POSET EDGE DENSITIES, NEARLY REDUCED WORDS, AND BARELY SET-VALUED TABLEAUX

VICTOR REINER, BRIDGET EILEEN TENNER, AND ALEXANDER YONG

Abstract. In certain finite posets, the expected down-degree of their elements is the same whether computed with respect to either the uniform distribution or the distribution weighting an element by the number of maximal chains passing through it. We show that this coincidence of expectations holds for Cartesian products of chains, connected minuscule posets, weak Bruhat orders on finite Coxeter groups, certain lower intervals in Young's lattice, and certain lower intervals in the weak Bruhat order below dominant permutations. Our tools involve formulas for counting nearly reduced factorizations in 0-Hecke algebras; that is, factorizations that are one letter longer than the Coxeter group length.

1. Introduction

The *edge density* of a finite poset P is the ratio of the number of its covering relations $q \lessdot p$ to its cardinality $\#P$. One can also interpret this ratio as the expectation $\mathbb{E}(X)$ of a random variable $X(p)$ counting the elements covered by $p \in P$. That is, the random variable $X(p)$ computes the down-degree of p in the Hasse diagram of P, with respect to the uniform distribution.

If, instead, one assigns to each $p \in P$ a probability proportional to the number of maximal chains through p in P, then one can define a random variable $Y(p)$ whose value is again the down-degree of p in the Hasse diagram, but now weighted by that probability.

Given the different distributions in play, one would generally not expect the expectations for $X(p)$ and $Y(p)$ to be equal. However, our observation is that, in a variety of interesting settings, one does indeed find equality.

Definition. A finite poset P has *coincidental down-degree expectations (CDE)* if $\mathbb{E}(X) = \mathbb{E}(Y)$.

We may also refer to P as *being* CDE. This terminology will be made more precise in Definition 2.1. To motivate our study, consider the following examples of CDE posets.

- *Disjoint unions of chains* are CDE because the two probability distributions are the same in this setting.
- *Cartesian products of chains* are CDE because Proposition 2.13 will show that CDE is preserved under Cartesian products of graded posets.
- *Weak Bruhat order* on a finite Coxeter group is CDE. In fact, any weak order on the chambers of a (central, essential) *simplicial hyperplane arrangement* in $\mathbb{R}^r$ (or, more generally, the topes of an *oriented matroid* of rank r) is CDE, as will be shown in Corollary 2.22.
- *Tamari lattices* on polygon triangulations are CDE, as will be shown in Corollary 2.23.
- *Connected minuscule posets* are CDE, as will be shown in Theorem 2.10. Also the distributive lattices $J(P)$ associated to arbitrary minuscule posets P are CDE, as will be shown Theorem 2.11.
- Our main result, Theorem 1.1, exhibits a rich class of lower intervals in *Young's lattice* and in weak Bruhat orders on permutations, all of which are CDE. (In fact, this paper grew from an attempt to understand Corollary 1.3 of Theorem 1.1 in two different ways.)

Before stating our main result, we recall a few definitions. *Young's lattice* is the partial order on integer partitions λ according to containment of their *Ferrers diagrams* $\mu \subset \lambda$. The *(right) weak Bruhat order* on permutations in the symmetric group $\mathfrak{S}_n$ is the transitive closure of the relation $u \lessdot w$ if $w = us$ for some adjacent transposition $s = \sigma_i = (i, i+1)$ with $u(i) < u(i+1)$. A permutation $w = w(1) \cdots w(n) \in \mathfrak{S}_n$ is *vexillary* if it is 2143-avoiding; that is, if there are no quadruples $i_1 < i_2 < i_3 < i_4$ with $w(i_2) < w(i_1) <$

2010 *Mathematics Subject Classification.* 05A99, 05E10.
Key words and phrases. reduced word, 0-Hecke, nilHecke, monoid, Grothendieck polynomial, tableau, dominant, rectangular shape, staircase shape, set-valued.

$w(i_4) < w(i_3)$. Such a vexillary permutation has *shape* λ if λ is the weakly decreasing rearrangement of its *Lehmer code* $c(w) := (c_1(w), c_2(w), \ldots)$, where $c_i(w) := \#\{j > i : w(i) \geq w(j)\}$. Within the class of vexillary permutations we will consider three subclasses.

- A permutation w is *dominant* if it is 132-avoiding; that is, if there are no triples $i_1 < i_2 < i_3$ with $w(i_1) < w(i_3) < w(i_2)$. If we regard the symmetric group $\mathfrak{S}_n$ as a subset of $\mathfrak{S}_{n+1}$ via the embedding $w \mapsto w'$, where $w'(i) = w(i)$ for $1 \leq i \leq n$ and $w(n+1) = n+1$, then there is a unique dominant permutation in $\bigcup_{n \geq 0} \mathfrak{S}_n$ of shape λ, characterized by $c(w) = \lambda$ (without rearrangement).
- A permutation w is *Grassmannian* if it has at most one *descent*; that is, if $w(i) > w(i+1)$ for at most one value of i.
- A permutation w is *inverse Grassmannian* if w^{-1} is Grassmannian; that is, if $w^{-1}(i) > w^{-1}(i+1)$ for at most one value of i.

We will also want to consider a family of partitions generalizing both

- *$a \times b$ rectangles* $b^a = (b, b, \ldots, b)$, and
- *staircases* $\delta_d = (d-1, d-2, \ldots, 2, 1)$.

As such, we study the *rectangular staircase partitions* $\delta_d(b^a)$, whose Ferrers diagrams are staircases δ_d in which each square is replaced by an $a \times b$ block. One such partition appears in Figure 1.

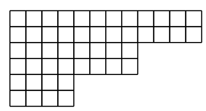


FIGURE 1. The rectangular staircase partition $\delta_4(4^2) = (12, 12, 8, 8, 4, 4)$. The dominant permutation with Lehmer code $c(w) = \delta_4(4^2)$ is $w = 13\ 14\ 9\ 10\ 5\ 6\ 1\ 2\ 3\ 4\ 7\ 8\ 11\ 12$ in $\mathfrak{S}_{14}$.

Theorem 1.1. *Let λ be a partition, and w a vexillary permutation of shape λ.*

(a) *The lower intervals $[\varnothing, \lambda]$ in Young's lattice and $[e, w]$ in weak Bruhat order have the same Y expectations; that is,*
$$\mathbb{E}(Y_{[\varnothing,\lambda]}) = \mathbb{E}(Y_{[e,w]}).$$
(b) *If w is either Grassmannian or inverse Grassmannian, then these intervals also have the same X expectations; that is,*
$$\mathbb{E}(X_{[\varnothing,\lambda]}) = \mathbb{E}(X_{[e,w]}).$$
(c) *If $\lambda = \delta_d(b^a)$ is a rectangular staircase and w is either*
- *dominant,*
- *Grassmannian, or*
- *inverse Grassmannian,*

then the posets $[\varnothing, \lambda]$ and $[e, w]$, and their duals $[\varnothing, \lambda]^*$ and $[e, w]^*$ are all CDE. Moreover,
$$\mathbb{E}(X) = \mathbb{E}(Y) = \frac{(d-1)ab}{a+b}$$
in each case.

Note that one always has $\mathbb{E}(X_P) = \mathbb{E}(X_{P^*})$ because the Hasse diagrams of P and P^* have the same edge densities, but $\mathbb{E}(Y_P)$ and $\mathbb{E}(Y_{P^*})$ need not be equal. Indeed, as will be described in Example 2.17, there are CDE posets P whose dual posets P^* are not CDE.

Conjecture 1.2. *When $\lambda = \delta_d(b^a)$, the conclusion in Theorem 1.1(c) holds for all vexillary w of shape λ.*

There is a close connection relating the instance of CDE given by Theorem 1.1(c) to recent work of Chan, Martín, Pflueger, and Teixidor i Bigas [CMPT15] and of Chan, Haddadan, Hopkins, and Moci [CHHM15]. The result [CMPT15, Corollary 2.15] (recapitulated as [CHHM15, Theorem 1.1]) calculates the expected "jaggedness" of a lattice path in an $a \times b$ grid under a certain probability distribution on paths. This is the central combinatorial fact used in [CMPT15] to reprove a formula of Eisenbud-Harris and of Pirola for the genera of Brill-Noether curves. Theorem 1.2 of [CHHM15] is a generalization of this jaggedness theorem to lattice paths in a general connected skew shape with respect to any "toggle symmetric" distribution. As is detailed further in Remark 2.6, Theorem 1.1(c) provides a different proof of [CMPT15, Corollary 2.15], whereas [CHHM15, Theorem 1.2] may be used to give a different proof of the assertion on $[\varnothing, \lambda]$ and $[\varnothing, \lambda]^*$ for $\lambda = \delta_d(b^a)$ in Theorem 1.1(c).

After covering the groundwork for CDE posets in Section 2, most of the paper is aimed toward proving the assertions of Theorem 1.1. We build up general techniques to compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ for $[\varnothing, \lambda]$ in Young's lattice using Young tableaux and set-valued tableaux (Section 3), and for $[e, w]$ in Coxeter groups involving reduced words and 0-Hecke words (Section 4). Tableaux reenter the discussion when we specialize to the symmetric group in Section 5, for reasons that we highlight now.

For a permutation w, maximal chains in the lower interval $[e, w]$ of weak Bruhat order correspond to *reduced words* for w. In particular, they describe factorizations $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_\ell}$ into adjacent transpositions σ_i having the minimum possible length ℓ, called $\ell(w)$. Stanley [St84] proved that the number of reduced words for any vexillary permutation w of shape λ is f^λ, the number of *standard Young tableaux* of shape λ, which has a simple product expression known as the Frame-Robinson-Thrall *hook-length formula* [St99, Corollary 7.21.6]. More generally, one can consider factorizations $T_w = T_{i_1}T_{i_2}\cdots T_{i_L}$ in the 0-*Hecke monoid* for permutations, with generators $T_1, \ldots, T_{n-1}$ satisfying the usual braid relations together with the quadratic relation $T_i^2 = T_i$. The 0-Hecke factorizations for w having the minimum length $L = \ell(w)$ correspond to reduced words as before. Using results from the theory of *Schubert* and *Grothendieck polynomials* (see Section 5.1), one can generalize Stanley's result to assert that the number of 0-Hecke words having length L for a vexillary permutation w of shape λ is the number of *standard set-valued tableaux* of shape λ having entries $1, 2, \ldots, L$, each appearing exactly once. Here a set-valued tableau has a subset of entries filling each square, but entries still increase from left-to-right in a row, and from top-to-bottom in a column. When $L = \ell(w) + 1$, we call the corresponding 0-Hecke words *nearly reduced* and the set-valued tableaux *barely set-valued*. This terminology will be made precise in Definition 3.2.

One no longer has a hook-length-style product formula for counting set-valued tableaux of any shape λ. However, we derive a general recurrence for counting these objects (Corollary 3.11), and use this to show that for dominant w whose shape is a rectangular staircase $\delta_d(b^a)$, one has this rephrasing of $\mathbb{E}(Y_{[e,w]}) = \frac{(d-1)ab}{a+b}$.

Corollary 1.3. *Let w be a dominant permutation of rectangular staircase shape $\lambda = \delta_d(b^a)$. Then the number of barely set-valued tableaux of shape λ (equivalently, the number of nearly reduced words for w) is*

$$(|\lambda| + 1)\,\frac{(d-1)ab}{a+b} f^\lambda.$$

Example 1.4. Taking $d = 3$ and $a = b = 1$, one has $\lambda = \delta_3(1^1) = (2, 1)$ and $w = 321$, with two reduced words: $(\sigma_1, \sigma_2, \sigma_1)$ and $(\sigma_2, \sigma_1, \sigma_2)$. Correspondingly, there are $f^\lambda = 2$ standard Young tableaux of shape λ:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

Meanwhile, there are eight 0-Hecke words of length 4 for w:

$$\{(T_1, T_1, T_2, T_1), (T_1, T_2, T_1, T_1), (T_1, T_2, T_1, T_2), (T_1, T_2, T_2, T_1),$$
$$(T_2, T_1, T_1, T_2), (T_2, T_1, T_2, T_1), (T_2, T_1, T_2, T_2), (T_2, T_2, T_1, T_2)\}\,.$$

These correspond to the eight barely set-valued tableaux of shape $\lambda = (2, 1)$:

$$\begin{array}{|c|c|}\hline 12 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 12 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 23 \\ \hline 4 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 23 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 24 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 24 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 34 \\ \cline{1-1}\end{array}\,, \text{ and } \begin{array}{|c|c|}\hline 1 & 34 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

This agrees with Corollary 1.3, which would have predicted this number to be

$$(|\lambda| + 1)\,\frac{(d-1)ab}{a+b} f^\lambda = (3+1)\frac{(3-1)\cdot 1\cdot 1}{1+1}\cdot 2 = 8.$$

Section 6 contains a conjecture (Conjecture 6.3) which is inspired both by Corollary 1.3 and by a formula of Fomin and Kirillov [FK97] (recapitulated here as Theorem 6.1). In fact, some support for our conjecture is derived from an extension (Theorem 6.7) of this Fomin-Kirillov formula, which we prove in Section 7.

We wish to highlight here one byproduct of our analysis. The calculation of $\mathbb{E}(X_{[\varnothing,\lambda]})$ for λ a rectangular staircase (Proposition 3.16) uses the $q = 1$ specialization of an easy recurrence for the *rank-generating function* $R(\lambda, q) := \sum_{\mu\subset\lambda} q^{|\mu|}$ of the interval $[\varnothing, \lambda]$ in Young's lattice. This recurrence encompasses

- the *q-Pascal recurrence* for *q-binomials* [St12, Equation (1.67)] when λ is a rectangle, and
- the recurrence for the *Carlitz-Riordan q-Catalan polynomial* which counts all Dyck paths by their enclosed area [Ha08, Proposition 1.6.1] when λ is a staircase,

but we were unable to find it in the literature.

Proposition 1.5. *For any partition λ,*

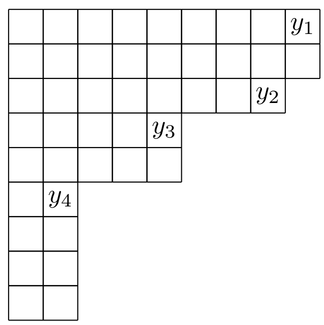
$$R(\lambda, q) = \sum_{x=(i,j)} q^{i(j-1)} \cdot R(\lambda_{(x)}, q) \cdot R(\lambda^{(x)}, q),$$

where x runs over all outside corner cells of λ, lying in row i and column j, and where

$$\lambda_{(x)} = (\lambda_{i+1}, \lambda_{i+2}, \ldots) \quad \textit{and} \quad \lambda^{(x)} = (\lambda_1 - j, \lambda_2 - j, \ldots, \lambda_{i-1} - j)$$

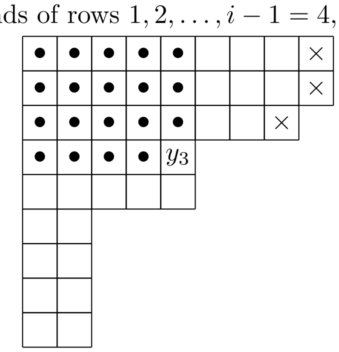
are the subshapes of λ in the rows strictly below x and the columns strictly to the right of x, respectively.

Proof. For $j > 1$, each outside corner cell $x = (i, j)$ has a cell $y = (i, j - 1)$ inside λ and directly to its left. For example, those neighboring cells are labeled $\{y_1, y_2, y_3, y_4\}$ in the shape below.



The recurrence in the theorem comes from classifying a shape $\mu \subset \lambda$ according to which, if any, is the northeasternmost cell y_i contained in μ. For example, with λ as above, consider the cell $y_3 = (i, j-1) = (4, 5)$, immediately to the left of the outside corner $x_3 = (i, j) = (4, 6)$. A partition $\mu \subset \lambda$ for which $y_3 \in \mu$ but $y_1, y_2 \notin \mu$ must contain

- all of the $i(j - 1) = 4 \cdot 5 = 20$ cells weakly northwest of y_3, labeled $\bullet$ in the figure below, and
- none of the cells at the east ends of rows $1, 2, \ldots, i - 1 = 4$, labeled $\times$ in the figure below.



Thus this μ is determined by its restrictions to the shapes in the unmarked cells of the figure above. The southwesternmost of these constitute exactly $\lambda_{(x_3)}$, and the northeasternmost form a copy of $\lambda^{(x_3)}$. $\square$

2. An overview of the CDE property

We now make precise the central theme of this paper, broached previously in Section 1.

Definition 2.1. Given a finite poset $(P, \leq)$, define two different probability spaces on the underlying set P.

- Let Ω_P^{unif} be the uniform distribution, assigning $\mathbf{Prob}(p) = 1/\#P$ for each $p \in P$.
- Let Ω_P^{chain} assign $\mathbf{Prob}(p)$ to be proportional to the number of maximal chains c in P containing p. That is, if $\mathcal{M}(P)$ is the set of maximal chains in P, then

$$\mathbf{Prob}(p) = \frac{\#\{c \in \mathcal{M}(P) : p \in c\}}{\#\{(c, q) \in \mathcal{M}(P) \times P : q \in c\}} = \frac{\#\{c \in \mathcal{M}(P) : p \in c\}}{\sum_{c \in \mathcal{M}(P)} \#c}.$$

Define random variables $X := X_P$ on Ω_P^{unif} and $Y := Y_P$ on Ω_P^{chain} via the same formula:

$$X(p) = Y(p) = \#\{q \in P : q \lessdot p\}.$$

A poset P has *coincidental down-degree expectations* (equivalently, P *is* CDE) if $\mathbb{E}(X) = \mathbb{E}(Y)$.

Whenever the poset P is graded, there is a natural way to interpolate between X and Y, pointed out to the authors by S. Hopkins, and suggested by the work in [CHHM15].

Definition 2.2. Given a finite poset P and a positive integer m, define a probability space $\Omega_P^{(m)}$ on the underlying set P, with $\mathbf{Prob}(p)$ proportional to the number of m-element *multichains* $p_1 \leq p_2 \leq \cdots \leq p_m$ in P that pass through p. On this probability space $\Omega_P^{(m)}$, define the random variable $X^{(m)} := X_P^{(m)}$ as before, where $X^{(m)}(p) = \#\{q \in P : q \lessdot p\}$ records the down-degree of the element p.

Two extreme cases are of particular interest. When $m = 1$, we have $(\Omega_P^{(1)}, X^{(1)}) = (\Omega_P^{\text{unif}}, X)$. On the other hand, if P is graded of rank r, that is, if all of its (inclusion-)maximal chains have exactly $r + 1$ elements, then it is not hard to see (cf. [CHHM15, Proposition 2.9]) that the pair $(\Omega_P^{(m)}, X^{(m)})$ approaches $(\Omega_P^{\text{chain}}, Y)$ in the limit as $m \to \infty$. Indeed, one can easily check (cf. [St12, §3.12]) that in this graded setting, the number of m-element multichains passing through p is a polynomial in m of degree r, and that the leading coefficient of this polynomial is $(1/r!) \cdot \#\{c \in \mathcal{M}(P) : p \in c\}$.

Definition 2.3. A finite poset P is *multichain-CDE* (written mCDE) if $\mathbb{E}(X^{(m)})$ is constant for $m \geq 1$.

In particular, observe that if P is both graded and mCDE, then P is also CDE; indeed, in that case we would have $\mathbb{E}(Y) = \lim_{m\to\infty} \mathbb{E}(X^{(m)}) = \mathbb{E}(X^{(1)}) = \mathbb{E}(X)$.

It will be helpful to know that for the distributive lattice $J(P)$ of order ideals I in P, the probability distribution $\Omega_{J(P)}^{(m)}$ is *toggle-symmetric*, a concept defined in [CHHM15] and which we explain now.

Definition 2.4. [CHHM15, Definition 2.2] Let P be a finite poset, and let I denote an order ideal in P. For a subset $A \subseteq P$, let $\max(A)$ (respectively, $\min(A)$) denote the subset of P-maximal (respectively, P-minimal elements) in A. A probability distribution on the finite distributive lattice $J(P)$ is *toggle-symmetric* if, for every $p \in P$, the distribution assigns the same probability to the event that $\max(I)$ contains p as it assigns to the event that $\min(P \setminus I)$ contains p.

Proposition 2.5. *For finite posets P, the distribution $\Omega_{J(P)}^{(m)}$ on $J(P)$ is toggle-symmetric.*

Proof. This is equivalent to showing that, for every $p \in P$, the following two sets have the same cardinality:

- the set of all pairs (I, c) in which I is an order ideal of P with $p \in \max(I)$, and $c = (I_1 \subseteq \cdots \subseteq I_m)$ is an m-element multichain in $J(P)$ that passes through I, and
- the set of all pairs (I', c') in which I' is an order ideal of P with $p \in \min(P \setminus I')$, and $c = (I'_1 \subseteq \cdots \subseteq I'_m)$ is an m-element multichain in $J(P)$ that passes through I'.

We provide a bijection between these two sets. Given (I, c), define two consecutive intervals of indices

$$[i_0, i_0 + a] := \{i : p \in \max(I_i)\}, \text{ and}$$
$$[i_0 + a + 1, i_0 + a + b] := \{i : p \in \min(P \setminus I_i)\},$$

so that $a, b \geq 0$, and one must have $I = I_{i_0+a_0}$ for some a_0 in the range $0 \leq a_0 \leq a$. To define the desired bijection, map $(I, c) \mapsto (I', c')$, where $I'_j \setminus \{p\} = I_j \setminus \{p\}$ for all j, and $I'_j = I_j$ if $j \notin [i_0, i_0 + a + b]$, but

$$[i_0, i_0 + b] := \{i : p \in \max(I'_i)\}, \text{ and}$$
$$[i_0 + b + 1, i_0 + a + b] := \{i : p \in \min(P \setminus I'_i)\},$$

and $I' := I'_{i_0+(a-a_0)+b}$. It is not hard to see that this map is a bijection. Its inverse is similarly defined. □

Remark 2.6. We can now further explain the connection between Theorem 1.1(c), and the two results [CMPT15, Corollary 2.15] and [CHHM15, Theorem 1.2] that were alluded to in Section 1. The result [CHHM15, Theorem 1.2] gives a vast generalization of [CMPT15, Corollary 2.15], which applies not only to Youngs lattice intervals $[\varnothing, \lambda]$ with $\lambda = b^a$ a rectangle, but also applies to arbitrary intervals $P = [\mu, \lambda]$. Such intervals are always finite distributive lattices, and the authors consider several families of toggle-symmetric probability distributions on them, including

- the uniform distribution Ω^{unif} used in defining X, and

- the distribution Ω^{chain} used in defining Y.

They show that for any toggle-symmetric distribution on the Young's lattice interval $[\mu, \lambda]$, if one defines A and B to be the number of nonempty rows and columns occupied by the skew shape λ/μ, then the down-degree random variable $d : P \to \mathbb{N}$ has expectation

- given by a formula [CHHM15, Theorem 1.2] showing it to be approximately equal to $\frac{AB}{A+B}$, and
- exactly equal to $\frac{AB}{A+B}$ when λ/μ satisfies a condition that they call *balanced* [CHHM15, Corollary 3.8].

We are lying slightly here, as the authors of [CHHM15] work not with down-degree, but with what they call *jaggedness*, which is down-degree plus up-degree. For toggle-symmetric probability distributions, this is equivalent to computing the expectation of down-degree: Definition 2.4 immediately implies that a toggle-symmetric probability distribution assigns down-degree and up-degree the same expectation, which must therefore be half the expectation that it assigns to the jaggedness statistic.

It is not hard to see that when $\mu = \varnothing$ and $\lambda = \delta_d \circ b^a$ is a rectangular staircase, then $\lambda/\mu = \lambda$ is balanced. Since, in this case, $A = (d-1)a$ and $B = (d-1)b$, their result not only predicts our formula from Theorem 1.1(c), but also shows that $[\varnothing, \lambda]$ is mCDE, with

$$\mathbb{E}(X) = \mathbb{E}(X^{(m)}) = \mathbb{E}(Y) = \frac{AB}{A+B} = (d-1)\frac{ab}{a+b}.$$

Remark 2.7. After seeing this, one might wonder whether some of the weak order intervals that our Theorem 1.1 asserts are CDE have the stronger mCDE property. However, this can fail even for the intervals $[e, w]$ where w is dominant of rectangular staircase shape $\lambda = \delta_d(b^a)$ when $d \geq 3$. For example, if $d = 3$, $a = 1$, and $b = 2$, so that $\lambda = \delta_3(2^1) = (4, 2)$, and $w = 53124 \in \mathfrak{S}_5$ is dominant of shape λ, then the weak order interval $[e, w]$ has

$$\mathbb{E}(X^{(m)}) = \frac{2(14m^3 + 111m^2 + 199m + 76)}{21m^3 + 168m^2 + 299m + 112}$$

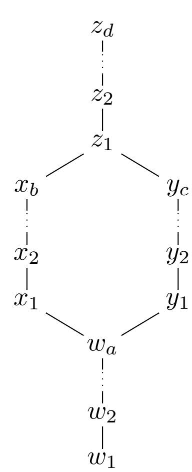
according to computations in SAGE.[1] As predicted by Theorem 1.1, this rational function has the correct value $(d-1)ab/(a+b) = 4/3$ at $m = 1$, and also in the limit as $m \to \infty$, but is not $4/3$ for integers $m \geq 2$.

2.1. **Examples of CDE posets.** We begin with some simple instances of CDE and mCDE posets.

Example 2.8. Finite disjoint unions of chains (that is, totally ordered sets) are CDE because each of their elements lie on exactly one maximal chain, and thus $(\Omega_P^{\text{unif}}, X) = (\Omega_P^{\text{chain}}, Y)$. If all of the chains have the same size, so that the poset is graded, then their union is also mCDE. This is because, similarly, $(\Omega_P^{\text{unif}}, X) = (\Omega_P^{(m)}, X^{(m)}) = (\Omega_P^{\text{chain}}, Y)$. On the other hand, one can check that when the chains have different sizes, the poset is CDE but might not be mCDE.

The following poset family is similarly straightforward, and will be used in the proof of Theorem 2.10.

Example 2.9. Consider the following poset $P_{a,b,c,d}$, parametrized by four positive integers a, b, c, d.



[1] `SAGE` code for calculating $\mathbb{E}(X^{(m)})$ as a rational function in m is available from the first author.

Fix $m \geq 1$, and denote by $f(p)$ the number of m-element multichains through p. One then computes

$$\begin{aligned} &f(w_i) = f(z_j) \text{ is constant for all } i = 1, 2, \dots, a, \text{ and } j = 1, 2, \dots, d, \\ &f(x_i) \text{ is constant for } i = 1, 2, \dots, b, \text{ and} \\ &f(y_i) \text{ is constant for } i = 1, 2, \dots, c. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}(X^{(m)}) &= \frac{(a-1)f(w_i) + bf(x_i) + cf(y_i) + (d-1)f(z_i) + 2f(z_i)}{af(w_i) + bf(x_i) + cf(y_i) + df(z_i)} \\ &= \frac{(a+d)f(w_i) + bf(x_i) + cf(y_i)}{(a+d)f(w_i) + bf(x_i) + cf(y_i)} \\ &= 1, \end{aligned}$$

and

$$\mathbb{E}(Y) = \frac{2 \cdot 0 + 2(a-1) + 1 \cdot b + 1 \cdot c + 2 \cdot 2 + 2(d-1)}{2(a+d) + b + c} = 1,$$

so every poset $P_{a,b,c,d}$ is both mCDE and CDE, whether it is graded (that is, whether $b = c$) or not.

The list of CDE posets in Section 1 mentioned another important family: the *minuscule posets*, which arise in the representation theory of Lie algebras, and have many amazing enumerative properties (see, for example, [Gr13, Chapter 11] and [Pr84a]). Up to poset isomorphism, the connected minuscule posets can be classified into three infinite families and two exceptional cases:

(a) the Cartesian product of two chains,
(b) the interval $[\varnothing, b^2]$ in Young's lattice,
(c) the special case $P_{a,1,1,a}$ of the posets $P_{a,b,c,d}$ from Example 2.9, and
(d) the posets $P(E_6)$ and $P(E_7)$ shown in Figure 2, with each element p labeled by $\#\{c \in \mathcal{M}(P) : p \in c\}$.

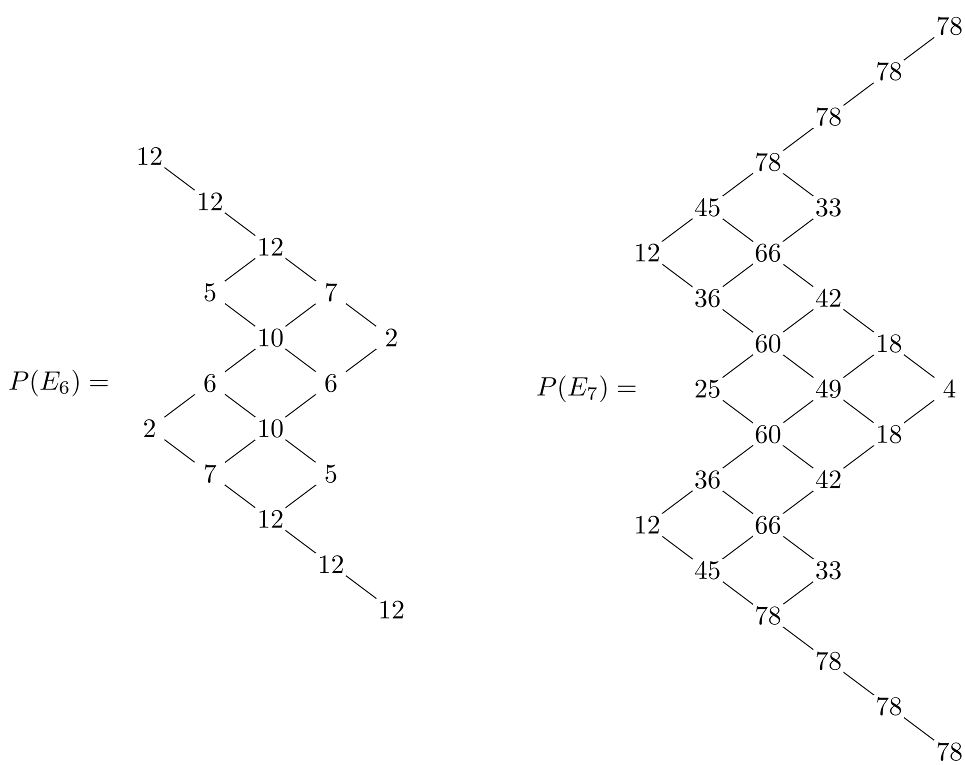


FIGURE 2. The minuscule posets $P(E_6)$ and $P(E_7)$, with elements labeled by the number of maximal chains passing through them.

Theorem 2.10. *Connected minuscule posets are mCDE, and, because they are graded, also CDE.*

Proof. The above classification lets one verify this case-by-case.

(a) Products of two chains will be shown to be mCDE in Proposition 2.18.
(b) Intervals $[\varnothing, b^2]$ in Young's lattice are mCDE by Proposition 2.5 and [CHHM15, Corollary 3.8].
(c) The posets $P_{a,1,1,a}$ of the family $P_{a,b,c,d}$ are mCDE by Example 2.9.
(d) For $P(E_6)$ and $P(E_7)$, calculations in `SAGE` showed that

$$\mathbb{E}(X_{P(E_6)}) = \mathbb{E}(X^{(m)}_{P(E_6)}) = \frac{5}{4} = \mathbb{E}(Y_{P(E_6)}), \text{ and}$$

$$\mathbb{E}(X_{P(E_7)}) = \mathbb{E}(X^{(m)}_{P(E_7)}) = \frac{4}{3} = \mathbb{E}(Y_{P(E_7)}).$$ □

Accompanying Theorem 2.10 is Theorem 2.11, concerning the distributive lattice of order ideals $J(P)$ when P is a minuscule poset. We can characterize this lattice $J(P)$ in terms of the root system Φ, the Weyl group W, and the minuscule dominant weight ω or simple root α corresponding to P (see [Pr84a]). In particular, one way to specify P is to pick a minuscule simple root α and take the restriction of the poset of positive roots Φ^+ to the positive roots lying weakly above α. Then $J(P)$ has the following two reinterpretations.

- $J(P)$ is the restriction of the (strong) Bruhat order to the set of minimum length coset representatives for W/W_ω, where W_ω is the maximal parabolic subgroup fixing ω.
- $J(P)$ is the weight poset on the W-orbit of ω, which indexes the weight spaces (all having multiplicity one) in the associated minuscule representation of the Lie algebra.

Theorem 2.11. *For (not necessarily connected) minuscule posets P, the distributive lattice $J(P)$ is CDE.*

Proof. Because disjoint unions affect the lattice of order ideals in a convenient way, namely

$$J(P_1 \sqcup \cdots \sqcup P_k) \cong J(P_1) \times \cdots \times J(P_k),$$

and finite distributive lattices $J(P)$ are always graded, we can apply Proposition 2.13 (below) to reduce to the case where P is connected. Now we again rely upon the classification of connected minuscule posets P preceding Theorem 2.10.

For the family (a), where $P = \boldsymbol{a} \times \boldsymbol{b}$ is the Cartesian product of two chains $\boldsymbol{a}$ and $\boldsymbol{b}$ having a and b elements, respectively, we have that $J(P) \cong [\varnothing, b^a]$ is CDE by Theorem 1.1 (and, in fact, mCDE by Proposition 2.5 and [CHHM15, Corollary 3.8]).

For the family (b), where $P = [\varnothing, b^2]$ in Young's lattice, we have $J(P) \cong [\varnothing, \delta_b]_{\text{shifted}}$ for the strict partition $\delta_{b+2} = (b+1, b, \ldots, 3, 2, 1)$. S. Hopkins [Ho16] has shown using the methods of [CHHM15] that $J(P)$ is not only CDE, but actually mCDE.

For the family (c) (that is, the special case $P = P_{a,1,1,a}$ among the posets $P_{a,b,c,d}$ from Example 2.9), one finds that $J(P) \cong P_{a+1,1,1,a+1}$, and hence it is also CDE (and, in fact, mCDE).

For the family (d) of Figure 2, one finds that $J(P(E_6)) \cong P(E_7)$, which we checked is CDE (and, in fact, mCDE) as part of Theorem 2.10. We have checked separately (both by hand and by computer) that $J(P(E_7))$ is CDE. □

In fact, all evidence points to the following strengthening of Theorem 2.11.

Conjecture 2.12. *For any minuscule poset P, the distributive lattice $J(P)$ is mCDE.*

If Question 2.20 below has an affirmative answer, then a case-by-case proof of Conjecture 2.12 is nearly within reach. In light of the proof of Theorem 2.11, where $J(P)$ was actually shown to be mCDE for almost all connected minuscule posets P, it would only remain to show that the distributive lattice $J(P(E_7))$ is mCDE. Nevertheless, we speculate that there should be a more uniform conceptual proof of Conjecture 2.12, interpreting multichains in $J(P)$ via standard monomial theory; see [Pr84a].

2.2. **CDE and poset operations.** Most poset operations do not consistently respect CDE. For example, disjoint union does not preserve the CDE property (Example 2.15), nor does ordinal sum (Example 2.16). The Cartesian product of graded posets, however, is an exception.

Proposition 2.13. *If two graded posets P and Q are CDE, then their Cartesian product $P \times Q$ is also CDE.*

As Example 2.14 will demonstrate, the "graded" assumption in Proposition 2.13 is essential, and thus the collection of all CDE posets is not closed under Cartesian product.

Before embarking on the proof of Proposition 2.13, we make an observation about graded posets. Recall that a finite poset P is graded with $\mathrm{rank}(P) = r$ if all maximal chains $c \in \mathcal{M}(P)$ contain $r+1$ elements; that is, each c has the form $\{p_0 \lessdot p_1 \lessdot \cdots \lessdot p_{r-1} \lessdot p_r\}$. Here are some straightforward reformulations of $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, the first of which was mentioned in Section 1:

$$\mathbb{E}(X) = \frac{\#\{(q,p) \in P \times P : q \lessdot p\}}{\#P}, \tag{1}$$
$$\mathbb{E}(Y) = \frac{\#\{(c,q,p) \in \mathcal{M}(P) \times P \times P : p \in c \text{ and } q \lessdot p\}}{\#\{(c,p) \in \mathcal{M}(P) \times P : p \in c\}},$$

and, in the case that P is graded, $\mathbb{E}(Y)$ can be rephrased as

$$\frac{\#\{(c,q,p) \in \mathcal{M}(P) \times P \times P : p \in c \text{ and } q \lessdot p\}}{(\mathrm{rank}(P)+1) \cdot \#\mathcal{M}(P)}. \tag{2}$$

Proof of Proposition 2.13. The down-degree function, $d_P : P \to \mathbb{N}$, satisfies $d_{P\times Q}(p,q) = d_P(p) + d_Q(q)$. Thus

$$\begin{aligned}
\mathbb{E}\left(X_{P\times Q}\right) &= \frac{1}{\#P \cdot \#Q} \sum_{(p,q)\in P\times Q} d_{P\times Q}(p,q) \\
&= \frac{1}{\#P \cdot \#Q} \sum_{p\in P} \sum_{q\in Q} \left(d_P(p) + d_Q(q)\right) \\
&= \frac{1}{\#P \cdot \#Q} \left(\#Q \sum_{p\in P} d_P(p) + \#P \sum_{q\in Q} d_Q(q) \right) \\
&= \frac{1}{\#P} \sum_{p\in P} d_P(p) + \frac{1}{\#Q} \sum_{q\in Q} d_Q(q) \\
&= \mathbb{E}(X_P) + \mathbb{E}(X_Q).
\end{aligned}$$

It therefore only remains to show that when P and Q are graded, one has

$$\mathbb{E}(Y_{P\times Q}) = \mathbb{E}(Y_P) + \mathbb{E}(Y_Q).$$

If the rank of P is r_P, then one can rephrase Expression (2) as

$$\mathbb{E}(Y_P) = \frac{1}{(r_P+1)\#\mathcal{M}(P)} \sum_{c_P\in\mathcal{M}(P)} \sum_{p\in c_P} d_P(p).$$

Thus, regarding $d_P(p)$ as a variable, its coefficient in $\mathbb{E}(Y_P)$ (and also in $\mathbb{E}(Y_P) + \mathbb{E}(Y_Q)$) is

$$\frac{\#\{c_P \in \mathcal{M}(P) : p \in c_P\}}{(r_P+1)\#\mathcal{M}(P)}. \tag{3}$$

We now argue that $d_P(p)$ has the same coefficient in $\mathbb{E}(Y_{P\times Q})$. Note that maximal chains in $P \times Q$ are chains that lie within the set $c_P \sqcup\!\sqcup\, c_Q$ of all shuffles of some pair (c_P, c_Q) in $\mathcal{M}(P) \times \mathcal{M}(Q)$. Therefore

$$\mathbb{E}\left(Y_{P\times Q}\right) = \frac{1}{N} \sum_{(c_P,c_Q)} \sum_{(p,q)} \sum_{\substack{c\in c_P \sqcup\!\sqcup\, c_Q: \\ (p,q)\in c}} \left(d_P(p) + d_Q(q)\right),$$

where (c_P, c_Q) runs over $\mathcal{M}(P) \times \mathcal{M}(Q)$ in the outer sum, (p,q) runs over $c_P \times c_Q$ in the inner sum, and

$$N := \#\mathcal{M}(P) \cdot \#\mathcal{M}(Q) \binom{r_P + r_Q}{r_P} (r_P + r_Q + 1). \tag{4}$$

The coefficient of $d_P(p)$ in $\mathbb{E}(Y_{P\times Q})$ is therefore

$$N^{-1} \cdot \#\big\{(c_P, c_Q, q, c) : (c_P, c_Q) \in \mathcal{M}(P) \times \mathcal{M}(Q), \text{ and } (p,q) \in c \in c_P \sqcup\!\sqcup\, c_Q\big\}. \tag{5}$$

A priori, because the posets are graded, the number of pairs (q,c) completing a quadruple (c_P, c_Q, q, c) as above should not depend on the chain c_P or c_Q, as long as p lies in c_P. Thus one might as well replace c_P

and c_Q by fixed chains $[0, r_P]$ and $[0, r_Q]$ of the appropriate ranks, and fix $i := \mathrm{rank}_P(p)$ in the chain $[0, r_P]$, while letting q vary over all values j in the chain $[0, r_Q]$. Then Expression (5) may be rewritten as

$$N^{-1} \cdot \#\{c_P \in \mathcal{M}(P) : p \in c_P\} \cdot \#\mathcal{M}(Q) \cdot \#\{(j,c) : (i,j) \in c \in [0, r_P] \sqcup [0, r_Q]\}. \tag{6}$$

The cardinality of the set of pairs (j,c) in this set is $\binom{r_P+r_Q+1}{r_P+1}$, via the bijection

$$\begin{array}{ccc} \{(j,c) : (i,j) \in c \in [0, r_P] \sqcup [0, r_Q]\} & \longrightarrow & [0, r_P+1] \sqcup [0, r_Q] \\ (j,c) & \longmapsto & c' \end{array}$$

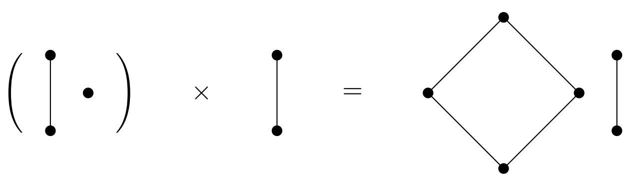
which forms c' from (j,c) by adding an extra step to c of the form $(i,j) \to (i+1,j)$, just after c passes through (i,j). The reverse bijection "contracts out" of c' its unique step of the form $(i,j) \to (i+1,j)$ for some j, producing c in the pair (j,c).

Plugging this and Equation (4) into Expression (6), yields the coefficient of $d_P(p)$ in $\mathbb{E}(Y_{P\times Q})$:

$$\frac{\#\{c_P \in \mathcal{M}(P) : p \in C_P\} \cdot \#\mathcal{M}(Q)\binom{r_P+r_Q+1}{r_P}}{\#\mathcal{M}(P) \cdot \#\mathcal{M}(Q)\binom{r_P+r_Q}{r_P}(r_P+r_Q+1)} = \frac{\#\{c_P \in \mathcal{M}(P) : p \in C_P\}}{(r_P+1)\#\mathcal{M}(P)}.$$

This is the same as its coefficient in $\mathbb{E}(Y_P) + \mathbb{E}(Y_Q)$, given in Expression (3), completing the proof. □

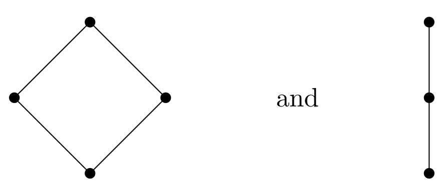
Example 2.14. Both P and Q must be graded in Proposition 2.13, as illustrated by the following non-CDE product of two CDE posets.



Note that Proposition 2.13 immediately implies that all finite products of chains are CDE, and, in particular, that finite Boolean algebras are CDE. Moreover, such products enjoy the stronger mCDE property, as we will show in Proposition 2.18. Proposition 2.18 is closely related to the Cartesian product operation, and its proof does bear some resemblance to the proof of Proposition 2.13, but we postpone it until the end of this section so as not to interrupt the discussion of poset operations more generally.

Example 2.14 also shows that disjoint unions $P_1 \sqcup P_2$ of CDE posets need not be CDE.

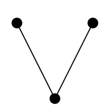
Example 2.15. The poset product in Example 2.14, is isomorphic to $P \sqcup Q$ for two Boolean algebras P and Q. Boolean algebras are CDE, by Proposition 2.13, but the disjoint union $P \sqcup Q$ depicted above is not. In fact, this can fail even when the two posets in the disjoint union are both graded and of the same rank. For example, both



are CDE, but their disjoint union is not.

The next example shows that ordinal sum, like disjoint union, does not always preserve the CDE property.

Example 2.16. Let P be a 1-element antichain and Q a 2-element antichain. Both of these posets are CDE because $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ in each case. However, their ordinal sum

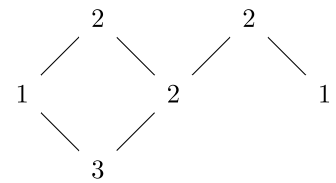
$P \oplus Q =$ 

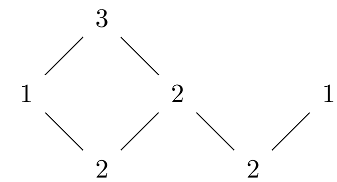
is not CDE because

$$\mathbb{E}(X_{P\oplus Q}) = 2/3 \text{ while } \mathbb{E}(Y_{P\oplus Q}) = 1/2.$$

We noted earlier that $\mathbb{E}(X_P) = \mathbb{E}(X_{P^*})$ for any finite poset P. However, there exist posets for which $\mathbb{E}(Y_P) \neq \mathbb{E}(Y_{P^*})$. Moreover, poset duality does not preserve the CDE property.

Example 2.17. Consider the following pair of dual posets P and P^*, with each element labeled by the number of maximal chains passing through it.

P =


P^* =


It is straightforward to calculate

$$\begin{aligned}\mathbb{E}(X_P) = \mathbb{E}(X_{P^*}) &= \frac{0+1+1+0+2+2}{6} = 1 \text{ and}\\ \mathbb{E}(Y_P) &= \frac{3\cdot 0 + 1\cdot 1 + 2\cdot 1 + 1\cdot 0 + 2\cdot 2 + 2\cdot 2}{11} = 1, \text{ while}\\ \mathbb{E}(Y_{P^*}) &= \frac{2\cdot 0 + 2\cdot 0 + 1\cdot 1 + 2\cdot 2 + 1\cdot 1 + 3\cdot 2}{11} = \frac{12}{11}.\end{aligned}$$

Thus P is CDE, and P^* is not.

Recall the discussion after Example 2.14, about finite products of chains being mCDE. We now substantiate that claim.

Given a finite poset P and a positive integer m, recall the probability space $\Omega_P^{(m)}$ used in Definition 2.3. For a random variable $Z : P \to \mathbb{R}$, let

$$\mathbb{E}(Z; m)$$

denote the expectation of Z with respect to $\Omega_P^{(m)}$. As before, we write $\boldsymbol{a}$ for a chain having a elements.

Proposition 2.18. *Given random variables $X_k : \boldsymbol{a_k} \to \mathbb{R}$, for $k = 1, \ldots, n$, define $Z : \boldsymbol{a_1} \times \cdots \times \boldsymbol{a_n} \to \mathbb{R}$ by*

$$Z(i_1, \ldots, i_n) := X_1(i_1) + \cdots + X_n(i_n).$$

Then

$$\mathbb{E}(Z; m) = \sum_{k=1}^{n} \mathbb{E}(X_k; m). \tag{7}$$

Proof. Consider expanding each side of Equation (7) in terms of the definition of expectation. We then check that, for each $k = 1, 2, \ldots, n$ and $i_k = 1, 2, \ldots, a_k$, the coefficient of $X_k(i_k)$ is the same on either side of Equation (7). By re-indexing, we may assume $k = 1$.

On the righthand side, the coefficient of $X_1(i_1)$ is $1/a_1$, because $\Omega_P^{(m)}$ and Ω_P^{unif} coincide when $P = \boldsymbol{a_1}$.

On the lefthand side, the coefficient of $X_1(i_1)$ is

$$\frac{g(i_1)}{g(1) + g(2) + \cdots + g(a_1)} \tag{8}$$

where $g(j)$ is the sum over all $(n-1)$-tuples $(j_2, \ldots, j_n)$ of the number of m-element multichains passing through $(j, j_2, \ldots, j_n)$ in $\boldsymbol{a_1} \times \cdots \times \boldsymbol{a_n}$. Therefore, it suffices to show that $g(1) = g(2) = \cdots = g(a_1)$, because that would mean that the ratio in Expression (8) would equal $1/a_1$.

We approach this goal bijectively. Let $G(j)$ consist of all pairs $((j_2, \ldots, j_n), c)$ where c is an m-element multichain $c := (p_1 \le \cdots \le p_m)$ in $\boldsymbol{a_1} \times \cdots \times \boldsymbol{a_n}$, passing through $(j, j_2, \ldots, j_n)$. Therefore $G(j)$ has cardinality $g(j)$. We will construct bijections $\psi : G(j) \to G(j+1)$, for $1 \le j \le n-1$, thus showing that $g(1) = \cdots = g(a_1)$, as desired.

Given $((j_2, \ldots, j_n), c) \in G(j)$ with $c = (p_1 \le \cdots \le p_m)$, there is a unique minimal i and maximal I such that in the sub-multichain

$$p_i \le p_{i+1} \le \cdots \le p_{I-1} \le p_I$$

of c, the first coordinate of each p_h is in the two-element set $\{j, j+1\}$. In particular, this sub-multichain is nonempty because $(j, j_2, \ldots, j_n) = p_{i_0}$ for some $i_0 \in [i, I]$. Let $\hat{p}_I$ be obtained by replacing the first coordinate of p_I by $j+1$; thus either $\hat{p}_I = p_I$ or $\hat{p}_I \gtrdot p_I$. Define an order-reversing involution φ on the subinterval $[p_i, \hat{p}_I]$ within the poset $\boldsymbol{a_1} \times \cdots \times \boldsymbol{a_n}$, sending $p \mapsto p_i + (\hat{p}_I - p)$.

Let c' be the m-element multichain

$$c' = (p_1 \le \cdots \le p_{i-1} \le \varphi(p_I) \le \varphi(p_{I-1}) \le \cdots \le \varphi(p_{i+1}) \le \varphi(p_i) \le p_{I+1} \le \cdots \le p_m)\,,$$

and write $\varphi(p_{i_0}) = (j+1, j_2', \ldots, j_n')$. Note that $\varphi(p_{i_0}) \in c'$.

Finally, define $\psi : G(j) \to G(j+1)$ by

$$(c, (j_2, \ldots . j_n)) \mapsto (c', (j_2', \ldots, j_n')).$$

It is not hard to check that $\psi : G(j) \to G(j+1)$ is bijective. In particular, the map ψ^{-1} identifies the sub-multichain of c' having first coordinates in $\{j, j+1\}$, and then applies the same involution φ to this sub-multichain, producing the multichain c.

Thus each set $G(j)$ is equinumerous, so

$$\frac{g(i_1)}{g(1) + g(2) + \cdots + g(a_1)} = \frac{1}{a_1},$$

as desired. □

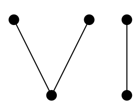
From Proposition 2.13, we obtain the following immediate corollary.

Corollary 2.19. *A product of chains $\boldsymbol{a_1} \times \cdots \times \boldsymbol{a_n}$ is mCDE and CDE, with*

$$\mathbb{E}(X) = \mathbb{E}(X^{(m)}) = \mathbb{E}(Y) = \sum_{k=1}^{n} \frac{a_k - 1}{a_k}.$$

By setting $a_1 = \cdots = a_n = 2$, we see that Boolean algebras $\mathbf{2}^n = \mathbf{2} \times \cdots \times \mathbf{2}$ of rank n are mCDE and CDE, with $\mathbb{E}(X) = \mathbb{E}(X^{(m)}) = \mathbb{E}(Y) = n/2$.

Propositions 2.13 and 2.18 raise some questions for which we have only partial answers. One might wonder, for example, whether for any finite posets $\{P_k\}_{k=1}^n$ and random variables $X_k : P_k \to \mathbb{R}$, the random variable $Z : P_1 \times \cdots \times P_n \to \mathbb{R}$ defined by $Z(p_1, \ldots, p_n) = \sum_{k=1}^n X_k(p_k)$ satisfies $\mathbb{E}(Z; m) = \sum_{k=1}^n \mathbb{E}(X_k; m)$. Note that this can fail even when the posets $\{P_k\}$ are graded; for example,

$P_1 =$  and $P_2 =$ 

do not produce the desired property.

On the other hand, we have encountered no examples to preclude an affirmative answer to the following question.

Question 2.20. *Is the Cartesian product $P_1 \times P_2$ of two mCDE posets P_1 and P_2, be they graded or not, always mCDE?*

An affirmative answer to this question would be useful in resolving Conjecture 2.12.

We now expand upon another topic related to poset operations, namely, duality. Despite Example 2.17, self-duality is relevant for the CDE property. The authors thank both S. Fishel and T. McConville for (independently) pointing out the CDE assertion in Proposition 2.21 below. Recall that a poset P is *self-dual* if one has a poset isomorphism $P \cong P^*$, and we will say that a poset P is *regular of valence* Δ if every element p in P has the same vertex degree Δ in the Hasse diagram.

Proposition 2.21. *A finite, self-dual poset P that is regular of valence Δ is always mCDE and CDE, with $\mathbb{E}(X) = \mathbb{E}(Y) = \Delta/2$.*

Proof. Given a poset isomorphism $\alpha : P \to P^*$, we will show a stronger assertion: for any probability distribution on the underlying set $P = P^*$ that is α-invariant in the sense that $\mathbf{Prob}(\alpha(p)) = \mathbf{Prob}(p)$ for all $p \in P$, the expected value of the down-degree random variable $d : P \to \mathbb{N}$ is $\Delta/2$. To see this, one calculates the expected value of d as follows:

$$\begin{aligned}\sum_{p\in P}\mathbf{Prob}(p)\cdot d(p) &= \frac{1}{2}\left(\sum_{p\in P}\mathbf{Prob}(p)\cdot d(p)+\sum_{p\in P}\mathbf{Prob}(\alpha(p))\cdot d(\alpha(p))\right)\\ &= \frac{1}{2}\left(\sum_{p\in P}\mathbf{Prob}(p)\cdot(d(p)+d(\alpha(p)))\right)\\ &= \frac{1}{2}\sum_{p\in P}\mathbf{Prob}(p)\cdot\Delta\\ &= \frac{\Delta}{2}.\end{aligned}$$

The penultimate equality used the fact that the down-degree $d(\alpha(p))$ of $\alpha(p)$ in P is the same as the up-degree of p in P, meaning that $d(p)+d(\alpha(p))$ is the sum of the up- and down-degrees of p, which is Δ.

Note that this then implies $\mathbb{E}(X)=\mathbb{E}(X^{(m)})=\mathbb{E}(Y)=\Delta/2$, because

- the uniform distribution Ω_P^{unif} on $P=P^*$ used for X is obviously α-invariant, while
- the distributions $\Omega_P^{(m)}$ and $\Omega_P^{\mathrm{chain}}$ on $P=P^*$ used for $X^{(m)}$ and Y are α-invariant because α bijects the chains (respectively, m-element multichains) through p in P with the same chains (respectively, multichains) through $\alpha(p)$ in P^*. □

Proposition 2.21 yields several interesting families of mCDE and CDE posets, many of them non-graded, which we briefly discuss here.

2.2.1. *Simplicial arrangements and oriented matroids.* The first are the weak orders on the chambers of a (central, essential) hyperplane arrangement in $\mathbb{R}^r$ (or, more generally, the topes of an oriented matroid of rank r). We will stick to the language of chambers and arrangements rather than the more general oriented matroid language here. Definitions and historical references can be found in [BLSWZ99, §4.2].

All such weak orders have the same underlying graph for their Hasse diagram, having vertices given by the chambers C (the maximal cones into which the arrangement dissects the space), and an edge $\{C,C'\}$ whenever two chambers C and C' are separated by exactly one hyperplane. When this graph is regular of valence r, the arrangement is called *simplicial*. In particular, this occurs for the arrangements of reflecting hyperplanes in a finite reflection group W of rank r, where chambers correspond to the group elements, and the weak orders are all isomorphic to what is called the *weak Bruhat order* on W. One defines one of the *weak orders* on the set of chambers generally by picking a base chamber C_0, and decreeing that $C\leq C'$ if every hyperplane separating C_0 from C also separates C_0 from C'. The map $C\mapsto -C$ is a poset anti-automorphism, showing that all weak orders are self-dual. Proposition 2.21 then immediately implies the following.

Corollary 2.22. *For a (central, essential) simplicial hyperplane arrangement in $\mathbb{R}^r$, or simplicial oriented matroid of rank r, any of its weak orders on chambers is both mCDE and CDE, with*

$$\mathbb{E}(X)=\mathbb{E}(X^{(m)})=\mathbb{E}(Y)=\frac{r}{2}.$$

In particular, this is true for the weak Bruhat order on any finite reflection group W.

2.2.2. *Tamari orders and some generalizations.* The set of all triangulations of an n-sided polygon carries a well-known partial order known as the *Tamari order* [MPS12]. The underlying graph for its Hasse diagram has an edge $\{T,T'\}$ if the triangulations T and T' differ only by a single *diagonal flip*, that is, from one diagonal to the other inside a quadrangle triangulated by both T and T'. After labeling the polygon vertices cyclically as $1,2,\ldots,n$, one has $T\lessdot T'$ if the diagonal flip exchanges the diagonal $\{i,k\}$ for the diagonal $\{j,\ell\}$ within a quadrangle $ijk\ell$ that has $(1\leq)i<j<k<\ell(\leq n)$. As an example, the special case of the CDE family $P_{1,1,2,1}$ from Example 2.9 is the *Tamari lattice* [MPS12] on triangulations of a pentagon.

The Tamari order on triangulations of an n-gon is regular of valence $n-3$, because each triangulation has $n-3$ internal diagonals that one can flip; in fact, it is also the 1-skeleton of a simple $(n-3)$-dimensional polytope, called the *associahedron* [Zi95, Example 9.11]. The Tamari order is self-dual, because the map on the vertices swapping $i\leftrightarrow n+1-i$ reverses the order.

Proposition 2.21 then immediately implies the following.

Corollary 2.23. *Tamari order on triangulations of an n-gon is mCDE and CDE, with*

$$\mathbb{E}(X) = \mathbb{E}(X^{(m)}) = \mathbb{E}(Y) = \frac{n-3}{2}.$$

The authors thank T. McConville for also pointing out the following generalizations of Tamari orders that are all valence-regular, and, in some cases, self-dual. Valence-regularity stems from the fact that, in each case, the object can be described as a partial order whose underlying Hasse diagram is the graph of all maximal simplices in a pure $(\Delta - 1)$-dimensional simplicial complex with the pseudomanifold property (that is, every $(\Delta - 2)$-dimensional simplex lies in exactly two maximal simplices).

- N. Reading [Re06] defined a *Cambrian lattice* P associated to each orientation of the Coxeter diagram of a finite Coxeter group (W, S). This P is always regular of valence $|S|$. It will be self-dual (and hence both mCDE and CDE by Proposition 2.21) whenever the opposite orientation corresponds to a diagram automorphism of (W, S); see [Re06, Theorem 3.5]. The Tamari order is the special case when the Coxeter system (W, S) is of type A, and its Coxeter diagram is a path that is equioriented (that is, the arrows all point in the same direction along the path).
- Derksen, Weyman, and Zelevinsky [DWZ10] introduced the notion of a *quiver with potential* (Q, W), and its associated *(complete) Jacobian algebra* $A := \hat{J}(Q, W)$ over a field k. The operation of mutation on (Q, W) gives rise to its exchange graph, which is regular of valence $|Q_0|$, the number of nodes in the quiver Q. When the k-algebra A has finite representation type (that is, only finitely many indecomposable modules up to isomorphism), this exchange graph is finite. Under this same representation-finite hypothesis, the exchange graph also carries an orientation that is acyclic and whose transitive closure is a poset P that coincides with both the poset P of support-tilting modules for A and the poset of torsion-free classes for A; see [BY13, §2, §3, and Theorem 3.6] and [GM15]. Additionally, the Hasse diagram of P is equal to the exchange graph; that is, none of the directed edges of the oriented exchange graph are implied transitively by others.

  If, furthermore, there is an algebra isomorphism $A^* \cong A$, then the poset P will be self-dual (and hence both mCDE and CDE by Proposition 2.21) [IRTT15, Proposition 1.3]. This occurs, for example, whenever the potential $W = 0$ and Q is a representation-finite quiver whose opposite orientation can be achieved by applying a graph automorphism. The Tamari order again corresponds to the special case when the quiver is an equioriented path of type A.
- Santos, Stump, and Welker [SSW14] introduced the *Grassman-Tamari orders* $GT_{k,n}$ on the set of all maximal noncrossing families of k element subsets of $\{1, 2, \ldots, n\}$. The Tamari order is the special case $GT_{2,n}$. McConville [McC15] generalized this further in his *grid orders* $GT(\lambda)$ where λ is any shape, meaning any finite induced subgraph of the $\mathbb{Z} \times \mathbb{Z}$ rectangular grid. When λ is a $k \times (n-k)$ rectangle, one has $GT(\lambda) = GT_{k,n}$.

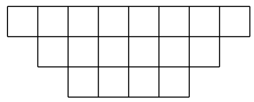
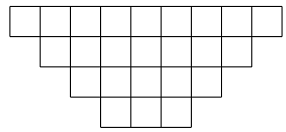
  Let λ^* be the result of rotating λ by 180°, and let λ^t denote the shape obtained from λ by transposing rows and columns. One can check that that $GT(\lambda^*) \cong GT(\lambda)^* \cong GT(\lambda^t)$ [SSW14, Proposition 2.19]. Therefore the poset $P = GT(\lambda)$ is self-dual (and hence both mCDE and CDE by Proposition 2.21) whenever λ is invariant under either 180° rotation as in the case of $GT_{k,n}$, or under transposition of rows and columns.
- Pilaud [Pil15] introduced the poset of (k, n)-*twists* on the set of all k-triangulations of a convex $(n + 2k)$-gon; the case $k = 1$ recovers the Tamari poset. One can check that this poset is always self-dual (and hence both mCDE and CDE by Proposition 2.21) using its description as a quotient of the weak Bruhat order on $W = \mathfrak{S}_n$ by a congruence that is preserved under the involutive anti-automorphism $w \mapsto w_0 w$ [Pil15, Definition 26].

2.3. **Further CDE conjectures and questions.**

2.3.1. *Intervals in the shifted version of Young's lattice.* For a *strict partition* $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell)$, the *shifted Ferrers diagram* for λ is drawn with each successive row indented one position further than its predecessor. Some examples are shown below. There is a shifted version of Young's lattice, which is simply its induced partial order on the subset of all strict partitions. In light of Theorem 1.1, one might ask if there exist some strict partitions λ whose interval $[\varnothing, \lambda]_{\text{shifted}}$ in the shifted version of Young's lattice is CDE. We offer here two conjectural families of such partitions.

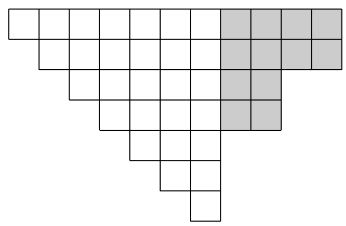
Conjecture 2.24. *For integers $\ell \geq 1$ and $0 \leq k < \ell/2$, the shifted Young's lattice interval $[\varnothing, \lambda]_{\text{shifted}}$ below $\lambda = (\ell, \ell - 2, \ell - 4, \ldots, \ell - 2k)$ is CDE, with $\mathbb{E}(X) = \mathbb{E}(Y) = |\lambda|/(\ell + 1)$.*

Note that this conjecture is independent of the parity of ℓ. For example, it would apply to both of the shifted shapes $(8, 6, 4)$ and $(9, 7, 5, 3)$ depicted here.

Conjecture 2.25. *For integers $a, d, e \geq 1$ with $d > a(e-1)+1$, the shifted Young's lattice interval $[\varnothing, \lambda]_{\text{shifted}}$ below $\lambda = \delta_d + \delta_e \circ a^a$ is CDE, with $\mathbb{E}(X) = \mathbb{E}(Y) = (d + a(e - 1))/4$.*

We depict here the shifted shape $\delta_8 + \delta_3 \circ 2^2$, with the cells of $\delta_3 \circ 2^2$ shaded.

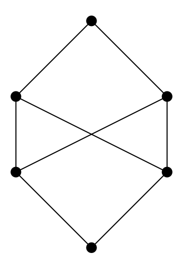


Remark 2.26. The last two conjectures overlap. That is, Conjecture 2.24 with $(\ell, k) = (2N - 1, N - 1)$ and Conjecture 2.25 with $(d, e, a) = (N + 1, N, 1)$ both assert that, for $\lambda = (2N - 1, 2N - 3, \cdots, 5, 3, 1)$, the interval $[\varnothing, \lambda]_{\text{shifted}}$ is CDE with $\mathbb{E}(X) = \mathbb{E}(Y) = N/2$.

Interestingly, this particular interval $[\varnothing, \lambda]_{\text{shifted}}$ is isomorphic to the distributive lattice of $J(\Phi_W^+)$ of order ideals in the usual *poset of positive roots* Φ_W^+ for the root systems of types $W = B_N$ or C_N. It should be noted that for the root system of type $W = A_{d-1}$, the same lattice $J(\Phi_W^+)$ is isomorphic to the usual Young's lattice interval $[\varnothing, \delta_d]$, and hence is shown to be CDE as part of Theorem 1.1. Unfortunately, for the root system of type D_4, it was checked that $J(\Phi_{D_4}^+)$ is not CDE.

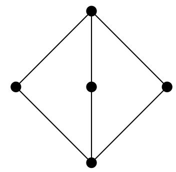
2.3.2. *A few negative examples.*

- Recall from Section 1 that weak Bruhat order on a finite Coxeter group is CDE (see Corollary 2.22). One might ask whether *strong Bruhat order* has the same property, but this fails already for the strong Bruhat order on the symmetric group $\mathfrak{S}_3$, shown here, because $\mathbb{E}(X) = 4/3$ and $\mathbb{E}(Y) = 5/4$.



- In light of Theorem 2.11, one might wonder whether to expect, more generally, that the distributive lattices $J(P \times \boldsymbol{k})$, with P minuscule, will always be CDE. However, this fails already for the first minuscule family, because $J(\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{k})$ is not CDE for $a = b = k = 2$.
- In light of Remark 2.26, one might ask whether the posets $P = \Phi_W^+$ of positive roots for W of types A or B/C might themselves be CDE. However, small examples show that this is not the case.
- One can easily check that the CDE fails for the five-element modular, non-distributive lattice depicted below, which happens to be both the lattice of partitions of the set $\{1, 2, 3\}$ and the $n = q = 2$ instance

of the lattice of subspaces of $(\mathbb{F}_q)^n$.



- Corollaries 2.23 and 2.22 might make one might wonder whether any of the following poset families
  - Bergeron and Préville-Ratelle's *m-Tamari* lattices [BP12],
  - Kapranov and Voevodsky's *higher Stasheff-Tamari* posets [KV91],
  - Manin and Schechtman's *higher Bruhat orders* [MS89], or
  - Law and Reading's lattice of *diagonal rectangulations* [LR12],

  all of which are related to Tamari and weak Bruhat orders, might be CDE. However, in each case, we found small counterexamples.

3. Young's lattice and tableaux

Computing $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ for Young's lattice intervals $[\varnothing, \lambda]$ and their duals involves various known flavors of tableaux. In this section, we review their definitions, provide formulas to count them generally, and then specialize to rectangular staircase shapes.

Definition 3.1. A *(set-valued) filling* T of *shape* λ is an assignment of a finite subset $T(x) \subset \{1, 2, \ldots\}$ to each cell x in the Ferrers diagram of λ. Define the monomial

$$\mathbf{x}^T := \prod_{j \in T(y)} x_j \tag{9}$$

as y runs through the cells of λ.

Several classes of fillings are of particular relevance to this work.

Definition 3.2. A *column-strict set-valued tableau* T of shape λ is a filling T in which

- $\max\ T(x) \leq \min\ T(x')$ when x is to the left of x' in the same row of λ, and
- $\max\ T(x) < \min\ T(x')$ when x is above x' in the same column of λ.

This T is a *standard set-valued tableau* if $\mathbf{x}^T = x_1 x_2 x_3 \cdots x_N$ for some integer N. A column-strict set-valued tableau T is a *(column-strict) tableau* if $\#T(x) = 1$ for every cell $x \in \lambda$, while T is *barely set-valued* if $\#T(x) = 1$ for all $x \in \lambda$ with the exception of a unique $x_0 \in \lambda$ for which $\#T(x_0) = 2$.

Also useful are tableaux with row-by-row bounds on their values.

Definition 3.3. A column-strict set-valued tableau T of shape λ is *flagged* by a sequence of positive integers $\varphi = (\varphi_1, \varphi_2, \ldots)$ if every cell x in row i of λ satisfies $\max\ T(x) \leq \varphi_i$. The sequence φ is called a *flag*.

Example 3.4.

123	3	35
46	67	
7		

column-strict set-valued

12	3	56
4	79	
8		

standard set-valued

1	3	5
2	6	
4		

standard tableau

1	1	3
2	3	
4		

column-strict tableau

1	13	3
3	4	
4		

barely set-valued

1	12	2
2	3	
3		

flagged by $\varphi = (2, 3, 4)$

Proposition 3.5. *Fix $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and set $\varphi := (2, 3, 4, \ldots)$. There are bijections between*

(a) *standard tableaux of shape λ, and maximal chains in $[\varnothing, \lambda]$, or in its dual $[\varnothing, \lambda]^*$;*
(b) *standard barely set-valued tableaux of shape λ, and triples (c, μ, ν) with $\mu \in c$ a maximal chain in $[\varnothing, \lambda]$ and $\nu \lessdot \mu$ a covering relation, not necessarily in c;*
(c) *standard barely set-valued tableaux of shape λ, and triples (c, μ, ν) with $\mu \in c$ a maximal chain in $[\varnothing, \lambda]^*$ and $\nu \lessdot \mu$ a covering relation, not necessarily in c;*
(d) *column-strict tableaux of shape λ flagged by φ, and elements of $[\varnothing, \lambda]$; and*
(e) *barely set-valued column-strict tableaux of shape λ flagged by φ, and covering relations $\nu \lessdot \mu$ in $[\varnothing, \lambda]$.*

Proof. For (a), the bijection sends T to the maximal chain whose ith step is the partition $\mu^{(i)}$ occupied by the values $\{1, 2, \ldots, i\}$ in T. For example,

$$\begin{ytableau} 1 & 3 & 5 \\ 2 & 6 \\ 4 \end{ytableau} \quad \longmapsto \left(\varnothing\ ,\ \ydiagram{1}\ ,\ \ydiagram{1,1}\ ,\ \ydiagram{2,1}\ ,\ \ydiagram{2,1,1}\ ,\ \ydiagram{3,1,1}\ ,\ \ydiagram{3,2,1} \right).$$

Maximal chains in the dual $[\varnothing, \lambda]^*$ are in bijection with those in $[\varnothing, \lambda]$, by reading the chain backwards.

For (b), assume we are given a barely set-valued standard tableau T of shape λ, with $n := |\lambda|$, and let $T(x_0) = \{a_0 < b_0\}$ for a unique cell x_0. Then the bijection sends $T \mapsto (c, \mu, \nu)$ in which the chain $c = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n)})$ has $\mu^{(i)}$ the shape occupied by the ith smallest values among $\{1, 2, \ldots, n+1\} \setminus \{b_0\}$, with $\mu := \mu^{(b_0-1)}$, and $\nu := \mu \setminus \{x_0\}$. For example, T shown below has $\{1, 2, \ldots, n+1\} \setminus \{b_0\} = \{1, 2, 3, 4, 6, 7\}$, and

$$\begin{ytableau} 1 & {\scriptstyle 25} & 6 \\ 3 & 7 \\ 4 \end{ytableau} \quad \longmapsto (c, \mu, \nu), \text{ with } c = \left(\varnothing, \ydiagram{1}\ ,\ \ydiagram{2}\ ,\ \ydiagram{2,1}\ ,\ \ydiagram{2,1,1}\ ,\ \ydiagram{3,1,1}\ ,\ \ydiagram{3,2,1} \right) \text{ and } \nu = \ydiagram{1,1,1} \lessdot \ydiagram{2,1,1} = \mu = \mu^{(5-1)}.$$

For (c), the bijection is similar to (b), except that now $T \mapsto (c, \mu, \nu)$ where $c = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n)})$ has $\mu^{(i)}$ as the shape occupied by the $(n-i)$th smallest values among $\{1, 2, \ldots, n+1\} \setminus \{a_0\}$, with $\mu := \mu^{(n-(a_0-1))}$ and $\nu := \mu \cup \{x_0\}$. For example, T shown below has $\{1, 2, \ldots, n+1\} \setminus \{a_0\} = \{1, 3, 4, 5, 6, 7\}$, and

$$\begin{ytableau} 1 & {\scriptstyle 25} & 6 \\ 3 & 7 \\ 4 \end{ytableau} \quad \longmapsto (c, \mu, \nu), \text{ with } c = \left(\ydiagram{3,2,1}\ ,\ \ydiagram{3,1,1}\ ,\ \ydiagram{2,1,1}\ ,\ \ydiagram{1,1,1}\ ,\ \ydiagram{1,1}\ ,\ \ydiagram{1}\ ,\ \varnothing \right) \text{ and } \nu = \ydiagram{2} \gtrdot \ydiagram{1} = \mu = \mu^{(7-(2-1))}.$$

For (d), the bijection sends the tableau T to the partition μ describing the cells x in T which are filled with their row index i, rather than with the flag upper bound $i + 1 = \varphi_i$. For example,

$$\begin{ytableau} \mathbf{1} & 2 & 2 \\ \mathbf{2} & 3 \\ 4 \end{ytableau} \quad \longmapsto \ydiagram{1,1}\ .$$

For (e), the bijection sends the tableau T to the covering relation $\nu \lessdot \mu$, where

- ν gives the cells x in T filled by their row index,
- the set difference (or *skew shape*) λ/μ gives the cells x filled by one more than their row index, and
- the unique cell x_0 of μ/ν is filled by $T(x_0) = \{i, i+1\}$, where i is its row index.

For example,

$$\begin{ytableau} \mathbf{1} & \mathbf{1} & 2 & 2 \\ \mathbf{2} & {\scriptstyle 23} & 3 \\ 4 \end{ytableau} \quad \longmapsto \nu = \ydiagram{2,1} \lessdot \ydiagram{2,2} = \mu$$

$\square$

Recall that f^λ is the number of standard tableau of shape λ. Let us now name the number of tableaux of each kind appearing in Proposition 3.5.

Definition 3.6. Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.

(a) Let $f^\lambda(+1)$ be the number of standard barely set-valued tableaux of shape λ.
(b) Let $R(\lambda)$ be the number of column-strict tableaux of shape λ flagged by $\varphi := (2, 3, 4, \ldots)$.
(c) Let $R^{(+1)}(\lambda)$ be the number of column-strict barely set-valued tableaux of shape λ flagged by φ.

Combining Proposition 3.5 with Equations (1) and (2) implies the following.

Corollary 3.7. *For any partition λ,*

$$\mathbb{E}(X_{[\varnothing,\lambda]}) = \frac{R^{(+1)}(\lambda)}{R(\lambda)} = \mathbb{E}(X_{[\varnothing,\lambda]^*}), \text{ and} \tag{10}$$

$$\mathbb{E}(Y_{[\varnothing,\lambda]}) = \frac{f^\lambda(+1)}{(|\lambda| + 1) f^\lambda} = \mathbb{E}(Y_{[\varnothing,\lambda]^*}). \tag{11}$$

To count barely set-valued tableaux, our strategy is to convert them to tableaux with extra data.

Definition 3.8. Given a column-strict barely set-valued tableau T, its *uncrowding* is the column-strict tableau T^+ obtained as follows: if $T(x_0) = \{a_0 < b_0\}$ for the unique cell x_0, then remove b_0 from x_0 and use *Robinson-Schensted-Knuth (RSK) row-insertion* (see, for example, [St99, §7.11]) to bump b_0 into the rows of T strictly below the row of x_0. For example,

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&3&34&4\\\cline{1-4} 4&5&5&7\\\cline{1-4} 5&6&6\\\cline{1-3} 6\\\cline{1-1}\end{array}^{\,+} = \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&3&3&4\\\cline{1-4} 4&\mathbf{4}&5&7\\\cline{1-4} 5&\mathbf{5}&6\\\cline{1-3} 6&\mathbf{6}\\\cline{1-2}\end{array},$$

where the bumped entries are in boldface.

Proposition 3.9. *The uncrowding operation $T \mapsto T^+$ gives a bijection between*

- *column-strict barely set-valued tableaux of shape λ, and*
- *triples (T^+, x, i_0) where*
  - *T^+ is a column-strict tableau,*
  - *x is one of its (inner) corner cells, and*
  - *i_0 is in the range $1, 2, \ldots, i-1$, where i is the row-index of x.*

Under this bijection, the shapes λ and λ^+ of T and T^+ are related by $\lambda \lessdot \lambda^+$ and $\lambda^+/\lambda = \{x\}$. Furthermore, i_0 is the row-index of the unique cell x_0 for which $\#T(x_0) = 2$.

Proof. Given (T^+, x, i_0), the inverse bijection ("crowding") starts by doing *reverse RSK row-insertion* out of the corner cell x in T^+. However, rather than stopping when it reverse-bumps an entry out of row 1, the procedure stops when an entry b_0 from row $i_0 + 1$ is about to bump an entry a_0 of row i_0, say in a cell x_0, and instead adds b_0 as an extra set-valued entry to make $T(x_0) = \{a_0, b_0\}$. □

Example 3.10. If one selects the boldface corner cell x in the figure below (row 5, column 2),

$$T^+ = \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&3&3&4\\\cline{1-4} 4&4&5&7\\\cline{1-4} 5&5&6\\\cline{1-3} 6&\mathbf{6}\\\cline{1-2}\end{array}$$

then crowding the triples (T^+, x, i_0) for $i_0 = 1, 2, 3, 4$ yields these barely set-valued tableaux, respectively.

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&3&3&4\\\cline{1-4} 4&4&5&7\\\cline{1-4} 5&56&6\\\cline{1-3} 6\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&3&3&4\\\cline{1-4} 4&45&5&7\\\cline{1-4} 5&6&6\\\cline{1-3} 6\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&3&34&4\\\cline{1-4} 4&5&5&7\\\cline{1-4} 5&6&6\\\cline{1-3} 6\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1&1&2&23&4\\\hline 2&3&4&4\\\cline{1-4} 4&5&5&7\\\cline{1-4} 5&6&6\\\cline{1-3} 6\\\cline{1-1}\end{array}$$

Proposition 3.9 yields useful recurrences for counting barely set-valued tableaux.

Corollary 3.11. *For any partition λ,*

$$f^\lambda(+1) = \sum_{x=(i,j)} (i-1) f^{\lambda \cup \{x\}},$$

where x runs through all outside corner cells of λ.

Proof. This is immediate from Proposition 3.9, restricting the uncrowding bijection to standard barely set-valued tableaux T of shape λ, so that T^+ in the triple (T^+, x, i_0) is a standard tableau. □

Remark 3.12. A second proof of Corollary 3.11 uses the fact that f^λ and $f^\lambda(+1)$ are the coefficients of $x_1x_2\cdots x_{|\lambda|}$ and $x_1x_2\cdots x_{|\lambda|+1}$ in the *Schur function* s_λ and the *stable Grothendieck polynomial* G_λ; see Definition 5.3 below. A formula of Lenart [Le99] gives the expansion

$$G_\lambda = \sum_{\mu\supset\lambda} (-1)^{|\mu/\lambda|} g_{\mu/\lambda} s_\mu, \tag{12}$$

where $g_{\mu/\lambda}$ is the number of *row-strict and column-strict* tableaux of the skew shape λ/μ with entries in row i in the range $1, 2, \ldots, i-1$. If $\mu = \lambda \cup \{x\}$ with $x = (i, j)$, then $g_{\mu/\lambda} = i - 1$. Thus Corollary 3.11 also follows by extracting the coefficient of the square-free monomial $x_1x_2\cdots x_{|\lambda|+1}$ in Equation (12).

Remark 3.13. Note that $f^{\lambda^t}(+1) = f^\lambda(+1)$ via the conjugation involution on standard set-valued tableaux. Hence Corollary 3.11 implies a second identity; namely,

$$f^\lambda(+1) = \sum_{x=(i,j)} (j-1) f^{\lambda\cup\{x\}},$$

where x still runs through all outside corner cells of λ. Concordance between these two identities requires the difference between their righthand sides to equal zero; that is, we need

$$\sum_{x=(i,j)} c(x) f^{\lambda\cup\{x\}} = 0, \tag{13}$$

where $c(x) = j - i$ is the *content* of the cell $x = (i, j)$. Indeed, dividing Equation (13) by $n \cdot f^\lambda$ gives a fact that was first shown by Kerov ([Ke96, Equation (10.6)] and [Ke93]): the content $c(x)$ has mean 0 when one considers it as a random variable on the outside corners x of a random partition λ, grown one box at a time using *Plancherel measure*.

We can now prove part of Theorem 1.1.

Proposition 3.14. *For the rectangular staircase $\lambda = \delta_d(b^a)$, the interval $[\varnothing, \lambda]$ has $\mathbb{E}(Y) = \frac{(d-1)ab}{a+b}$.*

Proof. Recall that Equation (11) says that $\mathbb{E}(Y) = \frac{f^\lambda(+1)}{(|\lambda|+1)f^\lambda}$. The outside corners of λ are

$$x_i := \big(1 + b(d-1-i), 1 + ai\big)$$

for $i = 0, 1, \ldots, d-1$, and the hook-length formula shows that for the cell x_i,

$$\frac{f^{\lambda\cup\{x_i\}}}{(|\lambda|+1)f^\lambda} = \frac{\left(\frac{b}{a+b}\right)_i}{i!} \cdot \frac{\left(\frac{a}{a+b}\right)_{d-1-i}}{(d-1-i)!},$$

where $(z)_j := z(z+1)\cdots(z+j-1)$ is the Pochhammer symbol.

Thus Corollary 3.11 implies

$$\begin{aligned}
\mathbb{E}(Y) &= \frac{f^\lambda(+1)}{(|\lambda|+1)f^\lambda} \\
&= \sum_{i=0}^{d-1} ai \cdot \frac{f^{\lambda\cup\{x_i\}}}{f^\lambda} \\
&= \sum_{i=1}^{d-1} ai \cdot \frac{\left(\frac{b}{a+b}\right)_i}{i!} \cdot \frac{\left(\frac{a}{a+b}\right)_{d-1-i}}{(d-1-i)!} \\
&= a \cdot \frac{\frac{b}{a+b}\left(\frac{a}{a+b}\right)_{d-2}}{(d-2)!} \cdot {}_2F_1\left(\begin{matrix} -(d-2) & \frac{b}{a+b}+1 \\ & -\left(d-3+\frac{a}{a+b}\right) \end{matrix}\middle|\, 1\right) \\
&= a \cdot \frac{\frac{b}{a+b}\left(\frac{a}{a+b}\right)_{d-2}}{(d-2)!} \cdot \frac{(-(d-1))_{d-2}}{\left(-\left(d-3+\frac{a}{a+b}\right)\right)_{d-2}} \\
&= \frac{(d-1)ab}{a+b},
\end{aligned}$$

where the penultimate equality uses the Chu-Vandermonde summation ${}_2F_1\left(\begin{matrix}-m & B \\ & C\end{matrix}\middle| 1\right) = \frac{(C-B)_m}{(C)_m}$. ☐

Recall from Section 1 that $R(\lambda, q) := \sum_{\mu \subset \lambda} q^{|\lambda|}$ is the rank-generating function for $[\varnothing, \lambda]$, and that for an outside corner cell $x = (i, j)$ of λ in row i and column j, we defined two subshapes:

$$\lambda_{(x)} := (\lambda_{i+1}, \lambda_{i+2}, \ldots) \text{ and}$$
$$\lambda^{(x)} := (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}) - (j, j, \ldots, j).$$

Recall also from Proposition 3.5 and Definition 3.6 that $R(\lambda) = \#[\varnothing, \lambda] = [R(\lambda, q)]_{q=1}$ is the same as the number of column-strict tableaux of shape λ flagged by $\varphi = (2, 3, 4, \ldots)$, while $R^{(+1)}(\lambda)$ is the number of column-strict barely set-valued tableaux of shape λ flagged by φ.

Corollary 3.15. *For any partition λ,*

$$R^{(+1)}(\lambda) = \sum_{x=(i,j)} (i-1) \cdot R(\lambda_{(x)}) \cdot R(\lambda^{(x)}),$$

where x runs through all outside corner cells of λ.

Proof. Restrict the domain of the uncrowding bijection to column-strict barely set-valued tableaux T of shape λ flagged by $\varphi = (2, 3, 4, \ldots)$. Then, during the uncrowding of T, the bumpings that occur are always to a value i in row $i-1$ trying to bump a value $i+1$ in row i. The bumping stops only when this value i comes to rest at a corner cell $x = (i, j)$ at the end of row i, because row i only contains the value i (in particular, it has no $i+1$). In this situation, the resulting column-strict tableau T^+ thus breaks into three pieces:

- an $i \times j$ rectangle that is weakly to the upper left of $x = (i, j)$, having every cell filled by its row-index,
- a column-strict tableau $T^{(x)}$ strictly north and east of x, filling $\lambda^{(x)}$, which is flagged by φ, and
- a column-strict tableau $T_{(x)}$ strictly south and west of x, filling $\lambda_{(x)}$, which, after reducing all of its entries by i, would be flagged by φ.

For example, here is such a T and T^+, along with $T^{(x)}$ and the version of $T_{(x)}$ with entries reduced by $i = 5$.

$T =$

1	1	1	1	1	2
2	2	2	2	3	
3	3	3	34	4	
4	4	5	5		
5	5				
6	7				
7					

$\longmapsto$ $T^+ =$

1	**1**	**1**	1	1	2
2	**2**	**2**	2	3	
3	**3**	**3**	3	4	
4	**4**	**4**	5		
5	**5**	**5**			
6	7				
7					

$\longmapsto$

			1	1	2
			2	3	
			3	4	
			5		
		•			
1	2				
2					

The 5×3 rectangle weakly to the upper left of cell $(5, 3)$ in T^+ has every cell filled by its row-index, in boldface. ☐

This lets us prove another part of Theorem 1.1.

Proposition 3.16. *For the rectangular staircase $\lambda = \delta_d(b^a)$, the interval $[\varnothing, \lambda]$ has $\mathbb{E}(X) = \frac{(d-1)ab}{a+b}$.*

Proof. Recall that Equation (10) asserts that $\mathbb{E}(X) = \frac{R^{(+1)}(\lambda)}{R(\lambda)}$. As observed earlier, λ has outside corners

$$x_k := \big(1 + b(d-1-k), 1 + ak\big) \text{ for } k = 0, 1, \ldots, d-1.$$

Therefore Corollary 3.15 implies

$$R^{(+1)}(\lambda) = \sum_{k=0}^{d-1} ak \cdot R(\lambda_{(x_k)}) \cdot R(\lambda^{(x_k)}).$$

After dividing by a, this yields

$$\frac{R^{(+1)}(\lambda)}{a} = \sum_{k=0}^{d-1} k \cdot R(\lambda_{(x_k)}) \cdot R(\lambda^{(x_k)}). \tag{14}$$

The conjugation involution $\mu \mapsto \mu^t$ gives a bijection between the outside corners of λ and of λ^t, giving rise to this counterpart for Equation (14):

$$\frac{R^{(+1)}(\lambda^t)}{b} = \sum_{k=0}^{d-1}(d-1-k)\cdot R(\lambda_{(x)})\cdot R(\lambda^{(x)}). \tag{15}$$

The conjugation also gives a poset isomorphism $[\varnothing,\lambda] \cong [\varnothing,\lambda^t]$ showing that $R(\lambda) = R(\lambda^t)$, and that these two posets have the same expectations $\mathbb{E}(X)$, meaning that $R^{(+1)}(\lambda) = R^{(+1)}(\lambda^t)$. Therefore, adding Equations (14) and (15) gives

$$\frac{R^{(+1)}(\lambda)}{a} + \frac{R^{(+1)}(\lambda)}{b} = (d-1)\sum_{k=0}^{d-1} R(\lambda_{(x)})\cdot R(\lambda^{(x)}) = (d-1)R(\lambda),$$

where the last equality used Proposition 1.5 specialized to $q = 1$. Hence one has

$$\mathbb{E}(X) = \frac{R^{(+1)}(\lambda)}{R(\lambda)} = (d-1)\cdot\frac{1}{\frac{1}{a}+\frac{1}{b}} = \frac{(d-1)ab}{a+b}.$$

□

Propositions 3.14 and 3.16 imply that $[\varnothing,\lambda]$ is CDE when $\lambda = \delta_d(b^a)$ is a rectangular staircase.

4. Coxeter groups and 0-Hecke monoids

Computing $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ for lower intervals in the weak Bruhat ordering involves descent sets and factorizations in Coxeter systems and 0-Hecke monoids. We begin this section by reviewing the relevant concepts from Coxeter groups, then develop several formulas and results, and finally do the same for 0-Hecke monoids.

4.1. **Coxeter groups.**

Definition 4.1. A *Coxeter matrix* is a finite set S and a choice of $m_{s,t} = m_{t,s}$ in $\{2,3,4,\ldots\}\cup\{\infty\}$ for $s \neq t$ in S. The corresponding *Coxeter system* (W,S) is the group W generated by S, subject to relations $s^2 = e$ for all $s \in S$, and

$$\underbrace{stst\cdots}_{m_{s,t}\text{ factors}} = \underbrace{tsts\cdots}_{m_{s,t}\text{ factors}} \quad \text{for } s \neq t \text{ in } S.$$

Definition 4.2. The *length function* $\ell : W \to \mathbb{N}$ for (W,S) is defined by

$$\ell(w) := \min\{\ell : w = s_1 s_2 \cdots s_\ell \text{ for } s_i \in S\}.$$

A word $\mathbf{s} := (s_1, s_2, \ldots, s_{\ell(w)})$, for which $w = s_1 s_2 \cdots s_{\ell(w)}$, is a *reduced word* for w.

Definition 4.3. The *(right) descent set* of w is

$$\mathrm{Des}(w) := \{s \in S : \ell(ws) < \ell(w)\}.$$

The set of *reflections* in W is $T := \{wsw^{-1} : w \in W, s \in S\}$. The *(left) inversion set* of w in W is

$$T_L(w) := \{t \in T : \ell(tw) < \ell(w)\},$$

which has size $\ell(w)$, and is computable from any $\mathbf{s} = (s_1,\ldots,s_{\ell(w)}) \in \mathrm{Red}(w)$ as follows [BB05, Corollary 1.4.4]:

$$T_L(w) = \left\{\begin{array}{c} s_1, \\ s_1 s_2 s_1, \\ s_1 s_2 s_3 s_2 s_1, \\ \vdots \\ s_1 s_2 \cdots s_{\ell(w)} \cdots s_2 s_1 \end{array}\right\}.$$

Definition 4.4. The *(right) weak Bruhat order* $u \leq_R w$ on W can be defined via any of the following equivalent conditions (see [BB05, Chapter 3]):

(a) $u \leq_R w$ is the transitive closure of the covering relation $u \lessdot_R w$, where $u = ws$ and $s \in \mathrm{Des}(w)$;
(b) there exist $u = u_0, u_1, \ldots, u_{\ell-1}, u_\ell = w$ in W and $s_i \in S$ such that $u_i s_i = u_{i+1}$ and $\ell(u_{i+1}) > \ell(u_i)$;

(c) there exists $(s_1, s_2, \ldots, s_{\ell(w)}) \in \mathrm{Red}(w)$ having a prefix $(s_1, s_2, \ldots, s_{\ell(u)}) \in \mathrm{Red}(u)$;
(d) $\ell(u) + \ell(u^{-1}w) = \ell(w)$; and
(e) $T_L(u) \subseteq T_L(w)$.

We note the following facts about weak Bruhat order intervals, which follow trivially from Definition 4.4.

Proposition 4.5. *In a Coxeter system (W, S),*
(a) *the weak Bruhat interval $[u, w]$ is isomorphic to the lower interval $[e, u^{-1}w]$, via $v \mapsto u^{-1}v$, and*
(b) *the dual poset $[e, w]^*$ to the lower interval $[e, w]$ is isomorphic to $[e, w^{-1}]$, via $u \mapsto w^{-1}u$.*

The definition of $\leq_R$ allows us to reformulate Equations (1) and (2).

Corollary 4.6. *For any Coxeter system (W, S) and $w \in W$,*

$$\mathbb{E}(X_{[e,w]}) = \frac{1}{\#[e,w]} \sum_{u \leq_R w} \#\,\mathrm{Des}(u) \quad \textit{and} \tag{16}$$

$$\mathbb{E}(Y_{[e,w]}) = \frac{1}{(\ell(w)+1) \cdot \#\,\mathrm{Red}(w)} \sum_{\substack{\mathbf{s}=(s_1,s_2,\ldots,s_{\ell(w)}) \\ \in \mathrm{Red}(w)}} \sum_{i=0}^{\ell(w)} \#\,\mathrm{Des}(s_1 s_2 \cdots s_i). \tag{17}$$

It is helpful to reformulate Equation (16) for later use.

Proposition 4.7. *For any Coxeter system (W, S) and any $w \in W$,*

$$\mathbb{E}(X_{[e,w]}) = \frac{1}{2}\left(\#S - \frac{1}{\#[e,w]} \sum_{u \leq_R w} \#\{s \in S : u \lessdot_R us \not\leq_R w\}\right).$$

Proof. Because posets P and P^* have the same expectation for the variable X, one always has

$$\mathbb{E}(X_P) = \frac{\mathbb{E}(X_P) + \mathbb{E}(X_{P^*})}{2} = \frac{\mathbb{E}(X_P + X_{P^*})}{2}.$$

For each p, the statistic $X_P(p)$ reports the down-degree of p in P. From the perspective of P^*, the statistic $X_{P^*}(p)$ computes the *up-degree* of p in P.

Specializing to $P = [e, w]$, the down-degree of an element $u \leq_R w$ is $\#\,\mathrm{Des}(u) = \#\{s \in S : us \lessdot_R u\}$ as before, and its up-degree is $\#\{s \in S : u \lessdot us \leq_R w\}$. Therefore the sum of an element's down- and up-degrees is

$$\#\{s \in S : us \lessdot_R u \text{ or } u \lessdot us \leq_R w\} = \#S - \#\{s \in S : u \lessdot us \not\leq_R w\}.$$

Halving the expectation of this random variable on $[e, w]$ yields the formula in the proposition. ☐

4.2. **0-Hecke monoids.** We wish also to reformulate Equation (17), this time via words in the 0-Hecke monoid.

Definition 4.8. Given a Coxeter matrix $(m_{s,t})$ and Coxeter system (W, S), the associated 0-*Hecke monoid* $\mathcal{H}_W(0)$ is the monoid generated by $\{T_s\}_{s \in S}$, subject to the quadratic relation $T_s^2 = T_s$ for $s \in S$ and

$$\underbrace{T_s T_t T_s T_t \cdots}_{m_{s,t} \text{ factors}} = \underbrace{T_t T_s T_t T_s \cdots}_{m_{s,t} \text{ factors}} \text{ for } s \neq t \text{ in } S.$$

It turns out (see, for example, [No79]) that any choice of reduced word $\mathbf{s} = (s_1, s_2, \ldots, s_{\ell(w)}) \in \mathrm{Red}(w)$ defines the same element $T_w := T_{s_1} \cdots T_{s_{\ell(w)}}$ in the monoid $\mathcal{H}_W(0)$. Furthermore, $T_w = T_{w'}$ in $\mathcal{H}_W(0)$ if and only if $w = w'$ in W, and thus, as a set, one has

$$\mathcal{H}_W(0) = \{T_w : w \in W\}.$$

This means that one can speak of a 0-*Hecke word* $(s_1, s_2, \ldots, s_L)$ for w, meaning that $T_w = T_{s_1} T_{s_2} \cdots T_{s_L}$ in $\mathcal{H}_W(0)$. It also implies that one has the following relations in $\mathcal{H}_W(0)$:

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \text{ and} \\ T_w & \text{if } \ell(ws) < \ell(w). \end{cases} \tag{18}$$

Factorizations $T_w = T_u T_v$ in $\mathcal{H}_W(0)$ give yet another characterization of the weak Bruhat order on W.

Proposition 4.9. *One has $u \leq_R w$ in W if only if there exists $v \in W$ with $T_u T_v = T_w$.*

Proof. If $u \leq_R w$ then $v = u^{-1}w$ will satisfy $T_u T_v = T_w$. Conversely, if $T_u T_v = T_w$, then pick reduced words

$$\mathbf{s} = (s_1, \ldots, s_{\ell(u)}) \in \mathrm{Red}(u) \quad \text{and} \quad (s'_1, \ldots, s'_{\ell(v)}) \in \mathrm{Red}(v).$$

Form a subword $(s'_{i_1}, \ldots, s'_{i_k})$ by omitting any s'_i for which $\ell(us'_1 \cdots s'_{i-1}s'_i) < \ell(us'_1 \cdots s'_{i-1})$. Using the relations in Equation (18) and the factorization

$$T_w = T_u T_v = T_u T_{s'_1} \cdots T_{s'_{\ell(v)}},$$

one deduces that $T_w = T_u T_{s'_{i_1}} \cdots T_{s'_{i_k}}$. Moreover, we have $\ell(us'_{i_1} \cdots s'_{i_j}) = \ell(u) + j$ for each j. Thus $(s_1, \ldots, s_{\ell(u)}, s'_{i_1}, \cdots, s'_{i_k})$ is a reduced word for w, and it contains $\mathbf{s} \in \mathrm{Red}(u)$ as a prefix. Hence $u \leq_R w$. ☐

Definition 4.10. If $T_w = T_{s_1} T_{s_2} \cdots T_{s_L}$ in $\mathcal{H}_W(0)$ with $L = \ell(w) + 1$, call $(s_1, s_2, \ldots, s_L)$ a *nearly reduced word* for w. Denote by $\mathrm{Red}(w)$ (resp., $\mathrm{Red}^{(+1)}(w)$) the set of reduced (resp., nearly reduced) words for w.

Proposition 4.11. *For any Coxeter system (W, S) and $w \in W$, one has a bijection*

$$\begin{array}{ccc} \left\{ \begin{array}{c} ((s_1, s_2, \ldots, s_{\ell(w)}), i, s) \in \mathrm{Red}(w) \times [0, \ell(w)] \times S : \\ s \in \mathrm{Des}(s_1 s_2 \cdots s_i) \end{array} \right\} & \longrightarrow & \mathrm{Red}^{(+1)}(w) \\ \\ \big((s_1, s_2, \ldots, s_{\ell(w)}), i, s\big) & \longmapsto & (s_1, s_2, \ldots, s_i, s, s_{i+1}, \ldots, s_{\ell(w)}). \end{array}$$

Proof. The given map is well-defined because $s \in \mathrm{Des}(s_1 s_2 \cdots s_i)$ implies that

$$\begin{aligned} T_{s_1} T_{s_2} \cdots T_{s_i} \cdot T_s \cdot T_{s_{i+1}} \cdots T_{s_{\ell(w)}} &= T_{s_1} T_{s_2} \cdots T_{s_i} \cdot T_{s_{i+1}} \cdots T_{s_{\ell(w)}} \\ &= T_w. \end{aligned}$$

It also surjects: given a nearly reduced word $\mathbf{s}^+ = (s_1^+, s_2^+, \ldots, s_{\ell(w)+1}^+)$ for w, if i is the smallest index with $(s_1^+, s_2^+, \cdots, s_{i+1}^+)$ not reduced, then $\mathbf{s} := (s_1^+, s_2^+, \cdots, s_i^+, s_{i+2}^+, s_{i+3}^+, \cdots, s_{\ell(w)+1}^+)$ has $(\mathbf{s}, i, s_{i+1}^+) \longmapsto \mathbf{s}^+$. To show injectivity, assume that the elements $(\mathbf{s}, i, s)$ and $(\mathbf{t}, j, t)$, with $i \leq j$, have the same image under the map. If one has strict inequality $i < j$, then the two words must match up as follows.

$$\begin{array}{ccccccccccccc} s_1 & s_2 & \ldots & s_i & s & s_{i+1} & \ldots & s_{j-1} & s_j & s_{j+1} & \ldots & s_{\ell(w)} \\ \| & \| & & \| & \| & \| & & \| & \| & \| & & \| \\ t_1 & t_2 & \ldots & t_i & t_{i+1} & t_{i+2} & \ldots & t_j & t & t_{j+1} & \ldots & t_{\ell(w)} \end{array}$$

The word $\mathbf{t}$ is reduced, so the same must be true of its prefix $(t_1, t_2, \cdots, t_i, t_{i+1}) = (s_1, s_2, \ldots, s_i, s)$, which contradicts $s \in \mathrm{Des}(s_1 s_2 \cdots s_i)$. Therefore $i = j$, which then implies that $(\mathbf{s}, i, s) = (\mathbf{t}, j, t)$. ☐

This result has two interesting corollaries, including a reformulation of Equation (17).

Corollary 4.12. *For any Coxeter system (W, S) and $w \in W$,*

$$\mathbb{E}(Y_{[e,w]}) = \frac{\#\,\mathrm{Red}^{(+1)}(w)}{(\ell(w) + 1) \cdot \#\,\mathrm{Red}(w)}.$$

In turn, because reversing a word gives bijections $\mathrm{Red}(w) \leftrightarrow \mathrm{Red}(w^{-1})$ and $\mathrm{Red}^{(+1)}(w) \leftrightarrow \mathrm{Red}^{(+1)}(w^{-1})$, Corollary 4.12 implies the following fact.

Corollary 4.13. *For any Coxeter system (W, S) and $w \in W$,*

$$\mathbb{E}(Y_{[e,w]}) = \mathbb{E}(Y_{[e,w^{-1}]}) = \mathbb{E}(Y_{[e,w]^*}).$$

5. Type A and vexillary permutations

There are special features to the weak Bruhat order intervals $[e, w]$ in the case of Coxeter systems (W, S) of type A_{n-1}. In this setting, W is the *symmetric group* $\mathfrak{S}_n$ of all permutations on $\{1, 2, \ldots, n\}$, and $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ with each σ_i being the adjacent transposition that swaps i and $i + 1$. In this section, we first explain how the theory of Schubert and Grothendieck polynomials let one recast $\mathbb{E}(Y_{[e,w]})$ for vexillary permutations. We then show how to recast $\mathbb{E}(X_{[e,w]})$ for *all* permutations. Finally, we specialize to dominant permutations, and then specialize even further to dominant permutations of rectangular staircase shape.

5.1. **Recasting $\mathbb{E}(Y_{[e,w]})$ for vexillary permutations.** Recall that Section 1 defined vexillary, dominant, Grassmannian, and inverse Grassmannian permutations.

Theorem 5.1. *Fix a shape λ. For all vexillary permutation w of shape λ,*

$$\mathbb{E}(Y_{[e,w]}) = \frac{\mathrm{Red}^{(+1)}(w)}{(\ell(w)+1)\,\mathrm{Red}(w)} = \frac{f^{\lambda}(+1)}{(|\lambda|+1)f^{\lambda}} = \mathbb{E}(Y_{[\varnothing,\lambda]}).$$

Furthermore, for those permutations that are Grassmannian or inverse Grassmannian,

$$\mathbb{E}(X_{[e,w]}) = \mathbb{E}(X_{[\varnothing,\lambda]}).$$

Example 5.2. We illustrate Theorem 5.1 for the partition $\lambda = (3,1,1)$.

(a) Figure 3 depicts the dual interval $[\varnothing,\lambda]^*$ and its isomorphic partner $[e, 236145]$, where 236145 is Grassmannian with code $(1,1,3)$. Both have $\mathbb{E}(X) = 13/10$ and $\mathbb{E}(Y) = 23/18$, as predicted by the theorem. These expectations, for both X and Y, would be shared by the interval $[e,(236145)^{-1}] \cong [\varnothing,\lambda]$, because $(236145)^{-1} = 412563$ is inverse Grassmannian.

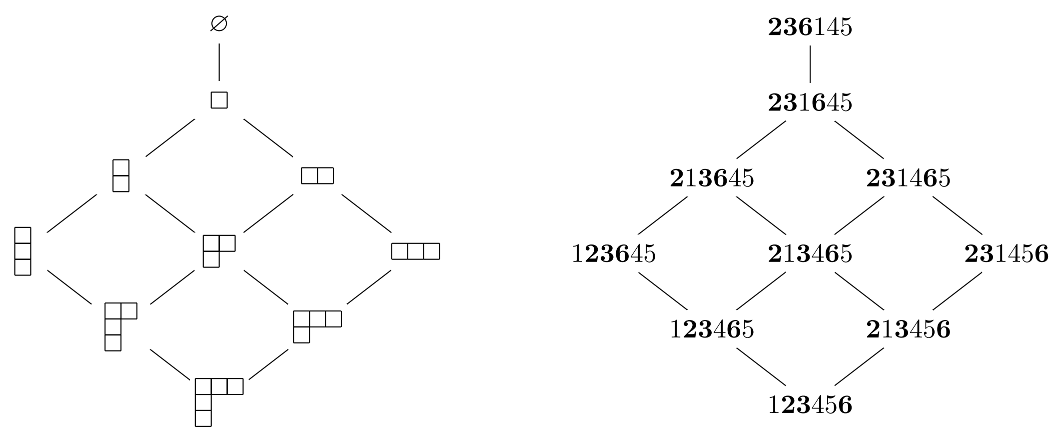


FIGURE 3. The interval $[\varnothing,(3,1,1)]^*$ and its isomorphic partner $[e, 23614]$ of Example 5.2.

(b) On the other hand, Figure 4 depicts the intervals $[e, 4231]$ and $[e, 25314]$. The permutation 4231 is dominant with code $\lambda = (3,1,1)$, while 25314 is vexillary with code $(1,3,1)$ but is neither dominant nor Grassmannian nor inverse Grassmannian.

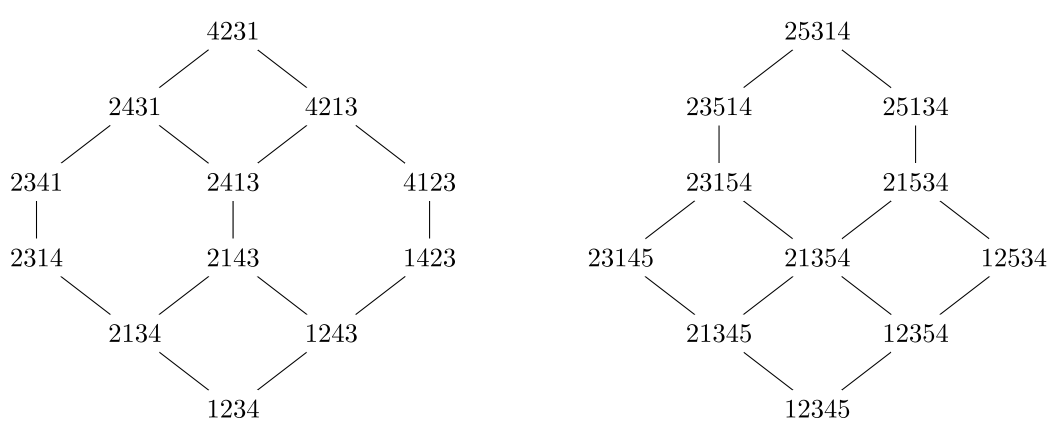


FIGURE 4. The intervals $[e, 4231]$ and $[e, 25314]$ of Example 5.2.

These intervals all have $\mathbb{E}(Y) = 23/18$, as predicted by the theorem because 236145, 4231, and 25314 are all vexillary of shape λ. However, $\mathbb{E}(X_{[e,4231]}) = 5/4$, while $\mathbb{E}(X_{[e,25314]}) = 14/11$, neither of which matches $\mathbb{E}(X_{[e,236145]}) = 13/10$.

The proof of Theorem 5.1 uses the relation between reduced words and 0-Hecke words in type A and the theory of Stanley symmetric functions and Lascoux and Schützenberger's theory of Schubert and Grothendieck polynomials (see, for example, [Bu02, BKSTY08, FG94, LS82, Man01, St84, La03]).

Definition 5.3. Given a partition λ, the *Schur function* s_λ and the *stable Grothendieck polynomial* (for partitions) G_λ are computed by

$$s_\lambda = \sum_T \mathbf{x}^T \quad \text{and}$$
$$G_\lambda = \sum_T (-1)^{|T|-|\lambda|}\mathbf{x}^T,$$

(see, for example, [Bu02, Theorem 3.1]) where the first (respectively, second) sum runs over all column-strict tableaux (respectively, column-strict set-valued tableaux) T of shape λ, and $\mathbf{x}^T$ is as defined in Equation (9). Given $w \in \mathfrak{S}_n$, the *stable Schubert polynomial* (or *Stanley symmetric function*) F_w and the *stable Grothendieck polynomial* (for permutations) G_w are defined via

$$F_w = \sum_{\substack{(\sigma_{a_1},\ldots,\sigma_{a_{\ell(w)}}),\\ (b_1,\ldots,b_{\ell(w)})}} x_{b_1}\cdots x_{b_{\ell(w)}} \quad \text{and}$$
$$G_w = \sum_{\substack{(\sigma_{a_1},\cdots,\sigma_{a_L}),\\ (b_1,\ldots,b_L)}} (-1)^{L-\ell(w)} x_{b_1}\cdots x_{b_{\ell(w)}}$$

(see, for example, [FG94, Examples 2.2 and 2.5]). In the first sum, $(\sigma_{a_1},\cdots,\sigma_{a_{\ell(w)}})$ ranges over all reduced words for w, while in the second sum, $(\sigma_{a_1},\cdots,\sigma_{a_L})$ ranges over all 0-Hecke words for w. In both cases, $(b_1, b_2, \ldots)$ are weakly increasing sequences of positive integers satisfying the compatibility condition that $b_i < b_{i+1}$ whenever $a_i \leq a_{i+1}$.

Although it is not obvious, the functions s_λ, G_λ, F_w, and G_w are all *symmetric functions* in the infinite variable set $\{x_1, x_2, \ldots\}$.

Finally for any $w \in \mathfrak{S}_n$, the (β)-*Grothendieck polynomial* is defined by

$$\mathfrak{G}_w^{(\beta)} = \sum_{\substack{(\sigma_{a_1},\cdots,\sigma_{a_L}),\\ (b_1,\ldots,b_L)}} \beta^{L-\ell(w)} x_{b_1}\cdots x_{b_{\ell(w)}}, \tag{19}$$

where the summation is over the same pairs of sequences as for G_w, with the additional condition that $b_i \leq a_i$. We also mention that the $\beta = 0$ and $\beta = -1$ specializations $\mathfrak{G}_w^{(0)}$ and $\mathfrak{G}_w^{(-1)}$ are called the *Schubert polynomial* and *Grothendieck polynomial* for w, respectively.

The relevance of these polynomials comes from their coefficients on certain squarefree monomials:

$$\begin{aligned} f^\lambda &\text{ is the coefficient of } x_1x_2\cdots x_{|\lambda|} \text{ in } s_\lambda,\\ \#\operatorname{Red}(w) &\text{ is the coefficient of } x_1x_2\cdots x_{\ell(w)} \text{ in } \mathfrak{G}_w^{(0)},\\ f^\lambda(+1) &\text{ is the coefficient of } -x_1x_2\cdots x_{|\lambda|}x_{|\lambda|+1} \text{ in } G_\lambda,\\ \#\operatorname{Red}^{(+1)}(w) &\text{ is the coefficient of } -x_1x_2\cdots x_{\ell(w)}x_{\ell(w)+1} \text{ in } \mathfrak{G}_w^{(-1)}. \end{aligned} \tag{20}$$

There are also various known relationships between them.

- Note that s_λ and F_w are the lowest-degree terms of G_λ and G_w, respectively.
- F_w and G_w are called *stable* Schubert and Grothendieck polynomials because

$$F_w = \lim_{N\to\infty} \mathfrak{G}^{(0)}_{1^N\times w}(x_1,\ldots,x_{N+n}) \text{ and}$$
$$G_w = \lim_{N\to\infty} \mathfrak{G}^{(-1)}_{1^N\times w}(x_1,\ldots,x_{N+n}),$$

where $1^N \times w := (1, 2, \ldots, N, N + w(1), N + w(2), \ldots, N + w(n))$ lies in $\mathfrak{S}_{N+n}$.

- For w a Grassmannian permutation of shape λ, one has

$$F_w = s_\lambda \quad \text{and} \qquad (21)$$
$$G_w = G_\lambda.$$

Our proof of Theorem 5.1 will rest on the following generalization of the relations in (21).

Lemma 5.4. *For a vexillary permutation w of shape λ, one has*

$$F_w = s_\lambda,$$
$$G_w = G_\lambda.$$

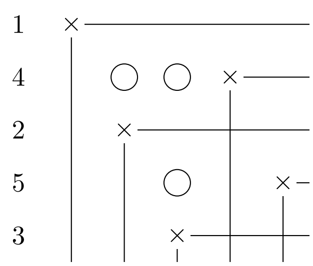
In order to prove this lemma, we will employ a tableau formula for $\mathfrak{G}_w^{(-1)}$ from [KMY09] that involves flagged set-valued tableaux. First recall the notion of a flag from Definition 3.3.

Suppose w is a vexillary permutation with shape λ having ℓ nonzero parts. One defines the flag $\varphi(w) = (\varphi_1, \varphi_2, \ldots, \varphi_\ell)$ as follows (see [KMY09, §5.2] for more details). Recall that the *Rothe diagram* of w is

$$D(w) := \{(i,j) : 1 \leq i, j \leq n, w(i) > j, w^{-1}(j) > i\} \quad \subset \quad \{1,2,\ldots,n\} \times \{1,2,\ldots,n\};$$

(see, for example, [Man01, §2.2.1]). Let $\mu(w)$ be the smallest Ferrers shape (northwest justified within the square shape n^n) that contains all the boxes of $D(w)$. Overlay the northwest corner of $\lambda(w)$ on the northwest corner of the square n^n. Let the *diagonal of row i* be the diagonal occupied by the rightmost box of $\lambda(w)$ in row i. (In fact it is true that $\lambda(w) \subseteq \mu(w)$.) Then set φ_i to be the row number of the southeastmost box of $\mu(w)$ in the diagonal of row i.

Example 5.5. If w is the vexillary permutation $14253 \in \mathfrak{S}_5$, then its Rothe diagram $D(w)$ is the set of row and column indices (i,j) corresponding to the circles in this picture:



Then $\lambda(w) = (2,1)$ and $\mu(w) = (3,3,3,3)$. Hence $\varphi(w) = (2,4)$.

The following case is especially important to this paper.

Example 5.6. If w is a dominant permutation, then $\lambda(w) = \mu(w)$ and therefore $\varphi(w) = (1,2,3,\ldots)$.

We now can state the following tableau formula, found in [KMY09] (up to minor notational conventions).

Theorem 5.7 ([KMY09, Theorem 5.8])**.** *Let w be vexillary. Then*

$$\mathfrak{G}_w^{(-1)}(x_1,\ldots,x_n) = \sum_T (-1)^{|T|-|\lambda|}\mathbf{x}^T,$$

where the sum is over all set-valued tableaux of shape $\lambda(w)$ flagged by $\varphi(w)$.

One checks that $w \mapsto \varphi(w)$ commutes as follows with the operation $w \mapsto 1^N \times w$ on vexillary permutations:

$$\varphi(1^N \times w) = \varphi(w) + (N,N,\ldots) = (\varphi_1 + N, \varphi_2 + N, \ldots) \quad \text{if } \varphi(w) = (\varphi_1, \varphi_2, \ldots). \qquad (22)$$

We can now complete the proof of Lemma 5.4, and then of Theorem 5.1.

Proof of Lemma 5.4. The equality $F_w = s_\lambda$ for vexillary is well-known [St84, Corollary 4.2], but will also follow once we show $G_w = G_\lambda$, since F_w and s_λ are the lowest-degree terms in G_w and G_λ, respectively.

To this end, note that when working in finitely many variables $x_1, x_2, \ldots, x_N$ for any positive integer N, Definition 5.3 implies that

$$G_w(x_1,\ldots,x_N) = \mathfrak{G}_{1^N \times w}^{(-1)}(x_1,\ldots,x_N,0,0,0,\ldots).$$

On the other hand, by Theorem 5.7 and Equation (22), one has

$$\mathfrak{G}^{(-1)}_{1^N \times w}(x_1, \dots, x_{N+n}) = \sum_T (-1)^{|T|-|\lambda|}\mathbf{x}^T, \tag{23}$$

where the sum is over all column-strict set-valued tableaux of shape λ flagged by $(\varphi_1+N, \varphi_2+N, \dots)$. Hence

$$\mathfrak{G}^{(-1)}_{1^N \times w}(x_1, \dots, x_N, 0, 0, 0, \dots) = \sum_T (-1)^{|T|-|\lambda|}\mathbf{x}^T,$$

where the sum is over column-strict set-valued tableaux with entries from $1, 2, \dots, N$ (that is, the flagging condition on each row becomes redundant). Therefore, for any positive integer N,

$$G_w(x_1, \dots, x_N) = G_\lambda(x_1, \dots, x_N).$$

Because G_w and G_λ are both symmetric functions this suffices to show $G_w = G_\lambda$. □

Proof of Theorem 5.1. To prove the first assertion in the theorem, note that Lemma 5.4 together with Equation (20) show that when w is vexillary of shape λ, one has $\mathrm{Red}(w) = f^\lambda$ and $\mathrm{Red}^{(+1)}(w) = f^\lambda(+1)$. Together with the fact that $\ell(w) = |\lambda|$, this gives the middle equality here

$$\mathbb{E}(Y_{[\varnothing,\lambda]}) = \frac{f^\lambda(+1)}{(|\lambda|+1)f^\lambda} = \frac{\#\,\mathrm{Red}^{(+1)}(w)}{(\ell(w)+1)\cdot \#\,\mathrm{Red}(w)} = \mathbb{E}(Y_{[e,w]}),$$

while the first equality is Equation (11) and the last equality is Corollary 4.12.

For the theorem's second assertion, use Equation (10), Proposition 4.5, and Proposition 5.12 below. □

Remark 5.8. In fact, Lemma 5.4 also shows that any two 0-Hecke words for a vexillary permutation are *K-Knuth equivalent* in the sense defined by Buch and Samuel [BS13, §5]. We will not go into the details, but this can be deduced by combining Lemma 5.4, along with properties of the *K-theoretic jeu de taquin* introduced by Thomas and Yong [TY09], together with results on the *Hecke insertion* introduced in [BKSTY08], [TY11, Theorem 4.2] and [BS13, Theorem 6.2].

5.2. **Evaluating $\mathbb{E}(X_{[e,w]})$ via noninversion posets.** In type A_{n-1}, the set of reflections T for the Coxeter system (W, S) is equal to all (not necessarily adjacent) *transpositions* $T = \{\tau_{ij} : 1 \le i < j \le n\}$, and we have $S = \{\sigma_1, \dots, \sigma_{n-1}\}$ where $\sigma_i := \tau_{i,i+1}$. Furthermore, the (left) inversion set of a permutation w is

$$\#T_L(w) := \{\tau_{ij} : 1 \le i < j \le n \text{ and } w^{-1}(i) > w^{-1}(j)\}$$

and the number of inversions $\#T_L(w)$ is the same as the Coxeter group length $\ell(w)$.

Definition 5.9. For $w \in \mathfrak{S}_n$, the *noninversion poset* $P_{\mathrm{ninv}}(w)$ is the partial order on $\{1, 2, \dots, n\}$ in which $i <_{P_{\mathrm{ninv}}(w)} j$ if and only if $i <_{\mathbb{Z}} j$ and $(i,j) \notin T_L(w)$; that is, in which $i < j$ and $w^{-1}(i) < w^{-1}(j)$.

Definition 5.10. For a poset P on $\{1, 2, \dots, n\}$, a *linear extension* of P is a permutation $w = w(1)\cdots w(n) \in \mathfrak{S}_n$ for which $i <_P j$ implies $w^{-1}(i) < w^{-1}(j)$; that is, $w(1) < w(2) < \cdots < w(n)$ extends P to a linear order. Denote by $\mathcal{L}(P)$ the set of all linear extensions of P.

The following may then be viewed as the rephrasing in type A of the characterization of the weak order that asserted $u \le_R w$ if and only if $T_L(u) \subset T_L(w)$.

Proposition 5.11. *For any $w \in \mathfrak{S}_n$, one has $[e, w] = \mathcal{L}(P_{\mathrm{ninv}}(w))$.*

This reformulation allows us to prove the following.

Proposition 5.12. *If w is Grassmannian of shape λ, then $[e, w] \cong [\varnothing, \lambda]^*$.*

Proof. If $\lambda = (\lambda_1, \dots, \lambda_\ell)$ with $\lambda_\ell > 0$, then the one-line notation for w is a concatenation of two increasing sequences; namely,

$$w(1) = \lambda_\ell + 1 < w(2) = \lambda_{\ell-1} + 2 < \cdots < w(\ell-1) = \lambda_2 + \ell - 1 < w(\ell) = \lambda_1 + \ell$$

concatenated with the sequence $w(\ell+1) < \cdots < w(n)$. Therefore the noninversion poset $P_{\mathrm{ninv}(w)}$ contains these two sequences as chains, along with some extra order relations between them. Thus any element u in $[e, w] = \mathcal{L}(P_{\mathrm{ninv}(w)})$ is a shuffle of these two increasing sequences, and hence is completely determined by the positions $u^{-1}(w(1)) < \cdots < u^{-1}(w(\ell))$ occupied by the initial increasing sequence in the one-line notation for w. This produces a poset isomorphism $[e, w] \to [\varnothing, \lambda]^*$ defined by

$$u \longmapsto \mu = (\mu_1, \ldots, \mu_\ell) = (u^{-1}(w(\ell)) - \ell, \ldots, u^{-1}(w(1)) - 1).$$

□

Proposition 5.11 lets us reinterpret the denominator $\#[e, w]$ of $\mathbb{E}(X_{[e,w]})$. We next work on the numerator.

Definition 5.13. Given a covering relation $i \lessdot_P j$ in a poset P on $\{1, 2, \ldots, n\}$, define a quotient poset $P/\{i, j\}$ that "sets i equal to j." More formally, consider the equivalence relation $\equiv_{ij}$ that has $n-1$ blocks by merging i and j into a single block, and check that the (reflexive, symmetric) transitive closure of the union of the two binary relations $\leq_P$ and $\equiv_{ij}$ gives a poset structure on the $n-1$ blocks of $\equiv$.

Proposition 5.14. *Fix a permutation $w \in \mathfrak{S}_n$ and set $P := P_{\mathrm{ninv}(w)}$. Then*

$$\sum_{u \leq_R w} \#\{s \in S : u \lessdot_R us \not\leq_R w\} = \sum_{i \lessdot_P j} \#\mathcal{L}(P/\{i, j\}),$$

and therefore

$$\mathbb{E}(X_{[e,w]}) = \frac{1}{2}\left(\#S - \sum_{i \lessdot_P j} \frac{\#\mathcal{L}(P/\{i, j\})}{\#\mathcal{L}(P)}\right). \tag{24}$$

Proof. Given an element $u \leq_R w$ and $s = \sigma_k = (k, k+1)$ in S for which $u \lessdot_R us \not\leq_R w$, let $i := u(k)$ and $j := u(k+1)$. Then $u \lessdot_R us$ implies $i < j$. Furthermore, $u \in \mathcal{L}(P)$ but $us \notin \mathcal{L}(P)$ implies that $i <_P j$ must be a covering relation in P, and one can regard $u/\{i, j\}$ as an element of $\mathcal{L}(P/\{i, j\})$.

Conversely, given a covering relation $i \lessdot_P j$ and an element $\hat{u}$ of $\mathcal{L}(P/\{i, j\})$, say with $\{i, j\} = \hat{u}_k$, one can recover from it an element $u \leq_R w$ with $u \lessdot u\sigma_k \not\leq_R w$ by replacing the block $\hat{u}_k$ by $(u(k), u(k+1)) = (i, j)$. □

5.3. **Dominant permutations and the forest hook-length formula.** Arbitrary permutations $w \in \mathfrak{S}_n$ have no nice product formula to compute $\#[e, w] = \#\mathcal{L}(P_{\mathrm{ninv}}(w))$, but dominant permutations do.

Definition 5.15. A finite poset P is a *forest poset* if each element is covered by at most one other element.

Proposition 5.16 ([BFLR12, Corollaries 5.3 and 5.4])**.** *The poset $P_{\mathrm{ninv}(w)}$ is a forest poset if and only if w is dominant.*

Forest posets have the following *hook-length formula* counting their linear extensions, first observed by Knuth.

Proposition 5.17 ([Kn73, §5.1.4 Exercise 20])**.** *Let P be a forest poset, and set $P_{\leq i} := \{j \in P : j \leq_P i\}$. Then*

$$\#\mathcal{L}(P) = \frac{\#P!}{\prod_{i \in P} \#P_{\leq i}}. \tag{25}$$

In computing $\mathbb{E}(X_{[e,w]})$ using Equations (24) and (25), the following reduction for forests will be useful.

Lemma 5.18. *Fix a covering relation $i \lessdot_P j$ in a forest poset P. Then*

$$\frac{\#\mathcal{L}(P/\{i, j\})}{\#\mathcal{L}(P)} = \frac{\#P_{\leq i}}{\#P} \cdot \prod_{k >_P i} \frac{\#P_{\leq k}}{(\#P_{\leq k} - 1)} = \frac{\#P_{\leq i}}{\#P} \cdot \frac{\prod_{k \in \beta(i)} \#P_{\leq k}}{\prod_{k \in \alpha(i)} (\#P_{\leq k} - 1)},$$

where the sets $\alpha(i)$ and $\beta(i)$ are defined by

$$\alpha(i) := \{k \in P : k >_P i \text{ and either } k \gtrdot_P i, \text{ or } k \text{ covers more than one element}\} \text{ and}$$

$$\beta(i) := \{k \in P : k >_P i \text{ and } k \text{ is either maximal, or } k \text{ is covered by an element of } \alpha(i)\}.$$

Figure 5 shows a schematic for the local structure above a node i in a forest poset P, with the nodes in $\alpha(i)$ circled and the nodes in $\beta(i)$ boxed. Note that the sets $\alpha(i)$ and $\beta(i)$ may intersect.

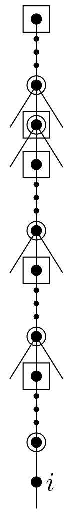


FIGURE 5. Local structure above a node i in a forest posets P. Elements of $\alpha(i)$ are circled and elements of $\beta(i)$ are boxed.

Proof. The first equality comes from Equation (25) via a calculation

$$\frac{\#\mathcal{L}(P/\{i,j\})}{\#\mathcal{L}(P)} = \frac{(\#P-1)! \prod_{k\in P} \#P_{\leq k}}{\#P! \prod_{k\in P/\{i,j\}} \#(P/\{i,j\})_{\leq k}} = \frac{\#P_{\leq i}}{\#P} \prod_{k>_P i} \frac{\#P_{\leq k}}{(\#P_{\leq k}-1)},$$

because if we label elements of $P/\{i,j\}$ by $k \in P \setminus \{i\}$, then $\#(P/\{i,j\})_{\leq k}$ is either $\#P_{\leq k}$ for $k \not>_P i$, or $\#P_{\leq k} - 1$ for $k >_P i$. The second equality comes from telescoping the factors in the rightmost product:

- for $k \notin \alpha(i)$, the denominator $\#P_{\leq k} - 1$ cancels with $\#P_{\leq \ell}$ for the unique $\ell \lessdot k$, and
- for $k \notin \beta(i)$, the numerator $\#P_{\leq k}$ is canceled by $\#P_{\leq \ell} - 1$ for the unique $\ell \gtrdot k$.

□

5.4. **Computing $\mathbb{E}(X_{[e,w]})$ for dominant permutations of rectangular staircase shape.** We now turn to the computation of $\mathbb{E}(X_{[e,w]})$ when w is a dominant permutation of rectangular staircase shape $\delta_d(b^a)$. As we will show, this has a very nice form.

Our strategy will approach this calculation via induction on d. To this end, throughout the remainder of this section, fix the rectangle dimensions $a, b \geq 1$, and assume, for convenience, that $a \leq b$. For each $d \geq 2$, define $w^{(d)}$ to be the dominant permutation of shape $\delta_d(b^a)$. One can check that $w^{(d)}$ lies in $\mathfrak{S}_N$ where $N := a + (d-1)b$, and that the one-line notation for $w^{(d)}$ is the following concatenation of contiguous intervals of integers:

$$w^{(d)} = I_{d-1} \cdots I_2\, I_1\, I_0\, J_1\, J_2 \cdots J_{d-1},$$

where $I_m := [mb+1, mb+a]$ and $J_m := [(m-1)b+a+1, mb]$. The noninversion poset for this permutation,

$$P^{(d)} := P_{\text{ninv}}(w^{(d)}),$$

is a forest poset with the schematic structure depicted in Figure 6, where each I_i and J_j is totally ordered in increasing order. By convention, set $J_0 := \varnothing$ and define the poset $P^{(1)} := I_0 = [1, a]$ totally ordered in increasing order.

Example 5.19. Fix $a = 3$ and $b = 7$. The posets for $P^{(d)}$ for $d = 1, 2, 3, 4$ are shown in Figure 7.

Recall that $w^{(d)} \in \mathfrak{S}_N$, where $N = a + (d-1)b$. Thus we can rewrite Equation (24) using Lemma 5.18 as

$$\mathbb{E}\left(X_{[e,w^{(d)}]}\right) = \frac{1}{2}\left(\#S - \sum_{i \lessdot j} \frac{\#\mathcal{L}(P^{(d)}/\{i,j\})}{\#\mathcal{L}(P^{(d)})}\right) = \frac{1}{2}\left(N - 1 - \frac{1}{N}\sum_{\ell=0}^{d-1} \theta_\ell^{(d)}\right), \tag{26}$$

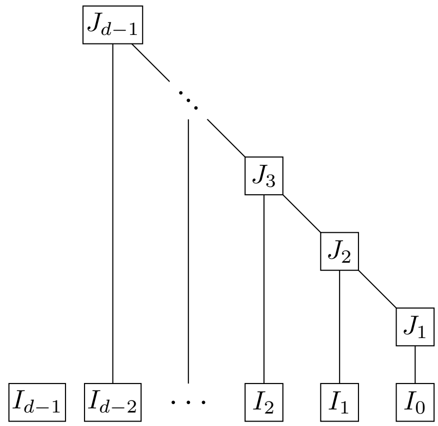


FIGURE 6. Structure of the noninversion poset for a dominant permutation of rectangular staircase shape $\delta_d(b^a)$.

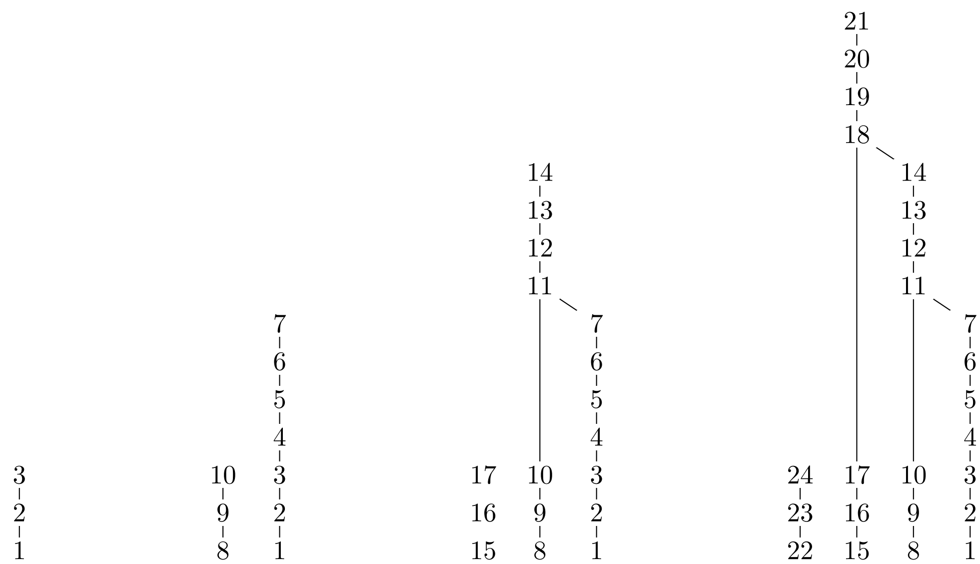


FIGURE 7. The noninversion posets $P^{(1)}, P^{(2)}, P^{(3)}, P^{(4)}$ for $a = 3$, $b = 7$, as described in Example 5.19.

where, for $\ell = 0, 1, 2, \ldots, d-1$, we introduce the sums

$$\theta_\ell^{(d)} := \sum_{\substack{i \in I_\ell \sqcup J_\ell \\ \text{not maximal} \\ \text{in } P^{(d)}}} f^{(d)}(i) \qquad \text{with} \qquad f^{(d)}(i) := \#P_{\leq i}^{(d)} \cdot \frac{\prod_{k \in \beta(i)} \#P_{\leq k}^{(d)}}{\prod_{k \in \alpha(i)} \left(\#P_{\leq k}^{(d)} - 1\right)}.$$

Note that the sum for $\theta_0^{(d)}$ is why we have made the convention $J_0 = \varnothing$.

For the sake of readability, we introduce the abbreviation

$$c_j := \frac{jb}{a + (j-1)b}.$$

Lemma 5.20. *The sums $\theta_\ell^{(d)}$ have these explicit formulas:*

$$\theta_\ell^{(d)} = \begin{cases} (a^2 + \ell b(b-a)) \cdot c_{\ell+1} c_{\ell+2} \cdots c_{d-1} & \text{if } \ell = 0, 1, 2, \ldots, d-2, \text{ and} \\ (a^2 + (d-1)b(b-a)) - N & \text{if } \ell = d-1. \end{cases}$$

Proof. For nonmaximal i in $P^{(d)}$, the elements of $\alpha(i)$ are $\min J_j$ for various j. Similarly, the elements of $\beta(i)$ are $\max J_j$ or $\max I_j$, for various j. Now observe that

$$\begin{aligned}\#P^{(d)}_{\leq \max J_j} &= jb,\\ \#P^{(d)}_{\leq \max I_j} &= a, \text{ and}\\ \#P^{(d)}_{\leq \min J_j} &= a+(j-1)b+1.\end{aligned}$$

From this, one can check that for nonmaximal i in $P^{(d)}$,

$$f^{(d)}(i) = c_{\ell+1}c_{\ell+2}\cdots c_{d-1}\cdot \begin{Bmatrix} a & \text{if } i\in I_\ell, \\ \ell b & \text{if } i \in J_\ell \end{Bmatrix}.$$

For $\ell = 0,1,2,\ldots,d-2$, the intervals I_ℓ and J_ℓ contain a and b elements, respectively, and all are nonmaximal in $P^{(d)}$. On the other hand, all but one element from each of I_{d-1} and J_{d-1} are nonmaximal. Therefore,

$$\theta^{(d)}_\ell = \begin{cases} c_{\ell+1}c_{\ell+2}\cdots c_{d-1}\cdot a\cdot a + c_{\ell+1}c_{\ell+2}\cdots c_{d-1}\cdot \ell b(b-a) & \text{if } \ell = 0,1,2,\ldots,d-2, \text{ and}\\ a\cdot(a-1)+(d-1)b\cdot(b-a-1) & \text{if } \ell = d-1.\end{cases}$$

This agrees with the formulas given in the statement of the lemma. □

Corollary 5.21. *For $a, b \geq 1$ and $d \geq 2$, and for w the dominant permutation of shape $\delta_d(b^a)$,*

$$\mathbb{E}(X_{[e,w]}) = \frac{(d-1)ab}{a+b}.$$

Proof. We may assume without loss of generality that $a \leq b$ by Proposition 4.5(b), and hence $w = w^{(d)}$. Set $N := a+(d-1)b$. By Equation (26), it suffices to show that

$$\frac{1}{N}\sum_{\ell=0}^{d-1}\theta^{(d)}_\ell = N-1-2(d-1)\frac{ab}{a+b} = a-1+(d-1)\frac{b(b-a)}{a+b} \tag{27}$$

for $d \geq 1$. We show the leftmost and rightmost sides of Equation (27) are equal via induction on d. In the base case $d = 1$,

$$\frac{1}{N}\sum_{\ell=0}^{d-1}\theta^{(d)}_\ell = \frac{1}{a}\theta^{(1)}_0 = \frac{1}{a}(a(a-1)) = a-1 = a-1+(d-1)\frac{b(b-a)}{a+b}.$$

In the inductive step, we use the following recursive reformulation of Lemma 5.20:

$$\theta^{(d)}_\ell = \begin{cases} c_{d-1}\theta^{(d-1)}_\ell & \text{if } \ell = 0,1,2,\ldots,d-3,\\ c_{d-1}\left(\theta^{(d-1)}_\ell + a+(d-2)b\right) & \text{if } \ell = d-2, \text{ and}\\ (a^2+(d-1)b(b-a))-N & \text{if } \ell = d-1.\end{cases} \tag{28}$$

Now assume the left and right sides of Equation (27) are equal for $d-1$, and use Equation (28) to compute

$$\begin{aligned}\frac{1}{N}\sum_{\ell=0}^{d-1}\theta^{(d)}_\ell &= \frac{c_{d-1}}{N}\left(\sum_{\ell=0}^{d-2}\theta^{(d-1)}_\ell + a+(d-2)b\right) + \frac{1}{N}\left((a^2+(d-1)b(b-a))-N\right)\\ &= \frac{c_{d-1}}{N}\left((a+(d-2)b)\left(a-1+(d-2)\frac{b(b-a)}{a+b}\right)+a+(d-2)b\right)\\ &\qquad + \frac{(a^2+(d-1)b(b-a))}{N}-1\\ &= a-1+(d-1)\frac{b(b-a)}{a+b},\end{aligned}$$

via straightforward algebra in the last step. □

Finally we can complete our goal.

Proof of Theorem 1.1. Combine Propositions 3.14, 3.16, Theorem 5.1, and Corollary 5.21. □

6. Macdonald and Fomin-Kirillov type formulas

This section presents a conjecture, Conjecture 6.3 below, inspired both by Corollary 1.3 and by a remarkable formula of Fomin and Kirillov [FK97] which we recall now. Let $w_0 := n(n-1)\cdots 21$ be the longest element of $\mathfrak{S}_n$, which is the dominant permutation of the staircase shape δ_n, having $\ell(w_0) = N := \binom{n}{2}$.

Theorem 6.1 ([FK97, Theorem 1.1])**.**

$$\sum_{(\sigma_{i_1},\sigma_{i_2},\ldots,\sigma_{i_N})} (x+i_1)(x+i_2)\cdots(x+i_N) = N! \prod_{1\le i<j\le n} \frac{2x+i+j-1}{i+j-1},$$

where the sum runs over all $(\sigma_{i_1},\sigma_{i_2},\ldots,\sigma_{i_N})$ *in* $\mathrm{Red}(w_0)$.

Extracting the coefficient of x^N in Theorem 6.1 gives Stanley's result [St84] that $\#\,\mathrm{Red}(w_0) = f^{\delta_n}$, while setting $x = 0$ recovers a result of Macdonald [Mac91, page 91].

To state our conjecture, we define a sum generalizing the left side in Theorem 6.1.

Definition 6.2. For a permutation w and a nonnegative integer L, define a polynomial in x of degree L by

$$FK(w,L) := \sum_{(\sigma_{i_1},\sigma_{i_2},\cdots,\sigma_{i_L})} (x+i_1)(x+i_2)\cdots(x+i_L),$$

where the sum runs over all 0-Hecke words $(\sigma_{i_1},\sigma_{i_2},\ldots,\sigma_{i_L})$ for w of length L.

In particular, $FK\left(w_0, \binom{n}{2}\right)$ is the sum in Theorem 6.1.

Conjecture 6.3. *Let w be the dominant permutation of rectangular staircase shape $\lambda = \delta_d(b^a)$. Then for $\ell := \ell(w) = |\lambda| = \binom{d}{2}ab$, the polynomial $FK(w,\ell)$ divides $FK(w,\ell+1)$ in $\mathbb{Q}[x]$, with quotient*

$$\frac{FK(w,\ell+1)}{FK(w,\ell)} = \binom{\ell+1}{2}\left(\frac{4x}{d(a+b)}+1\right).$$

Example 6.4. If $d = 3, a = b = 1$, then $\lambda = (2,1)$ with $\ell = |\lambda| = 3$ and $w = 321$. Using the two reduced words and eight nearly reduced words for w computed in Example 1.4, one finds that

$$\begin{aligned} FK(w,3) &= (x+1)(x+2)(x+1) + (x+2)(x+1)(x+2) \\ &= (x+1)(x+2)(2x+3), \end{aligned}$$

$$\begin{aligned} FK(w,4) &= 2(x+1)^3(x+2) + 4(x+1)^2(x+2)^2 + 2(x+1)(x+2)^2 \\ &= 2(x+1)(x+2)(2x+3)^2, \end{aligned}$$

so that $FK(w,3)$ does divide $FK(w,4)$, with quotient

$$\frac{FK(w,4)}{FK(w,3)} = 2(2x+3) = 4x+6.$$

This agrees with Conjecture 6.3, which predicts

$$\frac{FK(w,4)}{FK(w,3)} = \binom{4}{2}\left(\frac{4x}{3(1+1)}+1\right) = 4x+6.$$

The following relation was one of our motivations for Conjecture 6.3, and provides some evidence for it.

Proposition 6.5. *Conjecture 6.3 would imply Corollary 1.3.*

Proof. For any permutation w and any L, the (leading) coefficient c_L on x^L in $FK(w,L)$ counts the number of 0-Hecke words for w of length L. Therefore whenever w is vexillary of shape λ and $\ell = |\lambda|$, Lemma 5.4 implies $c_\ell = f^\lambda$ and $c_{\ell+1} = f^\lambda(+1)$.

On the other hand, for w dominant of rectangular staircase shape $\lambda = \delta_d(b^a)$ and $\ell = |\lambda|$, Conjecture 6.3 would imply that $FK(w,\ell+1)/FK(w,\ell)$ is a linear polynomial $rx+s$ whose leading coefficient r equals

$$\binom{\ell+1}{2}\frac{4}{d(a+b)} = \frac{c_{\ell+1}}{c_\ell} = \frac{f^\lambda(+1)}{f^\lambda}.$$

This is equivalent to the assertion of Corollary 1.3, using $\ell = \binom{d}{2}ab$. ☐

As further evidence in support of Conjecture 6.3, we will eventually verify it in the case $d = 2$; that is for rectangular shapes $\lambda = b^a$. In fact, it will turn out to be more convenient to work with an equivalent tableau version of the conjecture, whose statement requires some additional notation.

Definition 6.6. For a flag $\varphi = (\varphi_1, \varphi_2, \ldots)$, denote by $\mathtt{SSYT}(\lambda, \varphi, j)$ the collection of all column-strict set-valued tableaux of shape λ that are flagged by φ, and whose total number of entries is j (in other words, $j = \sum_y \#T(y)$ where y runs through the cells of λ). In particular, $\mathtt{SSYT}(\lambda, \varphi, j)$ is empty unless $j \geq |\lambda|$.

The equivalent tableau version of Conjecture 6.3 is the following.

Conjecture 6.3′. *Let w be the dominant permutation of rectangular staircase shape $\lambda = \delta_d(b^a)$. Then for $\ell := \ell(w) = |\lambda| = \binom{d}{2}ab$, and for any positive integer x, the flag $\varphi = (1, 2, 3, \ldots) + (x, x, x, \ldots)$ produces*

$$\frac{\#\mathtt{SSYT}(\lambda, \varphi, \ell + 1)}{\#\mathtt{SSYT}(\lambda, \varphi, \ell)} = \frac{2\ell \cdot x}{d(a + b)}.$$

The equivalence of Conjectures 6.3 and 6.3′ will follow from the next theorem, proven in Section 7, combining ideas of [FS94, FK94, FK96, FK97] with Equation (23). To state it, recall that the *Stirling number of the second kind* $S(L, j)$ counts partitions of $\{1, 2, \ldots, L\}$ into j blocks.

Theorem 6.7. *For vexillary w of shape λ and positive integer x, the flag $\varphi := \varphi(w) + (x, x, x, \ldots)$ satisfies*

$$FK(w, L) = \sum_{j=|\lambda|}^{L} \#\mathtt{SSYT}(\lambda, \varphi, j) \cdot j!S(L, j). \tag{29}$$

Remark 6.8. Theorem 6.7 is an extension of Theorem 6.1 in the following sense. If $w = w_0$ then $\lambda = \delta_n$. Setting $L = |\lambda| = \binom{n}{2} = N$ in Equation (29) gives

$$FK(w_0, N) = \#\mathtt{SSYT}(\delta_n, (1 + x, 2 + x, \ldots)) \cdot N!.$$

Theorem 6.1 is then derived from this in [FK97] by an application of a formula of Proctor [Pr84b], for plane partitions of staircase shape with largest bounded part, which implies

$$\#\mathtt{SSYT}(\delta_n, (1 + x, 2 + x, \ldots) = \prod_{1 \leq i < j \leq n} \frac{2x + i + j - 1}{i + j - 1}.$$

In contrast, for arbitrary j, even when $w = w_0$, we know of no such product formulas for $\#\mathtt{SSYT}(\lambda, \varphi, j)$.

Let us assume the validity of Theorem 6.7 for the moment, and check the following.

Corollary 6.9. *Conjectures 6.3 and 6.3′ are equivalent.*

Proof. Let w be a dominant permutation of shape λ, and set $\ell := |\lambda| = \ell(w)$. Dominant permutations w always have flag $\varphi(w) = (1, 2, 3, \ldots)$, which explains the choice $\varphi := (1, 2, 3, \ldots) + (x, x, x, \ldots)$. Now applying Theorem 6.7 twice, with $L = \ell$, and $L = \ell + 1$, and using the facts that $S(\ell, \ell) = 1 = S(\ell + 1, \ell + 1)$ and $S(\ell + 1, \ell) = \binom{\ell+1}{2}$, one obtains

$$\frac{FK(w, \ell + 1)}{FK(w, \ell)} = \frac{\ell! \cdot \binom{\ell+1}{2} \cdot \#\mathtt{SSYT}(\lambda, \varphi, \ell) \quad + \quad (\ell + 1)! \cdot \#\mathtt{SSYT}(\lambda, \varphi, \ell + 1)}{\ell! \cdot \#\mathtt{SSYT}(\lambda, \varphi, \ell)} = \binom{\ell + 1}{2} + (\ell + 1) \cdot \rho$$

where $\rho := \#\mathtt{SSYT}(\lambda, \varphi, \ell + 1)/\#\mathtt{SSYT}(\lambda, \varphi, \ell)$ is an abbreviaton for the ratio that appears in Conjecture 6.3′. Thus Conjecture 6.3 holds if and only if

$$\frac{FK(w, \ell + 1)}{FK(w, \ell)} = \binom{\ell + 1}{2} + (\ell + 1)\rho = \binom{\ell + 1}{2} \left(\frac{4x}{d(a + b)} + 1 \right).$$

Upon division by $\binom{\ell+1}{2}$, this assertion is equivalent to

$$1 + \frac{2}{\ell}\rho = \frac{4x}{d(a + b)} + 1, \qquad \text{and also} \qquad \rho = \frac{2\ell x}{d(a + b)}$$

which is exactly Conjecture 6.3′. ☐

Corollary 6.10. *Conjectures 6.3 and 6.3′ hold when $d = 2$; that is, for dominant w of rectangle shape $\lambda = b^a$.*

Proof. It will be most convenient to work with Conjecture 6.3′. Our strategy will once again use the uncrowding map in Definition 3.8 to convert barely set-valued tableaux to ordinary tableaux.

For a partition λ, let $\mathtt{CST}_{\leq t}(\lambda)$ denote the set of column-strict (ordinary) tableaux of shape λ with all entries in the range $\{1, 2, \ldots, t\}$. Note that column-strictness implies for rectangular shapes b^a that

$$\mathtt{SSYT}(b^a, \varphi, ab) = \mathtt{CST}_{\leq x+a}(b^a),$$

where $\varphi = (1, 2, 3, \ldots, a) + (x, x, x, \ldots, x)$.

On the other hand, consider the restriction of the uncrowding map to the domain $\mathtt{SSYT}((b^a), \varphi, ab + 1)$. Since the rectangle b^a has only outer corner cell in a row below 1, namely in row $a+1$, the result always has shape $(b^a, 1)$. Thus, one obtains a bijection

$$\begin{array}{rcl} \mathtt{SSYT}(b^a, \varphi, ab+1) & \longrightarrow & \mathtt{CST}_{\leq x+a}((b^a, 1)) \times \{1, 2, \ldots, a\}, \\ T & \longmapsto & (T^+, i_0) \end{array}$$

which shows $\#\mathtt{SSYT}(b^a, \varphi, ab + 1) = a \cdot \#\mathtt{CST}_{\leq x+a}((b^a, 1))$, and hence

$$\frac{\#\mathtt{SSYT}(b^a, \varphi, \ell+1)}{\#\mathtt{SSYT}(b^a, \varphi, \ell)} = \frac{a \cdot \#\mathtt{CST}_{\leq x+a}((b^a, 1))}{\#\mathtt{CST}_{\leq x+a}(b^a)}. \tag{30}$$

The numerator and denominator here are calculable via the *hook-content formula* ([St71, Theorem 15.3])

$$\#\mathtt{CST}_{\leq t}(\lambda) = \prod_y \frac{t + c(y)}{h(y)}$$

where the product runs over the cells $y = (i, j)$ of λ, with $c(y) = j - i$ its *content*, and $h(y) = \lambda_i + \lambda^t_j - (i+j) + 1$ its *hook-length*. Here are the relevant values of $t + c(y) = (x + a) + c(y)$ and $h(y)$ for cells y of $(b^a, 1)$:

$x+a$	$x+a+1$	$x+a+2$	$\cdots$	$x+a+b-1$
$x+a-1$	$x+a$	$x+a+1$	$\cdots$	$x+a+b-2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$x+1$	$x+2$	$x+3$	$\cdots$	$x+b-1$
$\boldsymbol{x}$				

$a+b$	$a+b-1$	$a+b-2$	$\cdots$	a
$a+b-1$	$a+b-2$	$a+b-3$	$\cdots$	$a-1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$1+b$	$b-1$	$b-2$	$\cdots$	1
1				

On the other hand, for cells y of b^a, the relevant values of $t + c(y) = (x + a) + c(y)$ are precisely the same except that the boldfaced value $\boldsymbol{x}$ does not arise, and the values of $h(y)$ are the same except for the first column, which are $a + b - 1, a + b - 2, \ldots, b + 1, b$ in that setting. Therefore,

$$\frac{\#\mathtt{CST}_{\leq x+a}((b^a, 1))}{\#\mathtt{CST}_{\leq x+a}(b^a)} = \frac{xb}{a+b}.$$

Comparing this with Equation (30) proves the $d = 2$ case of Conjecture 6.3′, because then $\ell = |\lambda| = ab$. □

7. Proof of Theorem 6.7.

As mentioned earlier, our proof combines ideas of [FS94, FK96, FK94, FK97] with Equation (23).

Let R denote the 0-*Hecke algebra* of type A_{n-1}. That is, R is the monoid algebra for the 0-Hecke monoid $\mathcal{H}_W(0)$, where (W, S) is the Coxeter system $W = \mathfrak{S}_n$ with the adjacent transpositions $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ as Coxeter generators. Thus R has $\mathbb{Z}$-basis given by $\{T_w\}_{w \in \mathfrak{S}_n}$ and multiplication extended $\mathbb{Z}$-linearly from the 0-Hecke monoid $\mathcal{H}_W(0)$. Abbreviate here $T_i := T_{\sigma_i}$ for $i = 1, 2, \ldots, n - 1$, as was done in the Introduction.

We will also consider various rings obtained from R by extension of scalars, such as $R \otimes_{\mathbb{Z}} \mathbb{Q}[[t]]$ or $R \otimes_{\mathbb{Z}} \mathbb{Q}[[t_1, \ldots, t_n]]$. Within these larger rings, we will also use without further mention a common exponential change-of-variables in which $x := e^t - 1$ in $\mathbb{Q}[[t]]$, and analogously, $x_i := e^{t_i} - 1$ in $\mathbb{Q}[[t_1, \ldots, t_n]]$.

The following lemma is implicit in [FK94] (cf. [FS94]), which explicitly states a consequence of it (see Remark 7.7 below). Indeed, the result is known to the authors of [FK94]; see Lemma 5.6 of `hep-th/9306005`. However, for convenience we include a proof. Recall that for $w \in \mathfrak{S}_n$, the β-Grothendieck polynomial $\mathfrak{G}^{(1)} = \mathfrak{G}(x_1, \ldots, x_{n-1})$ defined in Equation (19) is a polynomial in $n - 1$ variables.

Lemma 7.1 (Fomin-Kirillov). *In the ring $R \otimes_{\mathbb{Q}} \mathbb{Q}[[t]]$,*

$$e^{t(T_1+2T_2+\cdots(n-1)T_{n-1})} = \sum_{w\in\mathfrak{S}_n} \mathfrak{G}_w^{(1)}(e^t-1,\ldots,e^t-1)\cdot T_w = \sum_{w\in\mathfrak{S}_n} \mathfrak{G}_w^{(1)}(x,\ldots,x)\cdot T_w.$$

Proof. As in [FK96], define $h_i(t) := e^{tT_i}$. Then following [FK94], since $T_i^2 = T_i$, one has

$$h_i(t) = e^{tT_i} = \sum_{k=0}^{\infty} \frac{(tT_i)^k}{k!} = 1 + tT_i + t^2T_i/2! + \cdots = 1 + (e^t-1)T_i = 1 + xT_i.$$

It is a main result of [FK94] that, with notations

$$\begin{aligned} A_i(t) &:= h_{n-1}(t)h_{n-2}(t)\cdots h_i(t) \quad \text{and} \\ \mathfrak{G}(t_1,t_2,\ldots,t_{n-1}) &:= A_1(t_1)A_2(t_2)\cdots A_{n-1}(t_{n-1}), \end{aligned} \tag{31}$$

the "Grothendieck element" $\mathfrak{G}(t_1,t_2,\ldots,t_{n-1})$ expands as follows in $R\otimes_{\mathbb{Z}}\mathbb{Q}[[t_1,\ldots,t_{n-1}]]$:

$$\mathfrak{G}(t_1,t_2,\ldots,t_{n-1}) = \sum_{w\in\mathfrak{S}_n} \mathfrak{G}_w^{(1)}(e^{t_1}-1,\ldots,e^{t_{n-1}}-1)\cdot T_w = \sum_{w\in\mathfrak{S}_n} \mathfrak{G}_w^{(1)}(x_1,\ldots,x_{n-1})\cdot T_w$$

In fact, this is equivalent to the definition of $\mathfrak{G}_w^{(1)}$ from Equation (19): working in the variables $x_1,\ldots,x_{n-1}$, when one expands $\mathfrak{G}(t_1,t_2,\ldots,t_{n-1})$ as defined in Equation (31), its coefficient of T_w is exactly the sum over pairs $((\sigma_{a_1},\ldots,\sigma_{a_L}),(b_1,\ldots,b_L))$ on the right side of Equation (19).

Thus, it remains to prove that specializing the variables $t_1 = t_2 = \cdots = t_{n-1}$ to the single variable t gives

$$\mathfrak{G}(t,t,\ldots,t) = A_1(t)A_2(t)\cdots A_{n-1}(t) = e^{t(T_1+2T_2+\cdots+(n-1)T_{n-1})}. \tag{32}$$

To this end, we employ a *mutatis mutandis* modification of an argument of [FS94]. For brevity, we refer the reader to [FS94] for those details that remain unchanged.

It is easy to check that the collection $\{h_i\}$ satisfies the relations

(I) $h_i(s)h_j(t) = h_j(t)h_i(s)$ if $|i-j|\geq 2$,
(II) $h_i(s)h_i(t) = h_i(s+t)$, $h_i(0) = 1$ (and therefore $h_i(s)h_i(-s) = 1$),

as well as the *Yang-Baxter equation* [FK96]

(III) $h_i(s)h_{i+1}(s+t)h_i(t) = h_{i+1}(t)h_i(s+t)h_{i+1}(s)$.

The following lemma is the analogue of [FS94, Lemma 2.1]. Its proof is exactly the same as that result's, because it only depends on the relations (I)–(III).

Lemma 7.2. *$A_i(s)$ and $A_i(t)$ commute.*

Define [FS94, §4]

$$\widetilde{A}_i(t) = h_i(t)h_{i+1}(t)\cdots h_{n-1}(t)$$

and let

$$\widetilde{\mathfrak{G}}(t_1,\ldots,t_{n-1}) := \widetilde{A}_{n-1}(t_{n-1})\widetilde{A}_{n-2}(t_{n-2})\cdots\widetilde{A}_1(t_1).$$

Lemma 7.3. *$A_i(s)$ and $\widetilde{A}_i(t)$ commute.*

Proof. The proof is the same as in [FS94, Lemma 4.1], except we use (II) and Lemma 7.2 (where the original uses the exact analogues, [FS94, Lemmas 3.1(ii) and 2.1]). ☐

Lemma 7.4. $\widetilde{A}_{n-1}(t_{n-1})\cdots\widetilde{A}_i(t_i)A_i(s) = h_{n-1}(t_{n-1}+s)\cdots h_i(t_i+s)\widetilde{A}_{n-1}(t_{n-2})\cdots\widetilde{A}_{i+1}(t_i)$.

Proof. One follows [FS94, Lemma 4.2] except to use Lemma 7.3 rather than their [FS94, Lemma 4.1]. ☐

Lemma 7.5.

$$\widetilde{\mathfrak{G}}(t_1,\ldots,t_{n-1})\mathfrak{G}(s_1,\ldots,s_{n-1}) = \prod_{c=2-n}^{n-2}\ \prod_{i-j=c,\, i+j\leq n} h_{i+j-1}(s_i+t_j).$$

Here the multiplication of the factors associated to $c = 2-n, 3-n, \ldots, n-2$ is done from left to right. The factors in the second product commute.

Proof. As in [FS94, Lemma 4.3], this follows from repeated application of Lemma 7.4 combined with rearrangement of factors. To see that the factors in the second product commute, note that if $(i,j),(i',j')\in\mathbb{N}\times\mathbb{N}$ satisfy $i-j = c = i'-j'$ and $i+j = i'+j'-1$, we would have $2i-2i' = -1$, which is impossible. ☐

The following is the analogue of [FS94, Lemma 5.1].

Lemma 7.6. $\mathfrak{G}(t,t,\ldots,t)\mathfrak{G}(s,s,\ldots,s) = \mathfrak{G}(t+s,\ldots,t+s)$.

Proof. Same as that of [FS94, Lemma 5.1], using Lemma 7.5 above in place of their [FS94, Lemma 4.3]. □

We are now ready to prove Equation (32) as in the proof of [FS94, Lemma 2.3]. Using Lemma 7.6, and the fact that $G(0,0,\ldots,0)=1$, one finds that

$$\begin{aligned}\frac{d}{dt}\mathfrak{G}(t,t,\ldots,t) &= \lim_{h\to 0}\frac{\mathfrak{G}(t+h,t+h,\ldots,t+h)-\mathfrak{G}(t,t,\ldots,t)}{h}\\ &= \mathfrak{G}(t,t,\ldots,t)\cdot\lim_{h\to 0}\frac{\mathfrak{G}(h,h,\ldots,h)-1}{h} = \mathfrak{G}(t,t,\ldots,t)\cdot F\end{aligned}$$

where $F := \left[\frac{d}{dt}\mathfrak{G}(t,t,\ldots,t)\right]_{t=0}$, an element of R. From this one concludes that, in $R\otimes_{\mathbb{Z}}\mathbb{Q}[[t]]$, one has

$$\mathfrak{G}(t,t,\ldots,t) = e^{tF}. \tag{33}$$

On the other hand, since $\mathfrak{G}(t,\ldots,t) := A_1(t)\cdots A_{n-1}(t)$, one can use the Leibniz rule repeatedly to compute

$$\left[\frac{d}{dt}h_i(t)\right]_{t=0} = T_i \qquad \text{and} \qquad \left[\frac{d}{dt}A_j(t)\right]_{t=0} = T_{n-1}+T_{n_2}+\cdots+T_j.$$

Therefore

$$\begin{aligned}F = \left[\frac{d}{dt}A_1(t)\cdots A_{n-1}(t)\right]_{t=0} &= \sum_{j=1}^{n-1}\left[A_1(t)A_2(t)\cdots A_{j-1}(t)\cdot\frac{d}{dt}A_j(t)\cdot A_{j+1}(t)\cdots A_{n-1}(t)\right]_{t=0}\\ &= \sum_{j=1}^{n-1}(T_{n-1}+T_{n_2}+\cdots+T_j) = T_1+2T_2+\cdots+(n-1)T_{n-1}.\end{aligned}$$

Plugging this into Equation (33) proves Equation (32), and completes the proof of Lemma 7.1. □

Proof of Theorem 6.7. By inspection, one has

$$e^{t(T_1+2T_2+\cdots(n-1)T_{n-1})} = \sum_{w\in\mathfrak{S}_n}\left(\sum_L\frac{t^L}{L!}\sum_{(\sigma_{a_1},\ldots,\sigma_{a_L})}a_1\cdots a_L\right)\cdot T_w,$$

where the innermost sum is over 0-Hecke words for w of length L. In view of Lemma 7.1, this means that

$$\mathfrak{G}^{(1)}_w(e^t-1,\ldots,e^t-1) = \sum_L\frac{t^L}{L!}\sum_{(\sigma_{a_1},\ldots,\sigma_{a_L})}a_1\cdots a_L. \tag{34}$$

For positive integers x, note that $(\sigma_{a_1},\ldots,\sigma_{a_L})$ is a 0-Hecke word for w if and only if $(\sigma_{x+a_1},\ldots,\sigma_{x+a_L})$ is a 0-Hecke word for $1^x\times w$. Therefore, one similarly has

$$\mathfrak{G}^{(1)}_{1^x\times w}(e^t-1,\ldots,e^t-1) = \sum_L\frac{t^L}{L!}\sum_{(\sigma_{a_1},\ldots,\sigma_{a_L})}(x+a_1)\cdots(x+a_L).$$

Equivalently, using $\left[\frac{t^L}{L!}\right]f(t)$ to denote the coefficient of $t^L/L!$ in $f(t)$, one has for any $w\in\mathfrak{S}_n$,

$$FK(w,L) = \left[\frac{t^L}{L!}\right]\mathfrak{G}^{(1)}_{1^x\times w}(e^t-1,\ldots,e^t-1). \tag{35}$$

For w vexillary of shape λ, and $\varphi=\varphi(w)+(N,N,N,\ldots)$, replacing x_i by $-x_i$ in Equation (23) gives

$$\mathfrak{G}^{(1)}_{1^N\times w}(x_1,\ldots,x_{N+n}) = \sum_T\mathbf{x}^T$$

where the sum runs over all column-strict set-valued tableaux T of shape λ which are flagged by φ. Substituting $N=x$ and $x_i=e^t-1$ for all i, this shows that

$$\mathfrak{G}^{(1)}_{1^x\times w}(e^t-1,\ldots,e^t-1) = \sum_{j=|\lambda|}^{L}(e^t-1)^j\cdot\#\mathtt{SSYT}(\lambda,\varphi,j)$$

and hence Equation (35) becomes

$$FK(w,L) = \sum_{j=|\lambda|}^{L} \left[\frac{t^L}{L!}\right] (e^t-1)^j \cdot \#\mathtt{SSYT}(\lambda,\varphi,j).$$

Lastly, the well-known exponential generating function [St12, Equation (1.94b)] for Stirling numbers

$$\sum_{L\geq j} S(L,j)\frac{t^L}{L!} = \frac{(e^t-1)^j}{j!}$$

shows that $\left[\frac{t^L}{L!}\right](e^t-1)^j = j!S(L,j)$, completing the proof. □

Remark 7.7. Since $\mathfrak{G}^{(1)}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$, the case $w = w_0$ of Equation (34) gives

$$\sum_{L}\sum_{(a_1,\ldots,a_L)} a_1\cdots a_L\frac{t^L}{L!} = (e^t-1)^{\binom{n}{2}}.$$

This formula was stated in [FK94, §3].

Acknowledgements

The authors thank Zach Hamaker for pointing out an earlier argument for Lemma 5.4 in terms of Lascoux's transition formula for Grothendieck polynomials [La01]. We thank Kári Ragnarsson and Cara Monical for providing computer code useful in the investigations of Sections 2 and 6, respectively. We thank Susanna Fishel and Thomas McConville for pointing out Proposition 2.21. We thank Thomas McConville for pointing out the examples at the end of Section 2.2.2, and thank Alex Garver for help on oriented exchange graphs. We thank Sam Hopkins for helpful discussions on the work in [CHHM15], and for sharing with us his preliminary results on shifted Young diagrams. We also thank Bruce Sagan for several helpful comments and questions. VR was partially supported by NSF RTG grant DMS-1148634. BT was partially supported by a Simons Foundation Collaboration Grant for Mathematicians. AY was partially supported by NSF grant DMS-1500691.

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