}, x_{d_2+1}, \ldots, x_{d_2}\}, \ldots, \quad X_k = \{x_{d_{k-1}+1}, x_{d_{k-1}+2}, \ldots, x_{d_k}\}.\]

(The variables $x_{d_{k-1}+1}, x_{d_{k-2}+2}, \ldots$ do not appear in $\kappa_{\alpha}$.) Therefore, uniquely:

\begin{equation}
\kappa_{\alpha}(X) = \sum_{\lambda_1, \ldots, \lambda_k} \mathcal{E}_{\lambda_1, \ldots, \lambda_k}^\alpha s_{\lambda_1}(X_1) \cdots s_{\lambda_k}(X_k),
\end{equation}

where each $\lambda_i$ is a partition. \textit{A priori} one only knows $\mathcal{E}_{\lambda_1, \ldots, \lambda_k}^\alpha \in \mathbb{Z}$.

The \textbf{Rothe diagram} of a permutation $w \in S_n$ is
\[
\text{Rothe}(w) = \{(x, y) \middle| y < w(x) \text{ and } x < w^{-1}(y)\} \subset [n] \times [n]
\]
indexed so that the southwest corner is labeled $(1, 1)$. The \textbf{code} of $w$, denoted $\text{code}(w) \in \mathbb{Z}_{\geq 0}^n$ counts the number of boxes in columns of $\text{Rothe}(w)$ (from left to right). Given $\alpha \in \mathbb{Z}_{\geq 0}^n$, there is a unique $w[\alpha] \in S_n$ such that $\text{code}(w[\alpha]) = \alpha$ (up to trailing $0$’s); see, e.g., [20, Proposition 2.1.2]. We will need a special tableau coming from [24, Section 4]:

The tableau $T[\alpha]$: Given $w[\alpha], i_1 < i_2 < \ldots < i_6$ in the first column of $T[\alpha]$ are given by having $i_j$ be the largest descent position smaller than $i_{j+1}$ in the permutation $ws_{i_5}s_{i_4-1}\cdots s_{i_1+1}$. The next column of $T[\alpha]$ is similarly determined, starting from $ws_{i_6}\cdots s_{i_2}, \text{ etc.}$

An \textbf{increasing tableau} $T$ of shape $\lambda$ is a filling with strictly increasing rows and columns. (In fact, $T[\alpha]$ is an increasing tableau.) Let $\text{row}(T)$ be the reading word of $T$, obtained by reading the entries of $T$ along rows, from right to left, and from top to bottom. Let $\text{min}(T)$ be the smallest label in $T$. Finally, given a reduced word $w = a_1a_2\ldots a_m$, let $\text{EGLS}(a)$ be the output of the \textit{Edelman-Greene correspondence} (see Section 2.1).

The following result shows $\mathcal{E}_{\lambda_1, \ldots, \lambda_k}^\alpha \in \mathbb{Z}_{\geq 0}$. It is analogous to one on Schubert polynomials [4, Corollary 3] (which our proof uses).

\textbf{Theorem 1.1.} The number $\mathcal{E}_{\lambda_1, \ldots, \lambda_k}^\alpha$ counts sequences of increasing tableaux $(T_1, T_2, \ldots, T_k)$ where

- $T_i$ is of shape $\lambda^i$;
- $\text{min} T_1 > 0, \text{min} T_2 > d_1, \text{min} T_3 > d_2, \ldots, \text{min} T_k > d_{k-1}$; and
- $\text{row}(T_1) \cdot \text{row}(T_2) \cdots \text{row}(T_k)$ is a reduced word of $w[\alpha]$ such that
  \[
  \text{EGLS}(\text{row}(T_1) \cdot \text{row}(T_2) \cdots \text{row}(T_k)) = T[\alpha].
  \]

When $d_j = j$ for all $j \geq 1$, Theorem 1.1 specializes to an instance of the monomial expansion formula [21, Theorem 5(1)] for $\kappa_{\alpha}$ (restated as Theorem 2.5 below). Also, when (1) holds, $k = 1, d_1 = t$ and thus Theorem 1.1 gives (2).

\textbf{Example 1.2.} The (strict) descents of $\alpha = (1, 3, 0, 2, 2, 1)$ are $d_1 = 2, d_2 = 5$, and

\[
\kappa_{1,3,0,2,2,1} = s_{3,2}(x_1, x_2)s_{2,1,1}(x_3, x_4, x_5) + s_{2,2}(x_1, x_2)s_{2,1}(x_3, x_4, x_5)s_1(x_6)
\]

\[
+ s_{3,1}(x_1, x_2)s_{2,2}(x_3, x_4, x_5)s_1(x_6) + s_{3,1}(x_1, x_2)s_{2,2,1}(x_3, x_4, x_5).
\]
exhibits the claimed non-negativity of Theorem 1.1.

Also, \( w[\alpha] = 2516743 \) (one line notation) and \( T[\alpha] = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 \\ 4 & 6 \\ 5 & 0 \end{bmatrix} \). Thus, \( \mathcal{E}^{(1,3,0,2,2,1)} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 \\ 4 & 6 \\ 5 & 0 \end{bmatrix} \).

\( \mathcal{E}^{(1,3,0,2,2,1)} = \mathcal{E}^{(1,3,0,2,2,1)} = \mathcal{E}^{(1,3,0,2,2,1)} = \mathcal{E}^{(1,3,0,2,2,1)} = 1 \) are respectively witnessed by

\[
\left( \begin{array}{cccc}
1 & 3 & 4 & 6 \\
2 & 5 & 6 & 0 \\
\end{array} \right), \quad \left( \begin{array}{cccc}
1 & 3 & 4 & 6 \\
2 & 5 & 6 & 0 \\
\end{array} \right), \quad \left( \begin{array}{cccc}
1 & 3 & 4 & 5 \\
2 & 5 & 6 & 0 \\
\end{array} \right), \quad \text{and} \quad \left( \begin{array}{cccc}
1 & 3 & 4 & 5 \\
2 & 5 & 6 & 0 \\
\end{array} \right).
\]

For example, for the leftmost sequence, \( \text{EGLS}(43152 \cdot 6456 \cdot \emptyset) = T[\alpha] \) holds. \( \square \)

1.3. The \( \Omega \) polynomials. A. Lascoux [13] defines \( \Omega_\alpha \) for \( \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0}^\infty \) by replacing \( \pi_i \) in the definition of the Key polynomials with the operator defined by

\( \bar{\pi}_i(f) = \partial_i(x_i(1 - x_{i+1})f). \)

(These operators also can be seen to satisfy the braid relations; cf. [11, Chapter 1.4].)

The initial condition is \( \Omega_\alpha = x^\alpha (= \kappa_\alpha) \), if \( \alpha \) is weakly decreasing. The \( \Omega \) polynomials deform the Key polynomials. While at present there is no known geometric or representation theoretic interpretation of the \( \Omega \) polynomials, as is pointed out in \textit{loc. cit.}, many of the known relationships between the Key and Schubert basis extend to ones between the \( \Omega \) and Grothendieck basis (the latter family is formally recalled in the next subsection).

The \textbf{skyline diagram} is \( \text{Skyline}(\alpha) = \{(i, y) : 1 \leq y \leq \alpha_i \} \subset \mathbb{N}^2 \). Graphically, it is a collection of columns \( \alpha_i \) high. For instance,

\( \text{Skyline}(1, 3, 0, 2, 2, 1) = \left( \begin{array}{cccc}
. & + & . & . \\
. & . & + & + \\
+ & + & . & + \\
\end{array} \right) \).

Beginning with \( \text{Skyline}(\alpha) \), \textbf{Kohnert’s rule} [10] generates diagrams \( D \) by sequentially moving any \( + \) at the top of its column to the rightmost open position in its row and to its left. (The result of such a move need not be the skyline of any \( \gamma \in \mathbb{Z}_{\geq 0}^\infty \).) Let \( x^D = \prod_i x_i^{d_i} \) be the column weight where \( d_i \) is the number of \( + \)’s in column \( i \) of \( D \). If the same \( D \) results from a different sequence of moves, it only counts once. Kohnert’s theorem states \( \kappa_\alpha = \sum x^D \), where the sum is over all such \( D \). Extending this, we introduce:

The \textbf{K-Kohnert rule:} Each \( + \) either moves as in Kohnert’s rule, or stays in place \textit{and} moves. That is, in the latter case, we mark the original position with a “\( g \)” and we place a \( + \) in the rightmost open position in its row and to the left of the original position. The \( g \)’s are unmovable, and a given \( + \) cannot move past a \( g \). Diagrams with the same occupied positions but different arrangements of \( + \)’s and \( g \)’s are counted separately.\(^1\)

\(^1\)Added March 10, 2017: The published version misstates this conjecture by allowing boxes to move past \( g \)’s. However, this version is consistent with the older report [23] (and was what was computationally checked there). The second author apologizes for this error.
Thus of w

Example 1.4. (Once again, these operators are known to satisfy the braid relations.) Declare G

1.3. Below, we give all K-Kohnert moves one step from D:

Thus

\[ D = (+ + + +) \mapsto (+ + + +) \mapsto (+ + + +) \mapsto (+ + + +) . \]

Thus

\[ J_{\alpha}^{(1)} = \Omega_\alpha . \]

Conjecture 1.4 has been checked by computer, for a wide range of cases up to \( \alpha \) being of size 12, leaving us convinced. Clearly, \( J_{\alpha}^{(0)} = \kappa_\alpha, \) by Kohnert’s theorem.

Example 1.5. Let \( \alpha = (1, 0, 2) \). Then the diagrams contributing to \( J_{(1,0,2)} \) are:

\[ \text{Skyline}(1,0,2) = \left( \begin{array}{c} . \times . + . \\
+ + + . \\
\times + + . \end{array} \right) , \left( \begin{array}{c} . \times . + . \\
+ + + . \\
\times + + . \end{array} \right) , \left( \begin{array}{c} . \times . + . \\
+ + + . \\
\times + + . \end{array} \right) ; \]

Thus

\[ J_{(1,0,2)} = (x_1x_3^2 + x_1x_2x_3 + x_1x_2^2 + x_1x_2) - (x_1^2x_2x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2x_3^2 + x_1x_2^2x_3^2) + (x_1^2x_2^2x_3 + x_1^2x_2x_3^2) . \]

The lowest degree homogeneous component of \( \Omega_\alpha \) is \( \kappa_\alpha \). Hence any \( f \in \text{Pol} \) is a possibly infinite linear combination of the \( \Omega_\alpha \)’s. Finiteness is asserted in [11, Chapter 5]. We show in Section 4.2 that the \( J_\alpha \)’s also form a (finite) basis.

1.4. Grothendieck polynomials. The Grothendieck polynomial [17] is defined using the isobaric divided difference operator whose action on \( f \in \text{Pol} \) is given by:

\[ \pi_i(f) = \partial_i((1 - x_{i+1})f) . \]

(Once again, these operators are known to satisfy the braid relations.) Declare \( \mathfrak{G}_{w_0}(X) = x_1^{n-1}x_2^{n-2} \cdots x_{n-1} \) where \( w_0 \) is the long element in \( S_n \). Set \( \mathfrak{G}_w(X) = \pi_i(\mathfrak{G}_{w_{s_i}}) \) if \( i \) is an ascent of \( w \). The Grothendieck polynomials are known to lift \( \{s_\lambda\} \) to Pol.

One has \( \mathfrak{G}_w = \mathfrak{G}_w + (\text{higher degree terms}) \). We now state the A. Kohnert’s conjecture [10] for \( \mathfrak{G}_w \). Starting with Rothe(\( w \)), the Kohnert’s rule generates diagrams \( D \) by applying the same rules as described for his rule for \( \kappa_\alpha \). Then \( \mathfrak{G}_w = \sum_x D_x \); the sum is over all such \( D \).

Analogously, we define

\[ K_{w}^{(\beta)} = \sum_{D} g(\# g's \text{ appearing in } D) x^D . \]
where the sum is over all diagrams $D$ generated by the $K$-Kohnert rule. For example, if $w = 3142$ the diagrams contributing to $K_{3142}^{(3)}$ are

\[
\begin{pmatrix}
\cdots \\
+ \\
. \\
\end{pmatrix}, \begin{pmatrix}
\cdots \\
+ + \\
. \\
\end{pmatrix}, \begin{pmatrix}
\cdots \\
+ + g \\
. \\
\end{pmatrix}
\]

and hence correspondingly, $K_{3142}^{(-1)} = (x_1^2 x_3 + x_2^2 x_3) - (x_1^2 x_2 x_3)$.

**Conjecture 1.6.** $K_w^{(-1)} = \mathcal{G}_w$.

Note, $K_w^{(0)} = \mathcal{G}_w$ is precisely Kohnert’s conjecture. Conjecture 1.6 has been checked by computer for $n \leq 7$, and extensively for larger $n$. While Kohnert’s rule for $\mathcal{G}_w$ is handy, it remains mysterious, even after [25]. Conjectures 1.4 and 1.6 represent a return to Kohnert’s conjecture (albeit after introducing a parameter $\beta$).

2. Proof of Theorem 1.1

2.1. Reduced word combinatorics. Given $w \in S_n$, let

\[
\mathbf{a} = (a_1, a_2, \ldots, a_{\ell(w)}) \quad \text{and} \quad \mathbf{i} = (i_1, i_2, \ldots, i_{\ell(w)}).
\]

In connection to [1], we say the pair $(\mathbf{a}, \mathbf{i})$ is a **stable compatible pair** for $w$ if $s_{a_1} \cdots s_{a_{\ell(w)}}$ is a reduced word for $w$ and the following two conditions on $\mathbf{i}$ hold:

1. $(\text{cs.1})$ $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{\ell(w)} < n$;
2. $(\text{cs.2})$ $a_j < a_{j+1} \implies i_j < i_{j+1}$.

We will identify $w$ with $\mathbf{a}$ and the associated reduced word.

The **Edelman-Greene correspondence** [6] (the same basic construction is used in [17]) is a bijection

\[
\text{EGLS} : (\mathbf{a}, \mathbf{i}) \mapsto (T, U)
\]

where

- $T$ is an increasing tableau such that $\tau w(T)$ is a reduced word for $\mathbf{a}$; and
- $U$ is a semistandard tableau whose multiset of labels is precisely those in $\mathbf{i}$, and which has the same shape as $T$.

**EGLS (column) insertion:** We insert $\mathbf{a}$ from left to right, starting with $a_1$. When we reach step $j$ of this process, we initially insert $a_j$ into the leftmost column (of what will be $T$). If there are no labels strictly larger than $a_j$, we place $a_j$ at the bottom of that column. If $a_j + t$ for $t > 2$ appears, we bump this $a_j + t$ to the next column to the right, replacing it with $a_j$. The same holds if $a_j + 1$ appears but not $a_j$. Finally, if both $a_j + 1$ and $a_j$ already appear, we insert $a_j + 1$ into the next column to the right. Since $\mathbf{a}$ is assumed to be reduced, the above enumerates all possibilities. Finally at step $j$ a new box is created at a corner; in what will be $U$ we place $i_j$.

Mildly abusing terminology, let $\text{EGLS}(\mathbf{a}) = T$.

We will need another standard notion in the subject. Two reduced words $\mathbf{a}$ and $\mathbf{a}'$ for the same permutation are in the same **Coxeter-Knuth class** if $\text{EGLS}(\mathbf{a}) = \text{EGLS}(\mathbf{a}') = T$.  

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This $T$ represents the class. This equivalence relation $\sim$ on reduced words is defined by the symmetric and transitive closure of the relations:

\begin{align*}
A_{i(i+1)}B & \sim A(i+1)i(i+1)B \\
AacbB & \sim AcabB \\
AbacB & \sim AbcaB
\end{align*}

where $a < b < c$. In particular, it is true that $a \sim \text{row}(\text{EGLS}(a))$.

2.2. Formulas for Schubert polynomials. A stable compatible pair $(a, i)$ is a compatible pair for $w$ if in addition to (cs.1) and (cs.2) the following holds:

(cs.3) $i_j \leq a_j$.

Let $\text{Compatible}(w)$ be the set of compatible sequences for $w$. A rule of [1] states:

\begin{equation}
\mathfrak{G}_w(X) = \sum_{(a, i) \in \text{Compatible}(w)} x^i.
\end{equation}

A descent of $w$ is an index $j$ such that $w(j) > w(j + 1)$. Let $\text{Descents}(w)$ be the set of descents of $w$. The following is [4, Corollary 3]:

**Theorem 2.1.** Let $w \in S_n$ and suppose $\text{Descents}(w) \subseteq \{d_1 < d_2 < \ldots < d_k\}$. Then

\begin{equation}
\mathfrak{G}_w(X) = \sum_{\lambda_1, \ldots, \lambda_k} c_{\lambda_1, \ldots, \lambda_k}^w s_{\lambda_1}(X_1) \cdots s_{\lambda_k}(X_k)
\end{equation}

where $c_{\lambda_1, \ldots, \lambda_k}^w$ counts the number of tuples of increasing tableaux $(T_1, \ldots, T_k)$ where

(i) $T_i$ has shape $\lambda_i$;
(ii) $\min T_1 > 0$, $\min T_2 > d_1, \ldots, \min T_k > d_{k-1}$; and
(iii) $\text{row}(T_1) \cdots \text{row}(T_k)$ is a reduced word of $w$.

Assume for the remainder of the proof that

\begin{equation}
\text{Descents}(w) \subseteq \{d_1 < d_2 < \ldots < d_k\}.
\end{equation}

Let

$$\text{Tuples}(w) = \{(T_1, U_1), (T_2, U_2), \ldots, (T_k, U_k)\}$$

where the $T_i$’s satisfy (i), (ii) and (iii) from Theorem 2.1, and each $U_i$ is a semistandard tableau of shape $\lambda_i$ using the labels $d_{i-1} + 1, d_{i-1} + 2, \ldots, d_i$ ($d_0 = 0$).

2.3. “Splitting” the EGLS correspondence. Assuming (7) we define:

$$\Phi : \text{Compatible}(w) \to \text{Tuples}(w).$$

Description of $\Phi$ (using EGLS): Uniquely split $(a, i) \in \text{Compatible}$ as follows

\begin{equation}
((a^{(1)}, i^{(1)}), (a^{(2)}, i^{(2)}), \ldots, (a^{(k)}, i^{(k)}))
\end{equation}

where

- $a = a^{(1)} \cdots a^{(k)}$ and $i = i^{(1)} \cdots i^{(k)}$ (“...” means concatenation); and
- the entries of $i^{(j)}$ are contained in the set $\{d_{j-1} + 1, d_{j-1} + 2, \ldots, d_j\}$.

Now define

$$\Phi((a, i)) := (\text{EGLS}(a^{(1)}, i^{(1)}), \ldots, \text{EGLS}(a^{(k)}, i^{(k)})).$$
Proposition 2.2. The map \( \Phi : \text{Compatible}(w) \to \text{Tuples}(w) \) is well-defined and a bijection.

Proof. \( \Phi \) is well-defined: The condition (i) is just says \( T_j \) and \( U_j \) have the same shape, which is true by EGLS’ s description. For (ii), the splitting says each label in \( \mathcal{V}(j) \) is strictly bigger than \( d_{j-1} \). Now by (cs.3), each label in \( a(j) \) is strictly bigger than \( d_{j-1} \) as well. By EGLS’s definition, the set of labels appearing in \( T_j \) is the same as that of \( a(j) \); hence (ii) holds. Lastly, \( \text{row}(T_j) \) is a reduced word for \( a(j) \). Then (iii) is clear.

\( \Phi \) is a bijection: Since EGLS is a bijective correspondence, clearly \( \Phi \) is an injection. Consider the weight function on \( \text{Compatible}(w) \) that assigns \( (a,i) \) weight \( x^i \) and assigns \( (T_1,U_1), \ldots, (T_k,U_k) \) the weight \( x^{U_1} \cdots x^{U_k} \), where \( x^{U_i} \) is the usual monomial associated to the tableau \( U_i \). Then clearly \( \Phi \) is a weight-preserving map (since EGLS is similarly weight-preserving). Hence the surjectivity of \( \Phi \) holds by (5) and Theorem 2.1. \( \square \)

Proposition 2.2. The map \( \Phi : \text{Compatible}(w) \to \text{Tuples}(w) \) is well-defined and a bijection.

Proof. \( \Phi \) is well-defined: The condition (i) is just says \( T_j \) and \( U_j \) have the same shape, which is true by EGLS’s description. For (ii), the splitting says each label in \( \mathcal{V}(j) \) is strictly bigger than \( d_{j-1} \). Now by (cs.3), each label in \( a(j) \) is strictly bigger than \( d_{j-1} \) as well. By EGLS’s definition, the set of labels appearing in \( T_j \) is the same as that of \( a(j) \); hence (ii) holds. Lastly, \( \text{row}(T_j) \) is a reduced word for \( a(j) \). Then (iii) is clear.

\( \Phi \) is a bijection: Since EGLS is a bijective correspondence, clearly \( \Phi \) is an injection. Consider the weight function on \( \text{Compatible}(w) \) that assigns \( (a,i) \) weight \( x^i \) and assigns \( (T_1,U_1), \ldots, (T_k,U_k) \) the weight \( x^{U_1} \cdots x^{U_k} \), where \( x^{U_i} \) is the usual monomial associated to the tableau \( U_i \). Then clearly \( \Phi \) is a weight-preserving map (since EGLS is similarly weight-preserving). Hence the surjectivity of \( \Phi \) holds by (5) and Theorem 2.1. \( \square \)

See [18, Section 5] for a proof of Theorem 2.1 which is close to the study of the split EGLS correspondence (the argument constructs certain crystal operators).

2.4. The tableau \( T[\alpha] \). Recall \( w[\alpha] \in S_\infty \) satisfies \( \text{code}(w[\alpha]) = \alpha \). Let \( \prec \) be the pure reverse lexicographic total ordering on monomials. The Schubert polynomial \( \mathcal{S}_{w[\alpha]} \) has leading term \( x^\alpha \) (with respect to \( \prec \)). The same is true of \( \kappa_\alpha \) (see [21, Corollary 7]) so

\[
\mathcal{S}_{w[\alpha]} = \kappa_\alpha + \text{linear combination of other Key polynomials.}
\]

Given an increasing tableau \( U \), the nil left Key \( \mathcal{K}[U] \) is defined by [16] (cf. [21, p.111–114]). Let \( \text{sort}(\alpha) \) be the partition obtained by rearranging \( \alpha \) into weakly decreasing order. Also let \( \text{content}(T) \) the usual content vector of a semistandard tableau \( T \). This is a result of A. Lascoux-M.-P. Schützenberger (cf. [21, Theorem 4]):

Theorem 2.3.

\[
\mathcal{S}_{w[\alpha]}(X) = \sum_{\text{sort}(\alpha) \in \mathcal{K}[U]} \kappa_{\text{content}(\alpha)}(\mathcal{K}[U])
\]

where the sum is over all increasing tableaux \( U \) of shape \( \text{sort}(\alpha) \) with \( \text{row}(U) = w \).

Thus, by (9) combined with Theorem 2.3 there exists a unique increasing tableau \( U[\alpha] \) of shape \( \text{sort}(\alpha) \) with \( \text{row}(U[\alpha]) = w[\alpha] \) and such that \( \alpha = \text{content}(\mathcal{K}[U[\alpha]]) \).

Let \( F_w = \lim_{k \to \infty} \mathcal{S}_{1^k \times \mu(w)} \) be the stable Schubert polynomial associated to \( w \). This is a symmetric polynomial in infinitely many variables. So therefore one has an expansion

\[
F_w = \sum_{\lambda} a_{w,\lambda}s_\lambda,
\]

where the \( a_{w,\lambda} \) are counted by increasing tableaux \( A \) of shape \( \lambda \) with \( \text{row}(A) = w \). We mention that (6) is derived from (10) in [4]; thereby, (10) may be seen as a specialization of (6).

In [24, Theorem 4.1], it is shown \( a_{w,\mu(w)} = 1 \) for a certain explicitly described “maximal” \( \mu'(w) \). Moreover a simple description of the witnessing tableau \( A[\alpha] \) is given. Straightforwardly, \( \mu'(w[\alpha]) = \text{sort}(\alpha) \). Then \( T[\alpha] \) is precisely the witnessing tableau \( A[\alpha] \) for \( a_{w[\alpha],\mu(w[\alpha])} \) (after accounting for the fact that [24]'s conventions use \( F_{w[\alpha]} \) for what we call \( F_{w[\alpha]-1} \)). We leave the details to the reader.

Finally since the expansion of Theorem 2.3 refines (10); see, e.g., [21], we have:

\[
T[\alpha] = A[\alpha] = U[\alpha].
\]
So, $T[\alpha]$ is an increasing tableau of shape $\text{sort}[\alpha]$ with the properties that $\text{row}(T[\alpha]) = w[\alpha]$ and $\text{content}(K_{\alpha}(T[\alpha])) = \alpha$.

2.5. **Conclusion of the proof of Theorem 1.1:** From the definition of $\text{Rothe}(w[\alpha])$:

**Lemma 2.4.** The descents of $w[\alpha]$ are contained in the set of descents $d_1 < d_2 < \ldots < d_k$ of $\alpha$.

By Lemma 2.4 combined with Theorem 2.1 we obtain:

\[
\mathcal{G}_{w[\alpha]}(X) = \sum_{(a,i)} x^i = \sum_{\lambda_1, \ldots, \lambda_k} c_{\lambda_1, \ldots, \lambda_k}^{w[\alpha]} s_{\lambda_1}(X_1) \cdots s_{\lambda_k}(X_k).
\]

We recall the following formula [21, Theorem 5]:

**Theorem 2.5.** Fix an increasing tableau $T$ with $\text{content}(K_0^0(T)) = \alpha$. Then

\[
\kappa_\alpha = \sum_{(a,i)} x^i
\]

where the sum is over compatible sequences $(a, i)$ satisfying (cs.1), (cs.2), (cs.3) and $\text{EGLS}(a) = T$.

In view of (11) and the properties about $T[\alpha]$ stated immediately after said equation, we may set $T = T[\alpha]$ in Theorem 2.5 to obtain a monomial expansion formula for $\kappa_\alpha$ in terms of compatible pairs. Thus, our theorem statement is that $\kappa_\alpha$ is precisely equal to a prescribed subset of the summands of (12).

Thus to complete the proof, restrict $\Phi$ to those $(a, i) \in \text{Compatible}(w[\alpha])$ such that $\text{EGLS}(a) = T[\alpha]$. Consider $\Phi(a, i) = [(T_1, U_1), \ldots, (T_k, U_k)]$. Since $a^{(i)} \sim \text{row}(T_i)$, by (4) we see

\[
\text{row}(T_1) \cdots \text{row}(T_k) \sim a^{(1)} \cdots a^{(k)} = a.
\]

However, since we have assumed $\text{EGLS}(a) = T[\alpha]$, therefore:

\[
\text{EGLS}(\text{row}(T_1) \cdots \text{row}(T_k)) = T[\alpha],
\]

The other two requirements on $(T_1, \ldots, T_k)$ hold since $\Phi$ is well-defined (Proposition 2.2). The desired conditions on $(U_1 \ldots, U_k)$ follow from $\Phi$’s well-definedness and Lemma 2.4.

Conversely, suppose $[(T_1, U_1), \ldots, (T_k, U_k)]$ has $(T_1, \ldots, T_k)$ satisfying Theorem 1.1’s conditions. Since $\Phi$ is a bijection (Proposition 2.2), $\Phi^{-1}([(T_1, U_1), \ldots, (T_k, U_k)]) = (a, i) \in \text{Compatible}(w[\alpha])$. Also, by (13), $a \sim \text{row}(T_1) \cdots \text{row}(T_k)$. Now, we assumed (14) holds. Hence, $\text{EGLS}(a) = T[\alpha]$ as desired. 

\[\square\]

3. **Additional remarks**

3.1. **Comments on Theorem 1.1.** Since $\kappa_\alpha$ specialize non-symmetric Macdonald polynomials (see, e.g., [8, Section 5.3]), can one extend Theorem 1.1 in that direction?

Theorem 1.1 implies that the Key module of [21, Section 5] should have an action of $GL(d_1) \times GL(d_2 - d_1) \times \cdots \times GL(d_k - d_{k-1})$ such that the character is $\kappa_\alpha$.

V. Reiner suggests a variation of Theorem 1.1 using the plactic theory. The derivation should be similar, using formulas from [22]. However we are missing the analogue of [4, Corollary 4]; cf. [9, Sections 7, 8]. Theorem 1.1 naturally generalizes to Grothendieck polynomials, using [3, 2]; details may appear elsewhere.
3.2. \( J_\alpha \)'s form a (finite) basis of \( \text{Pol} \). Clearly, \( J_\alpha(X) = x_\alpha + \sum_{\beta \prec \alpha} c_\beta x_\beta \). One decomposes \( f \in \text{Pol} \) into a possibly infinite sum of \( J_\alpha \)’s:

(15) \[ f = \sum_{\alpha} g_\alpha J_\alpha \]

That is, find the \( \prec \) largest monomial \( x_{\theta_0} \) appearing in \( f^{(0)} := f \) (say with coefficient \( c_{\theta_0} \)) and let \( f^{(1)} := f - c_{\theta_0} J_{\theta_0} \). Thus \( f^{(1)} \) only contains monomials strictly smaller in the \( \prec \) ordering. Now repeat, defining \( f^{(t+1)} := f^{(t)} - c_{\theta_t} J_{\theta_t} \) where \( x_{\theta_t} \) is the \( \prec \)-largest monomial appearing in \( f^{(t)} \) etc. Since \( J_\alpha \) is not homogeneous, each step \( t \) potentially introduces \( \prec \)-smaller monomials but of higher degree. However, we claim:

**Proposition 3.1.** The expansion (15) is finite.

**Proof.** By the K-Kohnert rule, each \( \beta \) that appears in \( J_\alpha \) is contained in the smallest rectangle \( R \) that contains \( \alpha \). So the above procedure only involves the finitely many diagrams contained in \( R \) for one of the finitely many initial \( \alpha \in \mathbb{Z}_{\geq 0}^\infty \) such that \( x_\alpha \) is in \( f \). \( \square \)

3.3. **More on the interplay of Grothendieck and the \( \Omega \) polynomials.** M. Shimozono has suggested that the expansion of \( \mathcal{G}_w \) into \( \Omega_\alpha \) should alternate in sign, by degree. An explicit rule exhibiting this has been conjectured by V. Reiner and the second author.

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