MATH 2243 — FALL 2007 FINAL EXAM
DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA, MINNEAPOLIS

NAME: ______________________________

ID NUMBER: ______________

(1) Do not open this exam until you are told to begin.
(2) This exam has 14 pages including this cover and two intentionally blank pages for your use. There are 10 problems total. You have 3 hours.
(3) No notes or books are permitted.
(4) Only non-graphing calculators are permitted.
(5) Please turn off all cell phones.
(6) Place your ID card on your desk for inspection.
(7) Good luck!

PROBLEM	POINTS	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. Solve the following differential equation:

$$\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}.$$

You may leave your answer in implicit form.

2. Consider a population $P(t)$ satisfying the logistic equation $P'(t) = aP - bP^2$, where $\alpha = a$ is the constant birth rate per month per individual, and $\beta = bP$ is the death rate per month per individual. Assume that the initial population is 1000 individuals and there are 50 births and 20 deaths per month occurring at time $t = 0$.

(a) Calculate the constants a and b from the data.
(b) What is the maximum population which can be attained?
(c) How many months does it take for the population to reach two third of the limit population?

3. Consider the differential equation $dx/dt = -x^2 + kx - 1$ containing the parameter k. Determine the number and stability or unstability of the critical points depending on the value of k. Construct the bifurcation diagram.

4. Determine whether the following statements are **true** or **false**. If you state the statement is **false** give a counterexample that demonstrates your claim. *You will be scored* $+2$ *points for a complete correct answer,* 0 *for no answer and* -2 *for an incorrect answer. Thus guessing will be penalized.*

(a) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices then $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

(b) Given a collection of linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in a vector space V, any vector $\mathbf{x}$ can be expressed as $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_k\mathbf{v}_k$.

(c) If two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, then $\det \mathbf{A} = \det \mathbf{B}$.

(d) Suppose V is a vector space, $S \subseteq V$ is a subspace of V, $\mathbf{v} \in S$ and $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in V$. Then $\mathbf{a}, \mathbf{b} \in S$.

(e) An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\det A \neq 0$.

5. Let

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

be a 3×3 matrix.

(a) Find a collection of linearly independent vectors such that solution space corresponding to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is the set of linear combinations of those vectors.

(b) Find all values of the coefficients k and c such that the following system has:

(i) no solution.

(ii) infinitely many solutions.

(iii) only one solution.

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & c \\ 3 & 2 & 1 & 5 \\ 0 & 1 & k & 3 \end{array}\right]$$

6. Find the eigenvalues and a full set of linearly independent eigenvectors of the two following matrices. Determine which of them is diagonalizable and find a diagonalizing matrix S and a diagonal matrix D such that it is equal to $S^{-1}DS$.

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 3 & 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 3 & -3 & 5 \end{bmatrix}.$$

7. Find the general solution to the differential equation:

$$y^{(3)} - 2y'' + y' = x^2 + xe^x.$$

8. Apply the eigenvalue method to solve the following system of differential equations:

$$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= -5x_1 - 3x_2 - x_3 \\ x_3' &= 5x_1 + 5x_2 + 3x_3 \end{aligned}$$

9. Solve the system of linear equations

$$\begin{aligned} 3x_1 + x_2 + x_3 + 6x_4 &= 14 \\ x_1 - 2x_2 + 5x_3 - 5x_4 &= -7 \\ 4x_1 + x_2 + 2x_3 + 7x_4 &= 17 \end{aligned}$$

10.

(a) Find the Laplace transform of $f(t) = \sin^2(t/2)$.

(b) Find the inverse Laplace transform of $F(s) = \dfrac{2s^2 + 1}{2s^2(s^2 + 1)}$.

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