

# POLYNOMIALS FOR $GL_p \times GL_q$ ORBIT CLOSURES IN THE FLAG VARIETY

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ABSTRACT. The subgroup  $K = GL_p \times GL_q$  of  $GL_{p+q}$  acts on the (complex) flag variety  $GL_{p+q}/B$  with finitely many orbits. We introduce a family of polynomials specializing to representatives for cohomology classes of the orbit closures in the Borel model. We define and study  $K$ -orbit determinantal ideals to support the geometric naturality of these representatives. Using a modification of these ideals, we describe an analogy between two local singularity measures: the  $H$ -polynomials and the Kazhdan-Lusztig-Vogan polynomials.

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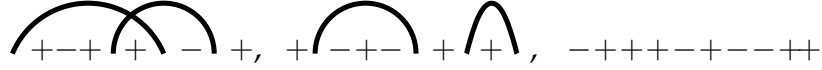
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# 1. INTRODUCTION

**1.1. Polynomial representatives in ordinary cohomology.** Consider the Levi subgroup  $K = GL_p \times GL_q$  of  $GL_n$  ( $n = p + q$ ). (Throughout, we consider only complex general linear groups.) By a general result of T. Matsuki [Mat79, Theorem 3], the flag variety  $GL_n/B$  decomposes as a disjoint union of finitely many  $K$ -orbits:

$$GL_n/B = \coprod_{\gamma} \mathcal{O}_{\gamma}.$$

The orbits  $\mathcal{O}_{\gamma}$  are parameterized by  $(p, q)$ -**clans**  $\gamma$ , as described first by T. Matsuki-T. Oshima [MaOs90, Theorem 4.1], and later elaborated upon by A. Yamamoto [Ya97, Theorem 2.2.8]. These clans are partial matchings of vertices  $\{1, 2, \dots, n\}$ , where unpaired vertices are assigned  $+$  or  $-$ ; the difference in the number of  $+$ 's and  $-$ 's must be  $p - q$ . Let  $\text{Clans}_{p,q}$  denote the set of all such clans. Three clans from  $\text{Clans}_{6,4}$  are shown below:



Let  $Y_{\gamma}$  be the Zariski closure of  $\mathcal{O}_{\gamma}$ . This is the union of  $\mathcal{O}_{\beta}$  for  $\beta \prec \gamma$ , where (by definition)  $\prec$  is the **closure order** on clans. It is an irreducible variety. By the formula of [Ya97, Proposition 2.3.8], its dimension is  $\binom{p}{2} + \binom{q}{2} + \ell(\gamma)$  where

$$(1) \quad \ell(\gamma) = \sum_{\substack{\text{vertices } i < j \text{ are} \\ \text{matched}}} j - i - \#\{\text{matchings of } s < t \text{ where } s < i < t < j\}.$$

$Y_{\gamma}$  admits a class in singular cohomology (with  $\mathbb{Z}$  coefficients):

$$[Y_{\gamma}] \in H^*(GL_n/B) \cong \mathbb{Z}[x_1, \dots, x_n]/I^{S_n},$$

where  $I^{S_n}$  is the ideal generated by symmetric polynomials without constant term. The above isomorphism, due to A. Borel [Bo53] (cf. [Fu99, Section 10.2]), is suggestive of the following problem:

Describe a choice of polynomial representatives  $\{\Upsilon_{\gamma}\}$  for the cosets associated to  $\{[Y_{\gamma}]\}$  under Borel's isomorphism.

One solution begins by assigning polynomials to the  $\binom{n}{p}$ -many **closed orbits**. These orbits are indexed by **matchless clans**  $\tau$ , i.e., those consisting of  $p$  many  $+$ 's and  $q$  many  $-$ 's (the third displayed clan above is an example). We will typically use  $\tau$  to denote a matchless clan, and  $\gamma$  to indicate an arbitrary clan. The **divided difference operator**  $\partial_i : \mathbb{Z}[x_1, \dots, x_n] \rightarrow \mathbb{Z}[x_1, \dots, x_n]$  is

$$\partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}}.$$

Representatives for all other orbits can be obtained by recursion using the  $\partial_i$ 's along a choice of path in *weak order* (defined in Section 2.1). This approach was used by the first author in [Wy13a].

We consider a different choice of polynomial representatives for the closed orbits than that found in *loc. cit.* From our perspective, this alternative choice of representatives is preferable for the following reasons:

- It is provably “self-consistent”, by which we mean that each  $\Upsilon_\gamma$  is a well-defined polynomial. Specifically,  $\Upsilon_\gamma$  depends neither on the choice of closed  $K$ -orbit  $\mathcal{O}_\tau$  at which we start the recurrence, nor on the aforementioned choice of path in weak order.
- Each  $\Upsilon_\gamma$  has nonnegative integer coefficients, and in many cases the geometric reason for this is transparent.
- Our choice extends simply to  $T$ -equivariant cohomology and ( $T$ -equivariant)  $K$ -theory, where  $T$  is the torus of diagonal matrices in  $GL_n$ . ([Wy13a] covers the case of  $T$ -equivariant cohomology, but neither ordinary nor  $T$ -equivariant  $K$ -theory are discussed.) We mostly suppress discussion of these refinements until Section 2.

To formulate our answer, we associate to a matchless  $(p, q)$ -clan  $\tau$  a partition, which we will denote  $\lambda(\tau)$ . We will also associate a sequence of nonnegative integers denoted by  $\vec{f}(\tau)$ ; this sequence is called a “flagging” in the context that we will use it below.

The partition  $\lambda(\tau)$  is formed as follows. Start from the upper-right corner of a  $p \times q$  rectangle, and trace a lattice path to the lower-left corner, by moving down at step  $i$  if the  $i$ th character of  $\tau$  is a  $+$ , and left if it is a  $-$ . Then  $\lambda(\tau)$  is the partition whose Young diagram is the portion of the  $p \times q$  rectangle northwest of this path. Clearly, the assignment of  $\lambda(\tau)$  to  $\tau$  defines a bijection between matchless  $(p, q)$ -clans (or, equivalently,  $p$ -element subsets of  $\{1, \dots, n\}$ ) and partitions whose Young diagrams fit within a  $p \times q$  rectangle.

Now,  $\vec{f}(\tau) = (f_1, \dots, f_p)$  for  $\lambda(\tau)$  is defined by  $f_i =$  index of  $i$ th  $+$  of  $\tau$ .

Next, let  $\hat{\tau}$  denote the  $(q, p)$ -clan obtained from  $\tau$  by flipping all signs. Then we can also form the partition  $\lambda(\hat{\tau})$  and the flagging  $\vec{f}(\hat{\tau})$ , as described above. Note that this partition has  $q$  parts, and its flagging is a  $q$ -tuple.

As an example, if  $\tau = ++--+-++$  then  $\lambda(\tau) = (3, 3, 1, 0, 0)$  and  $\vec{f}(\tau) = (1, 2, 5, 7, 8)$ , while  $\lambda(\hat{\tau}) = (3, 3, 2)$  and  $\vec{f}(\hat{\tau}) = (3, 4, 6)$ . The relevant pictures are as follows:

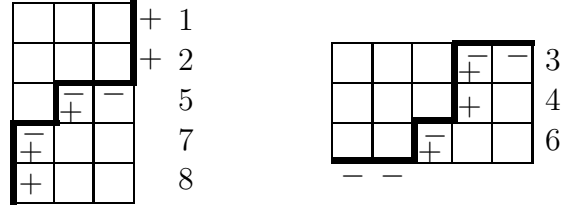


FIGURE 1.  $\lambda(\tau)$ ,  $\vec{f}(\tau)$  and  $\lambda(\hat{\tau})$ ,  $\vec{f}(\hat{\tau})$  for  $\tau = ++--+-++$ .

Now, given any partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0)$  and a sequence of nonnegative integers  $\vec{f} = (f_1, \dots, f_m)$  (a flagging), one defines the **flagged Schur polynomial** to be

$$s_{\lambda, \vec{f}}(X) = \sum_T \mathbf{x}^T,$$

where the sum is over all semistandard tableaux  $T$  of shape  $\lambda$  whose entries in row  $i$  are weakly bounded above by  $f_i$ ; see [Ma01, Section 2.6] for a textbook treatment of flagged Schur polynomials. So considering the partition  $\lambda(\tau) = (3, 3, 1, 0, 0)$  and the flagging  $\vec{f}(\tau) = (1, 2, 5, 7, 8)$  coming from the clan  $\tau$  in our example,

$$s_{(3,3,1,0,0),(1,2,5,7,8)}(x_1, x_2, x_3, x_4, x_5) = x_1^3 x_2^3 x_3 + x_1^3 x_2^3 x_4 + x_1^3 x_2^3 x_5.$$

The three monomials correspond to the tableaux

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 4 & & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 5 & & \\ \hline \end{array}.$$

Since  $\lambda(\tau)$  and  $\vec{f}(\tau)$  are determined by  $\tau$ , one can use the abbreviation  $\mathfrak{s}_\tau(X) = s_{\lambda(\tau), \vec{f}(\tau)}(X)$ , and similarly define  $\mathfrak{s}_{\hat{\tau}}(X)$ . For matchless  $\tau$ , define:

$$(2) \quad \Upsilon_\tau := \mathfrak{s}_\tau(X) \cdot \mathfrak{s}_{\hat{\tau}}(X).$$

Given a clan  $\gamma$  which is not matchless, by [RiSp90, Theorem 4.6] there exists a matchless clan  $\tau$  and a sequence  $s_1, \dots, s_l$  of simple transpositions such that  $\gamma = s_1 \cdot s_2 \cdot \dots \cdot s_l \cdot \tau$ . (This notation is explained in Section 2.1.) In general, neither  $\tau$  nor the permutation  $w = s_1 \dots s_l$  is uniquely determined by  $\gamma$ . Our wish is to define

$$\Upsilon_\gamma = \partial_1 \dots \partial_l \Upsilon_\tau.$$

However, in light of the preceding sentence, it is not at all clear that this is a valid “definition”. The main purpose of this paper is to present the following (and its refinements):

**Theorem 1.1.** *Each  $\Upsilon_\gamma$  is well-defined and represents  $[Y_\gamma]$  under Borel’s isomorphism.*

We now make a few easy observations about the  $\Upsilon_\gamma$ ’s.

The flagged Schur polynomials from (2) are Schubert polynomials (see Proposition 2.5). It is a standard fact that any product of Schubert classes expands as a nonnegative linear combination of Schubert classes, and moreover the Schubert polynomials represent the Schubert classes under the Borel isomorphism; see, e.g., Chapter 10 (and specifically Section 10.4) of [Fu99]. It follows that  $\Upsilon_\gamma$  is a nonnegative linear combination of Schubert polynomials. Since Schubert polynomials have nonnegative integer coefficients,

$$\Upsilon_\gamma \in \mathbb{Z}_{\geq 0}[x_1, \dots, x_n] \quad \text{for all } \gamma \in \text{Clan}_{p,q}.$$

We have emphasized the monomial expansion of  $\Upsilon_\gamma$  since this positivity should have a geometric explanation (see Section 3).

Finally, by our definition of  $\Upsilon_\tau$  for  $\tau$  matchless, it is easy to see that  $\Upsilon_\tau$  has degree  $\binom{n}{2} - \binom{p}{2} - \binom{q}{2} = pq$ . This reflects the fact that any closed  $K$ -orbit is isomorphic to the flag variety for the group  $K$ , and hence has dimension equal to  $\binom{p}{2} + \binom{q}{2}$ . Combining this with the aforementioned dimension formula of A. Yamamoto (cf. (1)), and with the fact that application of  $\partial_i$  lowers the degree of any polynomial by 1, it follows that the degree of  $\Upsilon_\gamma$  for arbitrary  $\gamma$  is  $pq - l(\gamma)$ , the codimension of  $\mathcal{O}_\gamma$  in the flag variety.

**1.2. Further results and comparisons to the literature.** For a reductive algebraic group  $G$  over  $\mathbb{C}$ , let  $B$  be a Borel subgroup and  $K \subset G$  be a **spherical** subgroup, i.e., one which acts by left translations on  $G/B$  with finitely many orbits.

The most widely analyzed case is when  $K = B$ , where the orbit closures are Schubert varieties. In this setting, the polynomial representatives problem was studied for Schubert varieties (in general type) by I. Bernstein-I. Gelfand-S. Gelfand [BeGeGe73]. In type  $A$ , this led to the development of *Schubert polynomials* by A. Lascoux-M.-P. Schützenberger [LaSh82]. Both papers begin with a choice of polynomial representative for the class of a point, with the remainder recursively obtained using  $\partial_i$ ’s. However, the salient feature of Schubert polynomials is the nonnegativity of their coefficients. Since their discovery, many nice combinatorial properties of Schubert polynomials have been found, including

combinatorial formulas for their expansion; see, e.g., the textbook [Ma01]. We will use properties of Schubert polynomials to establish our main results.

A spherical subgroup  $K$  is **symmetric** if  $K = G^\theta$  is the fixed point subgroup for a holomorphic involution  $\theta$  of  $G$ . The symmetric pairs  $(G, K)$  have a classification. For generalities, the reader may consult, e.g., [Mat79, Sp85, MaOs90, RiSp90]. The case of  $(GL_{p+q}, GL_p \times GL_q)$  corresponds to the involution

$$\theta(A) = I_{p,q} A I_{p,q}$$

where  $I_{p,q}$  is the diagonal  $\pm 1$  matrix with  $p$  many 1's followed by  $q$  many  $-1$ 's. For more details about this case, see, e.g., [Ya97, McGo09, McGoTr09, Wy13a].

The first author gave equivariant cohomology representatives for the closed orbits of cases of symmetric pairs  $(G, K)$  with  $G$  classical in [Wy13a, Wy13b]. For the case of  $(GL_{p+q}, GL_p \times GL_q)$ , small examples suggest that those representatives may also produce a self-consistent system, although we do not know a proof of this. At any rate, their ordinary cohomology specializations do not have nonnegative integer coefficients in general.

To our best knowledge, this paper provides the first self-consistency proof of its kind for any symmetric pair  $(G, K)$ . In the case of Schubert varieties, the divided difference recurrence has only one initial condition (the class of a point). Further, minimal paths in the weak Bruhat order of  $S_n$  correspond to reduced words of the same permutation. Since divided differences satisfy the braid relations, self-consistency is automatic for Schubert polynomials. As we have observed, neither of these two helpful properties hold for the symmetric pair we consider here.

For some other symmetric pairs (also defined over the complex numbers), such as  $(GL_{2n}, Sp_{2n})$  or  $(GL_n, O_n)$ , the property of having only one initial condition — that is, a unique closed  $K$ -orbit — *does* hold. However, even in such cases, minimal chains in weak order can again correspond to reduced words of *different* permutations, so self-consistency is not a given in these cases either. The two aforementioned additional cases are considered in a sequel [WyYo13].

There is further support for the choice of  $\Upsilon_\gamma$ . We use a geometric perspective originally applied by A. Knutson-E. Miller [KnMi05] to justify Schubert polynomials. For a variety  $X \subset GL_n/B$ , consider the preimage  $\pi^{-1}(X) \subset GL_n$  under the natural projection, and  $\overline{\pi^{-1}(X)} \subset \text{Mat}_{n \times n}$ . Because  $\pi^{-1}(X)$  is a union of left cosets of  $B$ ,  $\overline{\pi^{-1}(X)}$  is stable under right multiplication by  $B$ . Identifying

$$\overline{\pi^{-1}(X)}_B \in H_B^*(\text{Mat}_{n \times n}) \text{ with } \overline{\pi^{-1}(X)}_T \in H_T^*(\text{Mat}_{n \times n}) \cong \mathbb{Z}[x_1, \dots, x_n]$$

(see [KnMi05, Section 1.2]) uniquely picks out a polynomial representative for  $[X] \in H^*(GL_n/B)$ . In the case  $X = X_w := \overline{B_w B/B}$  of Schubert varieties, to actually compute  $\overline{\pi^{-1}(X_w)}_T$  they obtain, by Gröbner degeneration, the multidegree of Fulton's *Schubert determinantal ideal*  $I_w$ , whose generators scheme-theoretically cut out  $\overline{\pi^{-1}(X_w)}$ . Their conclusion is

$$\overline{\pi^{-1}(X_w)}_T = \mathfrak{S}_w(x_1, \dots, x_n),$$

the Schubert polynomial for  $X_w$  [KnMi05, Theorem A].

To study the case  $X = Y_\gamma$ , we define the  $K$ -orbit *determinantal ideal*  $I_\gamma$ , generated by minors of the generic  $n \times n$  matrix and certain auxiliary matrices. When  $\gamma$  is **non-crossing**,

i.e., no two arcs overlap (see the second of the displayed clans on page 2 for a non-example), these generators form a Gröbner basis with squarefree lead terms. The prime decomposition of the Gröbner limit is indexed by monomials of  $\Upsilon_\gamma$ . That is,  $I_\gamma$  scheme-theoretically cuts out  $\overline{\pi^{-1}(Y_\gamma)}$ , and we show

$$[\overline{\pi^{-1}(Y_\gamma)}]_T = \Upsilon_\gamma(x_1, \dots, x_n), \text{ for non-crossing } \gamma.$$

See Theorem 3.2, whose proof uses [KnMi05, KnMiYo09, Wy12, Wy13a]. This provides a geometric rationale for our choice of representatives, at least for the non-crossing case. Furthermore, we conjecture that the above equality holds for all  $\Upsilon_\gamma$ , whether  $\gamma$  has crossings or not (cf. Section 3.2).

The non-crossing condition is special because then  $Y_\gamma$  is a *Richardson variety* [Wy12], or the intersection of a Schubert variety with an opposite Schubert variety. Such varieties are so named because they were first studied by R. W. Richardson in [Ri92]. Properties of Richardson varieties can be transparently deduced from the two Schubert varieties involved [KnWoYo12]. These facts were our starting point for this project.

In [Br01], M. Brion proves (in a general setting, which applies in particular to the case at hand) a formula for  $[Y_\gamma]$  as a sum of Schubert *classes*. In our example, this sum turns out to be multiplicity-free, meaning that all Schubert classes occurring in the sum occur with coefficient 1. Thus taking Brion’s formula and replacing each Schubert *class* with its corresponding Schubert *polynomial* gives a cohomological representative of the type we are seeking. Indeed, our arguments will make it apparent that the representative so obtained is in fact equal to  $\Upsilon_\gamma$ . However, while Brion’s formula applies in both (ordinary) cohomology and  $K$ -theory, it does *not* apply  $T$ -equivariantly in either theory. Thus our representatives in the  $T$ -equivariant setting are truly “new”, in the sense that they cannot be easily be deduced from Brion’s formula.

Finally, we consider a modification of the  $K$ -orbit determinantal ideal which we conjecture provides local equations of  $Y_\gamma$ , cf. Conjecture 4.4. Having such equations allows us to study the singularities of the orbit closures inside  $G/B$ . The *Kazhdan-Lusztig-Vogan polynomials* are one local measure of these singularities. We describe a conjectural analogy with another singularity measure, the  $H$ -*polynomials* of  $Y_\gamma$ , defined in Section 4.3. This analogy parallels that between *Kazhdan-Lusztig polynomials* and  $H$ -polynomials of Schubert varieties described by L. Li and the second author in [LiYo11, Section 2].

**1.3. Organization.** In Section 2, we introduce a family of polynomials in two sets of variables, with a deformation parameter. This family is defined using Schubert polynomials and divided difference operators. With this, we state our choice of polynomial representatives for equivariant cohomology and equivariant  $K$ -theory. We establish our main theorems (Theorems 1.1, 2.10 and 2.13) that they define a self-consistent system. In Section 3, we define the  $K$ -orbit determinantal ideal and establish our Gröbner basis theorem in the non-crossing case as well as formulate the more general conjectures. In Section 4, we use a modification of these ideals in our exploration of the singularities of  $Y_\gamma$ .

## 2. MORE POLYNOMIAL FAMILIES AND COHOMOLOGY THEORIES

**2.1. Definition of  $\Upsilon_\gamma^{(\beta)}$ .** For non-crossing  $\gamma$ , define  $u(\gamma) \in S_n$  by assigning

- $-$ ’s and left endpoints of arcs the labels  $1, 2, \dots, q - 1, q$  from left to right, and

- +’s and right endpoints of arcs the labels  $q + 1, q + 2, \dots, n$  from left to right.

Define  $v(\gamma) \in S_n$  by assigning

- +’s and left endpoints of arcs the labels  $1, 2, \dots, p - 1, p$  from left to right, and
- -’s and right endpoints of arcs the labels  $p + 1, p + 2, \dots, n$  from left to right.

*Example 2.1.* For the second clan  $\gamma \in \text{Clan}_{6,4}$  shown on page 2,  $u(\gamma) = 512637849\ 10$  and  $v(\gamma) = 127389456\ 10$ .

*Example 2.2.* We are especially interested in matchless clans, which we typically denote by  $\tau$ . If  $\tau = ++--+-++$  (as in Section 1) then  $u(\tau) = 45126378 \in S_8$  (in one-line notation) and  $v(\tau) = 12673845$ .  $\square$

The discussion that follows freely uses facts about Schubert varieties, flag varieties and Schubert polynomials. Material on Schubert varieties and flag varieties may be found in Chapters 9 and 10 of [Fu99]. Material about Schubert polynomials appears in Chapter 10.4 of *loc. cit* as well as Chapter 2 of [Ma01].

Let  $X = \{x_1, x_2, \dots, x_n\}$  and  $Y = \{y_1, y_2, \dots, y_n\}$  be independent and commuting indeterminates. The  $\beta$ -**double Schubert polynomial**  $\mathfrak{S}_w^{(\beta)}(X; Y)$  is defined by setting

$$\mathfrak{S}_{w_0}^{(\beta)}(X; Y) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-i} (x_i - y_j + \beta x_i y_j)$$

where  $w_0$  is the long element of  $S_n$ . Define  $\partial_i^{(\beta)}$  by

$$\partial_i^{(\beta)}(f) = \partial_i((1 - \beta x_{i+1})f).$$

Now, if  $i$  is any position such that  $w(i) < w(i + 1)$  then

$$\mathfrak{S}_w^{(\beta)}(X; Y) = \partial_i^{(\beta)} \mathfrak{S}_{ws_i}^{(\beta)}(X; Y)$$

where  $s_i$  is the simple reflection transposing  $i$  and  $i + 1$ . Recall that

$$\mathfrak{S}_w(X; Y) = \mathfrak{S}_w^{(0)}(X, Y)$$

is the **double Schubert polynomial** and

$$\mathfrak{S}_w(X) = \mathfrak{S}_w^{(0)}(X; 0)$$

is the **single Schubert polynomial**. Also,

$$\mathfrak{S}_w(X; Y) = \mathfrak{S}_w^{(1)}\left(x_i \mapsto 1 - x_i; y_j \mapsto \frac{1 - y_j}{y_j}\right)$$

is the **double Grothendieck polynomial**  $\mathfrak{G}_w(X; Y)$  and finally,

$$\mathfrak{G}_w(X) = \mathfrak{G}_w(X; y_j \mapsto 1)$$

is the **single Grothendieck polynomial**. The use of a deformation parameter  $\beta$  in Schubert polynomial theory is found in [FoKi94]. Below we remind the reader in what sense the above substitutions give representatives of the Schubert classes.

When  $\tau$  is matchless, define

$$(3) \quad \Upsilon_\tau^{(\beta)}(X, Y) = \mathfrak{S}_{u(\tau)}^{(\beta)}(X; y_n, y_{n-1}, \dots, y_1) \cdot \mathfrak{S}_{v(\tau)}^{(\beta)}(X; Y).$$

For clans  $\gamma$  which are not matchless,  $\Upsilon_\gamma$  will be defined using divided difference operators according to the weak order on  $K$ -orbits, which we now define. Geometrically, we say that an orbit closure  $Y_\gamma$  covers another orbit closure  $Y_{\gamma'}$ , and write  $\gamma = s_i \cdot \gamma'$ , if

$$Y_\gamma = \pi_i^{-1}(\pi_i(Y_{\gamma'})),$$

where  $\pi_i : G/B \rightarrow G/P_{s_i}$  is the natural projection. Here,  $P_{s_i}$  is the standard minimal parabolic subgroup  $B \cup Bs_iB$  of  $G$ . Note that this definition makes sense not only in our current example, but in any situation where we are dealing with varieties  $Y$  which are closures of orbits of a spherical subgroup acting on  $G/B$ . Indeed, this is the appropriate definition of weak order in all such settings.

In our example, the weak order has the following combinatorial description [Mat79, Ya97]. The **weak Bruhat order** on  $\text{Clans}_{p,q}$  is the transitive closure of the covering relation  $s_i \cdot \gamma \succ \gamma = (c_1, \dots, c_n)$  if either:

- (a)  $s_i \cdot \gamma = (\dots, c_{i+1}, c_i, \dots)$  and
  - $c_i$  is a sign and  $c_{i+1}$  is the end of an arc matching with a vertex to its right;
  - $c_i$  is the end of an arc matching with a vertex to its left and  $c_{i+1}$  is a sign; or
  - $c_i$  and  $c_{i+1}$  are endpoints of different arcs, and the mate of  $c_i$  is left of the mate of  $c_{i+1}$
- (b)  $s_i \cdot \gamma$  is obtained from  $\gamma$  by replacing  $c_i = \pm$  and  $c_{i+1} = \mp$  by an arc.

If  $\gamma$  is not matchless, it follows from [RiSp90, Theorem 4.6] that there is a matchless clan  $\tau$  and a sequence of the form

$$\gamma = s_1 \cdot s_2 \cdots s_l \cdot \tau.$$

(Here,  $l = l(\gamma)$  in the notation of Section 1.) In this event, let

$$\Upsilon_\gamma^{(\beta)}(X; Y) = \partial_1^{(\beta)} \cdots \partial_l^{(\beta)} \Upsilon_\tau^{(\beta)}(X; Y).$$

Just as representatives of Schubert classes are specializations of  $\mathfrak{S}_w^{(\beta)}(X; Y)$ , we will see that the same specializations of  $\Upsilon^{(\beta)}(X; Y)$  give representatives of the classes of  $Y_\gamma$ 's:

$$\begin{aligned} \Upsilon_\gamma(X; Y) &:= \Upsilon_\gamma^{(0)}(X; Y) \\ \Upsilon_\gamma(X) &:= \Upsilon_\gamma(X; 0) \\ \Upsilon_\gamma^K(X; Y) &:= \Upsilon_\gamma^{(1)} \left( x_i \mapsto 1 - x_i; y_j \mapsto \frac{1 - y_j}{y_j} \right) \\ \Upsilon_\gamma^K(X) &:= \Upsilon_\gamma^K(X; y_j \mapsto 1) \end{aligned}$$

**2.2. Some combinatorial properties of  $\Upsilon_\gamma^{(\beta)}$ .** We assume familiarity with standard permutation combinatorics such as the Rothe diagram, essential set, code of a permutation and pattern avoidance; see, e.g., [Ma01, Sections 2.1-2.2].

A permutation is **vexillary** if it is 2143-avoiding.

**Lemma 2.3.** *If  $\gamma$  is non-crossing, then  $u(\gamma)$  and  $v(\gamma)$  are vexillary permutations. In addition  $u(\gamma)$  and  $v(\gamma)$  are inverse to Grassmannian permutations with descents at  $q$  and  $p$  respectively.*

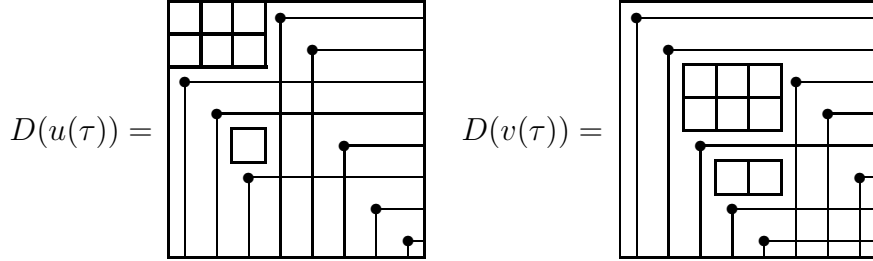
*Proof.* Consider  $u := u(\gamma)$  and suppose  $i_1 < i_2 < i_3 < i_4$  where  $u(i_1), u(i_2), u(i_3), u(i_4)$  are in the relative order 2143. Then since  $1, 2, \dots, q$  and  $q + 1, q + 2, \dots, p + q$  form rising sequences in  $u$ ,  $\gamma(i_1), \gamma(i_3) \in \{q + 1, q + 2, \dots, p + q\}$  and  $\gamma(i_2), \gamma(i_4) \in \{1, 2, \dots, q\}$ . Hence  $\gamma(i_1) > \gamma(i_4)$ , a contradiction. Thus  $u$  is vexillary.



It is straightforward to see that the essential set of  $u$  (provided  $u$  is not the identity) must all lie in column  $q$ . This is equivalent to the inverse Grassmannian claim.

The arguments for  $v(\gamma)$  are similar.  $\square$

*Example 2.4.* Continuing Example 2.2, where  $\tau = ++--+-++$ , the diagrams of  $u(\tau)$  and  $v(\tau)$  are given below. (The  $\bullet$ 's of  $D(\pi)$  are in positions  $(i, \pi(i))$ .)



The essential set boxes of  $u(\tau)$  all lie in column  $q = 3$  while the essential set boxes of  $v(\tau)$  lie in column  $p = 5$ , in agreement with Lemma 2.3.  $\square$

We now define **pipe diagrams** associated to  $u(\gamma)$  for non-crossing  $\gamma$ . (The nomenclature alludes to the “pipe dreams” terminology of [KnMi05].) To start, replace each box of  $D(u(\gamma))$  by a  $+$ . The result is one of the pipe diagrams. All other pipe diagrams are obtained from this first one by iterating the use of the local operation

$$(4) \quad \begin{array}{ccc} \cdot & \cdot & + \\ \cdot & + & \cdot \end{array} \mapsto \begin{array}{ccc} + & & \\ & \cdot & \cdot \\ & & \cdot \end{array}$$

with the additional restriction that no  $+$ 's appear in columns  $q + 1, q + 2, \dots, n$ . The collection of all such pipe diagrams is denoted  $\text{Pipe}(u(\gamma))$ . We define  $\text{Pipe}(v(\gamma))$  in the same way but using  $D(v(\gamma))$  and requiring that there are no  $+$ 's in columns  $p + 1, p + 2, \dots, n$ . In addition, given any configuration  $\mathcal{P}$  of  $+$ 's in the  $n \times n$  grid define

$$\text{wt}^{(\beta)}(\mathcal{P}) = \prod_{+ \text{ in position } (i,j)} x_i - y_j + \beta x_i y_j.$$

We now explain why the initial conditions (3) defining  $\Upsilon_\tau(X)$  agree with the ones from Section 1. Actually, we have an extension. For  $\gamma$  non-crossing, let  $\tau^-$  be the matchless clan obtained by replacing each left end of an arc by  $-$  and any right end of an arc by  $+$ . Also, let  $\tau^+$  be the matchless clan obtained by replacing each left end of an arc by  $+$  and each right end of an arc by  $-$ . Define  $\lambda(\gamma)$  to be  $\lambda(\tau^-)$ , and  $\lambda(\hat{\gamma})$  to be  $\lambda(\tau^+)$ , in the notation of the introduction. Define also flaggings  $\vec{f}(\gamma)$  and  $\vec{f}(\hat{\gamma})$  to be  $\vec{f}(\tau^-)$  and  $\vec{f}(\tau^+)$ , respectively. The following result is straightforward from the results of [KnMiYo09, Section 5] (see specifically Theorem 5.8) and the definitions of  $u(\gamma)$ ,  $v(\gamma)$ ,  $\lambda(\gamma)$ , and  $\lambda(\hat{\gamma})$ :

**Proposition 2.5.** *For non-crossing  $\gamma$  we have*

$$\mathfrak{S}_{u(\gamma)}^{(\beta)}(X; Y) = \sum_{\mathcal{P} \in \text{Pipe}(u(\gamma))} \text{wt}^{(\beta)}(\mathcal{P}) \text{ and } \mathfrak{S}_{v(\gamma)}^{(\beta)}(X; Y) = \sum_{\mathcal{P} \in \text{Pipe}(v(\gamma))} \text{wt}^{(\beta)}(\mathcal{P}).$$

*There is a (weight preserving) bijection between  $\text{Pipe}(u(\gamma))$  and semistandard set-valued Young tableaux of shape  $\lambda(\gamma)$  with flagging  $\vec{f}(\gamma)$ . The same holds for  $\text{Pipe}(v(\gamma))$  and semistandard set-valued Young tableaux of shape  $\lambda(\hat{\gamma})$  with flagging  $\vec{f}(\hat{\gamma})$ . In particular,*

$$\mathfrak{S}_{u(\gamma)}(X) = s_{\lambda(\gamma), \vec{f}(\gamma)}(X) \text{ and } \mathfrak{S}_{v(\gamma)}(X) = s_{\lambda(\hat{\gamma}), \vec{f}(\hat{\gamma})}(X).$$

**Proposition 2.6.** *Suppose  $\gamma$  is non-crossing and*

$$\mathfrak{S}_{u(\gamma)}^{(\beta)}(X; y_n, y_{n-1}, \dots, y_1) \mathfrak{S}_{v(\gamma)}^{(\beta)}(X; Y) = \sum_{\kappa \in \mathbb{Z}_{\geq 0}^{\infty}} c_{\kappa}^{(\beta)}(Y) \mathbf{x}^{\kappa},$$

where  $\mathbf{x}^{\kappa} = x_1^{\kappa_1} x_2^{\kappa_2} \dots$  and  $c_{\kappa}^{(\beta)}(Y) \in \mathbb{Z}[\beta][Y]$ . Then  $c_{\kappa}^{(\beta)}(Y) = 0$  unless  $\kappa \leq (n-1, n-2, \dots, 3, 2, 1, 0, 0, 0, \dots)$  (component-wise comparison).

*Proof.* Let us first show:

**Claim 2.7.** *If  $\mathbf{x}^{\kappa}$  appears in  $\mathfrak{S}_{u(\gamma)}(X) \mathfrak{S}_{v(\gamma)}(X)$  then  $\kappa \subseteq (n-1, n-2, \dots, 2, 1, 0) \in \mathbb{Z}_{\geq 0}^n$ .*

*Proof.* Suppose  $\mathbf{x}^{\kappa} = \dots x_i^m \dots$ . Let  $\omega$  be the width of the first non-empty row of  $D(u(\gamma))$  that occurs in some row  $s \geq i$  of  $n \times n$ . Let  $\omega'$  be the width of the first nonempty row of  $D(v(\gamma))$  that occurs in some row  $t \geq i$  of  $n \times n$ . It is easy to see from the definitions that

$$m \leq \omega + \omega'.$$

We may assume without loss that  $s$  and  $t$  exist and also  $t \geq s$  (the alternate cases are proved similarly).

Let  $A$  be the number of  $-$ 's or left ends of an arc occurring in the leftmost  $s$  positions of  $\gamma$ . Let  $B$  be the number of  $+$ 's or left ends of an arc occurring in the leftmost  $t$  positions of  $\gamma$ . Now

$$\omega = q - A \quad \text{and} \quad \omega' = p - B.$$

Since

$$\omega + \omega' = p + q - A - B$$

it suffices to show  $A + B \geq i$ . Now, because in any left initial segment of  $\gamma$ , the number of right ends of an arc is at most the number of left ends of an arc, we have:

$$\begin{aligned} A + B &\geq A + \#\{+ \text{ or left end of an arc in first } s \text{ positions of } \gamma\} \\ &\geq A + \#\{+ \text{ or right end of an arc in first } s \text{ positions of } \gamma\} \\ &= s \geq i, \end{aligned}$$

as desired. □

Suppose the proposition is not true and there are set-valued tableaux  $T$  and  $U$  that contribute to  $\mathfrak{S}_{u(\gamma)}^{(\beta)}(X; y_n, y_{n-1}, \dots, y_1)$  and  $\mathfrak{S}_{v(\gamma)}^{(\beta)}(X; Y)$  respectively (under the bijection of Proposition 2.5) such that the number of  $i$ 's in  $T$  and  $U$  combined strictly exceeds  $n - i$ , for some  $i$ . Now let  $T'$  be the ordinary tableau that picks each of those  $i$ 's as the representative of its box and picks any entry from the remaining boxes. Since  $T$  is semistandard,  $T'$  is semistandard as well and contributes to  $\mathfrak{S}_{u(\gamma)}(X)$ . Similarly, define  $U'$ , contributing to  $\mathfrak{S}_{v(\gamma)}(X)$ . Then in  $\mathfrak{S}_{u(\gamma)}(X) \mathfrak{S}_{v(\gamma)}(X)$  the monomial  $\mathbf{x}^{T'} \mathbf{x}^{U'}$  appears, contradicting Claim 2.7. □

It is well known (see, e.g., [Ma01, Proposition 2.5.4]) that the single Schubert polynomials  $\{\mathfrak{S}_w(X) : w \in S_n\}$  form a  $\mathbb{Z}$ -linear basis of the vector space  $\Gamma(X)$  of polynomials in  $X$  using only monomials  $\mathbf{x}^{\kappa}$  where  $\kappa \leq (n-1, n-2, \dots, 3, 2, 1)$ . Now,  $\mathfrak{S}_w(X)$  has the same lead term as  $\mathfrak{S}_w(X)$  under the reverse lexicographic order, namely  $\mathbf{x}^{\text{code}(w)}$ . In addition, it is known (from [FoKi94]) that  $\mathfrak{S}_w(X) \in \Gamma(X)$ . Thus  $\{\mathfrak{S}_w(X) : w \in S_n\}$  also forms a basis of  $\Gamma(X)$ . Similarly,  $\{\mathfrak{S}_w^{(\beta)}(X; Y) : w \in S_n\}$  is a  $\mathbb{Z}[\beta][Y]$ -module basis of  $\mathbb{Z}[\beta][Y] \otimes_{\mathbb{Z}} \Gamma(X)$ .

This is since  $\mathfrak{S}_w^{(\beta)}(X; Y)$  also has leading term of  $\mathbf{x}^{\text{code}(w)}$  and if any term  $c_\kappa^{(\beta)}(Y)\mathbf{x}^\kappa$  is any monomial then  $\kappa \leq (n-1, n-2, \dots, 2, 1, 0)$ .

Therefore, by Proposition 2.6, when  $\gamma$  is matchless

$$\Upsilon_\gamma^{(\beta)}(X; Y) = \sum_{w \in S_n} c_{\gamma, w}^{(\beta)}(Y) \mathfrak{S}_w^{(\beta)}(X; Y).$$

Since  $\partial_i^{(\beta)}$  sends  $\beta$ -Schubert polynomials to  $\beta$ -Schubert polynomials (or zero), such an expression where the summation is over  $S_n$  holds for all clans.

Given a clan  $\gamma$  let  $-\gamma$  be the clan where the  $+$ 's of  $\gamma$  are replaced by  $-$ 's and the  $-$ 's are replaced by  $+$  (the arcs remain as is). We record the following property:

**Proposition 2.8** ( $\gamma \leftrightarrow -\gamma$  symmetry). *Let  $\gamma \in \text{Clan}_{p,q}$ . Then*

$$\Upsilon_{-\gamma}^{(\beta)}(X; Y) = \Upsilon_\gamma^{(\beta)}(X; y_n, y_{n-1}, \dots, y_2, y_1).$$

*Proof.* Let  $\tau$  be a matchless clan such that

$$\Upsilon_\gamma^{(\beta)}(X; Y) = \partial_{i_m}^{(\beta)} \dots \partial_{i_1}^{(\beta)} \Upsilon_\tau^{(\beta)}(X; Y),$$

for some chain in weak Bruhat order from  $\tau$  to  $\gamma$  defined by  $i_1, \dots, i_m$ . Now we are done since the same sequence defines a chain from  $-\tau$  to  $-\gamma$  and because the proposition is clear from the definitions for matchless  $\tau$ .  $\square$

In the ordinary cohomology, there is a further sense in which the choice of  $\Upsilon_\gamma$  is simple. Consider the degree lexicographic term order on polynomials in  $\mathbb{Q}[x_1, \dots, x_n]$ . The *Gröbner normal form* is a distinguished representative of any coset modulo  $I^{S_n}$ . The Schubert polynomials  $\mathfrak{S}_w$  for  $w \in S_n$  are the normal forms for their cosets; this is a fact due to [FoGePo97, Section 12.1]. Thus any linear combination of these Schubert polynomials is the normal form for its coset modulo  $I^{S_n}$ . Concluding:

**Proposition 2.9** (Gröbner normal form property).  *$\Upsilon_\gamma(X)$  is the Gröbner normal form representative for the class of  $[Y_\gamma]$  under the degree lexicographic term order. In other words, it is the unique representative that is a linear combination of  $\{\mathfrak{S}_w : w \in S_n\}$ .*

**2.3. Representatives in the Borel models.** We first explain our proof for equivariant cohomology (the argument in equivariant  $K$ -theory is completely analogous). Let  $T \subset GL_p \times GL_q$  be the torus of invertible diagonal matrices. Since each  $Y_\gamma$  is  $T$ -stable, it admits a class  $[Y_\gamma]_T \in H_T^*(GL_n/B)$ , a module over  $H_T^*(pt) \cong \mathbb{Z}[y_1, \dots, y_n]$ . The Borel-type model is

$$(5) \quad H_T^*(GL_n/B) \cong \mathbb{Q}[X; Y]/J,$$

where  $J$  is the ideal generated by  $e_i(X) - e_i(Y)$  and  $e_i(X)$  is the elementary symmetric function in  $X$ , etc.

**Theorem 2.10.**  *$\Upsilon_\gamma(X; Y)$  is well-defined and represents the coset of  $[Y_\gamma]_T$  under (5).*

(The forgetful map from  $H_T^*(GL_n/B) \twoheadrightarrow H^*(GL_n/B)$  in this context amounts to setting each  $y_i = 0$  and sends  $[Y_\gamma]_T$  to  $[Y_\gamma]$ . Thus Theorem 1.1 follows from Theorem 2.10 since the forgetful maps and the Borel isomorphisms commute.)

The following is essentially standard. We include a proof for sake of completeness.

**Proposition 2.11.** *Suppose  $f_1(X; Y)$  and  $f_2(X; Y)$  are representatives of  $[Y_\gamma]_T$  such that*

$$f_1(X; Y) = \sum_{w \in S_n} a_w(Y) \mathfrak{S}_w(X; Y) \quad \text{and} \quad f_2(X; Y) = \sum_{w \in S_n} b_w(Y) \mathfrak{S}_w(X; Y).$$

Then  $f_1(X; Y) = f_2(X; Y)$ .

*Proof.* We need that  $a_w(Y) = b_w(Y)$  for all  $w \in S_n$ .

Since  $f_1$  and  $f_2$  are equivariant cohomology class representatives of  $[Y_\gamma]$  any substitution of  $X$  by a permutation  $Y_\sigma = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$  gives  $f_1(Y_\sigma; Y) = f_2(Y_\sigma; Y)$  (this is where we need that  $\sigma \in S_n$ ). This follows from the localization theorem for equivariant cohomology, combined with the fact that restriction to the  $T$ -fixed point  $\sigma$  is given by

$$[Y_\gamma]_T|_\sigma = f_1(Y_\sigma; Y) = f_2(Y_\sigma; Y).$$

These are standard facts, but the reader seeking a reference may consult [Wy13b, Section 1.2] for an expository treatment. Also,

$$(6) \quad \mathfrak{S}_w(Y_\sigma; Y) = 0 \text{ if } \sigma \not\geq w \text{ in strong Bruhat order.}$$

Now, pick any linear extension

$$\pi^{(1)} = id, \pi^{(2)}, \dots, \pi^{(n!)} = w_0$$

of Bruhat order. Hence

$$a_{\pi^{(1)}}(Y) \mathfrak{S}_{\pi^{(1)}}(Y_{\pi^{(1)}}; Y) = f_1(Y_{\pi^{(1)}}; Y) = f_2(Y_{\pi^{(1)}}; Y) = b_{\pi^{(1)}}(Y) \mathfrak{S}_{\pi^{(1)}}(Y_{\pi^{(1)}}; Y).$$

Since  $\mathfrak{S}_w(Y_w; Y) \neq 0$ , dividing we conclude  $a_{\pi^{(1)}}(Y) = b_{\pi^{(1)}}(Y)$ .

Now set

$$f'_1(X; Y) = f_1(X; Y) - a_{\pi^{(1)}}(Y) \mathfrak{S}_{\pi^{(1)}}(X; Y),$$

and

$$f'_2(X; Y) = f_2(X; Y) - a_{\pi^{(1)}}(Y) \mathfrak{S}_{\pi^{(1)}}(X; Y).$$

Thus

$$a_{\pi^{(2)}}(Y) \mathfrak{S}_{\pi^{(2)}}(Y_{\pi^{(2)}}; Y) = f'_1(Y_{\pi^{(2)}}; Y) = f'_2(Y_{\pi^{(2)}}; Y) = b_{\pi^{(2)}}(Y) \mathfrak{S}_{\pi^{(2)}}(Y_{\pi^{(2)}}; Y),$$

and so  $a_{\pi^{(2)}}(Y) = b_{\pi^{(2)}}(Y)$ .

Repeating, set

$$f''_1(X; Y) = f'_1(X; Y) - a_{\pi^{(2)}}(Y) \mathfrak{S}_{\pi^{(2)}}(X; Y),$$

and

$$f''_2(X; Y) = f'_2(X; Y) - a_{\pi^{(2)}}(Y) \mathfrak{S}_{\pi^{(2)}}(X; Y).$$

In this manner, we conclude all  $n!$  desired equalities. □

We will establish the assumption of the following claim at the end of this section, and in a different way, in the next section.

**Claim 2.12.** *Assuming  $\Upsilon_\tau(X; Y)$  represents  $[Y_\tau]_T$  when  $\tau$  is matchless, then  $\{\Upsilon_\gamma(X; Y)\}$  is self-consistent.*

*Proof.* Pick a (non-matchless) clan

$\gamma$  and suppose there are two matchless clans  $\tau_1$  and  $\tau_2$  (possibly with  $\tau_1 = \tau_2$ ) such that

$$[Y_\gamma]_T = \partial_{i_m} \cdots \partial_{i_1} [Y_{\tau_1}]_T \quad \text{and} \quad [Y_\gamma]_T = \partial_{j_m} \cdots \partial_{j_1} [Y_{\tau_2}]_T;$$

where we have mildly abused  $\partial_i$  to mean the geometrically defined (equivariant) push-pull operator on classes. We need to establish the *polynomial* equality:

$$\partial_{i_m} \cdots \partial_{i_1} \Upsilon_{\tau_1}(X; Y) = \partial_{j_m} \cdots \partial_{j_1} \Upsilon_{\tau_2}(X; Y).$$

Since we know  $\Upsilon_{\tau_1}(X; Y)$  and  $\Upsilon_{\tau_2}(X; Y)$  expand into double Schubert polynomials (from  $S_n$ ), the claim follows from Proposition 2.11.  $\square$

Following [KnMi05, Section 2.3], the  $K$ -cohomology ring  $K^\circ(GL_n/B)$  has the presentation

$$K^\circ(GL_n/B) \cong \mathbb{Z}[x_1, \dots, x_n]/K$$

where  $K$  is the ideal generated by  $e_d(x_1, \dots, x_n) - \binom{n}{d}$  for  $d \leq n$ ; here  $e_d(x_1, \dots, x_n)$  is the elementary symmetric function of degree  $d$ . Next, following [FuLa94] if we let  $K_T^\circ(GL_n/B)$  denote the  $T$ -equivariant  $K$ -theory ring of  $GL_n/B$  then

$$K_T^\circ(GL_n/B) = K_T^\circ(pt)[x_1, \dots, x_n]/J \cong \mathbb{Z}[y_1^{\pm 1}, \dots, y_n^{\pm 1}][x_1, \dots, x_n]/J,$$

where  $J$  is as in (5). In these senses, one can speak of a (Laurent) polynomial “representing” the class of a structure sheaf of a ( $T$ -stable) variety in  $GL_n/B$ .

**Theorem 2.13.** *The families  $\{\Upsilon_\gamma^K(X)\}$  and  $\{\Upsilon_\gamma^K(X; Y)\}$  are well-defined. Moreover,  $\Upsilon_\gamma^K(X)$  represents  $[\mathcal{O}_{Y_\gamma}] \in K^\circ(GL_n/B)$  and  $\Upsilon_\gamma^K(X; Y)$  represents  $[\mathcal{O}_{Y_\gamma}]_T \in K_T^\circ(GL_n/B)$ .*

The proof is exactly the same as in equivariant cohomology, except one must use equivariant  $K$ -theory localization. This requires the now standard fact that, in equivariant  $K$ -theory,  $[\mathcal{O}_{Y_\gamma}]_{T|\sigma} = f(Y_\sigma; Y)$  when  $f(X; Y)$  is a representative of  $[\mathcal{O}_{Y_\gamma}]_T$  in the Borel model. We are unaware of a specific reference for it in the literature, so we remark here that the argument of [Wy13b, Proposition 1.3] can be modified to apply to  $K$ -theory simply by replacing the first Chern classes of the tautological line bundles by (the classes of) the bundles themselves. A recent reference for equivariant localization in  $K$ -theory is [HaLa07]. The analogues of the vanishing conditions on Schubert classes (6) also hold. One also needs the following, which should also be straightforward to experts, but for which we are also not aware of a proof in the literature:

**Proposition 2.14.** *The isobaric divided difference operator  $\pi_i = \partial_i^{(1)}$  takes a representative of the class of  $Y_\gamma$  to one for  $Y_{s_i\gamma}$  in (equivariant)  $K$ -theory of  $GL_n/B$ .*

*Proof.* Let  $Y = Y_\gamma$ , and  $Y' = Y_{s_i\gamma}$ . First, recall that all orbit closures for this case are multiplicity-free, meaning that their cycle classes in the Chow ring can be expressed in the Schubert basis with all coefficients 0 or 1. This is noted in [Br01] and further elaborated upon in [Wy12]. Thus by [Br01, Theorem 6],  $Y$  has rational singularities. Let  $\pi : G/B \rightarrow G/P_{\alpha_i}$  be the natural projection where  $P_{\alpha_i}$  is the minimal parabolic associated to  $\alpha_i$ . Since  $Y' = \pi^{-1}(\pi(Y))$  is a  $\mathbb{P}^1$ -bundle over  $\pi(Y)$ , and since  $Y'$  has rational singularities (being another multiplicity-free  $K$ -orbit closure),  $\pi(Y)$  has rational singularities as well.

Now we note that the proof of [KoKu90, Lemma 4.12] or [FuLa94, Theorem 3], given there for (equivariant)  $K$ -classes of Schubert varieties, applies to the case at hand.  $\square$

*Conclusion of proof of Theorems 1.1, 2.10 and 2.13:* It remains to show that the proposed representatives are indeed representatives for the closed orbits. This follows from three facts. First, by [Wy12], when  $\gamma$  is non-crossing,

$$Y_\gamma = X_{v(\gamma)}^{w_0u(\gamma)} := \overline{B_-v(\gamma)B/B} \cap \overline{Bw_0u(\gamma)B/B}.$$

Second, in the case of equivariant  $K$ -theory, the representative of the Schubert variety  $X_w$  is  $\mathfrak{S}_w(X; Y)$ ; this is proved in [FuLa94, Theorem 3]. It also follows from *loc. cit.* that  $\mathfrak{S}_{w_0w}(X; y_n, y_{n-1}, \dots, y_1)$  represents the opposite Schubert variety  $X^w = \overline{BwB/B}$ . Similarly, it is known that  $\mathfrak{S}_w(X)$ ,  $\mathfrak{S}_w(X; Y)$  and  $\mathfrak{S}_w(X)$  represent the Schubert classes in the corresponding cohomology theories, and  $\mathfrak{S}_{w_0w}(X)$ ,  $\mathfrak{S}_{w_0w}(X; y_n, \dots, y_1)$  and  $\mathfrak{S}_{w_0w}(X)$  represent the opposite Schubert classes. Finally,  $[X_u^v] = [X_u][X^v]$  (interpreted in any of the cohomology theories we are using).  $\square$

*Remark 2.15 (Positivity).* The argument of the introduction that  $\Upsilon_\gamma(X) \in \mathbb{Z}_{\geq 0}[x_1, \dots, x_n]$  extends to prove appropriate notions of “positivity” for each of the given representatives associated to the other three cohomology theories. This is since in each case there is an available notion of positivity of Schubert calculus. See [AnGrMi08] and the references therein.  $\square$

*Remark 2.16.* Consider  $X_{2143}^{3412}$ . It is true that  $\mathfrak{S}_{2143}(x_1, x_2, x_3, x_4)^2 = x_1^4 + \dots$  represents the class of the Richardson variety. However, this polynomial is not the normal form representative of its coset because it involves  $x_1^4$  (cf. Proposition 2.9 and see also [LeSo03]). This emphasizes the role of Proposition 2.6 in our proofs.  $\square$

*Remark 2.17.* Our arguments show that if any collection of varieties in  $GL_n/B$  have their classes related by (isobaric) divided difference operators then any choice of polynomial representatives for their minimal elements that expand into Schubert polynomials from  $S_n$  gives a self-consistent family of representatives. In particular this can also be applied to the cases where  $(G, K) = (GL_{2n}, Sp_{2n})$  and  $(G, K) = (GL_n, O_n)$ ; cf. [Wy13b] and [WyYo13].

### 3. THE $K$ -ORBIT DETERMINANTAL IDEAL

**3.1. Geometric naturality of  $\Upsilon_\gamma^{(\beta)}$ .** The  $K$ -orbit determinantal ideal  $I_\gamma$  is defined as follows. Fix  $\gamma \in \text{Clan}_{p,q}$ . For  $i = 1, \dots, n$ , let:

- $\gamma(i; +)$  = the total number of  $+$ 's and matchings in the first  $i$  vertices, and
- $\gamma(i; -)$  = the total number of  $-$ 's and matchings in the first  $i$  vertices.

For  $1 \leq i < j \leq n$ , define

- $\gamma(i; j) = \#\{k \in [1, i] \mid k \text{ and } l \text{ are matched and } l > j\}$ .

Let  $R_+(\gamma)$  be the vector with  $i$ -th entry equal to  $i + 1 - \gamma(i; +)$  and  $R_-(\gamma)$  be the vector with  $i$ -th entry  $i + 1 - \gamma(i; -)$ . Also, let  $W(\gamma)$  be the  $n \times n$  matrix whose  $(i, j)$ -th entry is  $j + \gamma(i; j) + 1$  if  $i < j$  and is zero otherwise.

Identify  $\text{Fun}(\text{Mat}_{n \times n})$  with  $\mathbb{C}[z_{i,j}]$  where  $z_{i,j}$  is the coordinate function of matrix coordinate  $(i, j)$ . Let  $M_n$  be the generic  $n \times n$  matrix with entry  $z_{i,j}$ . Now define  $I_\gamma$  to have the following generators:

- (i) For each  $i = 1, \dots, n$ , the minors of size  $R_+(\gamma)_i$  of the lower-left  $q \times i$  submatrix of  $M_n$ .

- (ii) For each  $i = 1, \dots, n$ , the minors of size  $R_-(\gamma)_i$  of the upper-left  $p \times i$  submatrix of  $M_n$ .
- (iii) For each  $1 \leq i < j \leq n$ , the minors of size  $W(\gamma)_{ij}$  of the following  $n \times (i + j)$  matrix  $P_{i,j}$ : The upper-left  $p \times i$  block coincides with the upper-left  $p \times i$  block of  $M_n$ , the lower-left  $q \times i$  block is zero, and the last  $j$  columns coincide with the first  $j$  columns of  $M_n$ .

*Example 3.1.* Let  $\gamma = \wedge + -$ . Then

$$\begin{array}{l} \gamma_+ = (0, 1, 2, 2) \\ R_+(\gamma) = (2, \underline{2}, \underline{2}, 3) \end{array}, \quad \begin{array}{l} \gamma_- = (0, 1, 1, 2) \\ R_-(\gamma) = (2, \underline{2}, 3, 3) \end{array}, \quad W(\gamma) = \begin{array}{c|cccc} i/j & 1 & 2 & 3 & 4 \\ \hline 1 & & \underline{3} & \underline{4} & 5 \\ 2 & & & \underline{4} & 5 \\ 3 & & & & 5 \\ 4 & & & & \end{array}$$

Not all rank conditions give rise to non-trivial minors; we have underlined those that do. Specifically  $R_+$  demands that the  $2 \times 2$  minors of the southwest  $2 \times 3$  submatrix of

$$M_4 = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{pmatrix}$$

be among the generators.  $R_-$  contributes the  $2 \times 2$  northwest minor of this matrix. Here,

$$P_{1,2} = \begin{pmatrix} z_{11} & z_{11} & z_{12} \\ z_{21} & z_{21} & z_{22} \\ 0 & z_{31} & z_{32} \\ 0 & z_{41} & z_{42} \end{pmatrix}, \quad P_{1,3} = \begin{pmatrix} z_{11} & z_{11} & z_{12} & z_{13} \\ z_{21} & z_{21} & z_{22} & z_{23} \\ 0 & z_{31} & z_{32} & z_{33} \\ 0 & z_{41} & z_{42} & z_{43} \end{pmatrix}, \quad P_{2,3} = \begin{pmatrix} z_{11} & z_{12} & z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{21} & z_{22} & z_{23} \\ 0 & 0 & z_{31} & z_{32} & z_{33} \\ 0 & 0 & z_{41} & z_{42} & z_{43} \end{pmatrix}.$$

The conditions from  $W(\gamma)$  say that we add the  $3 \times 3$  minors of  $P_{1,2}$  and the  $4 \times 4$  minors of  $P_{1,3}$  and of  $P_{2,3}$ .

Actually, the rank conditions from  $R_+$  and  $R_-$  already imply the minors from  $W(\gamma)$ . This is true for all non-crossing  $\gamma$ , as explained below.  $\square$

Our reference for combinatorial commutative algebra, specifically the notion of multi-grading, multidegree and  $K$ -polynomial is [MiSt04, Chapter 8] as well as the connection to equivariant cohomology. For brevity, we refer the reader to that textbook for basic definitions and notions.

Let  $\prec_{p,q}$  be the lexicographic term order on monomials in  $\{z_{i,j}\}$  that orders the variables by reading the bottom  $q$  rows, from left to right and from bottom to top, followed by the top  $p$  rows from left to right and from top to bottom. The  $T \times T$  action on  $M_n$  restricts to an action on  $M_\gamma = \overline{\pi^{-1}(Y_\gamma)}$ . The associated grading associated to multidegrees assigns the variable  $z_{ij}$  the weight  $x_j - y_i$ . For  $K$ -polynomials, the grading assigns  $z_{ij}$  the weight  $1 - \frac{x_j}{y_i}$ .

The following result explains the geometric naturality of our choices for representatives of the closed orbits. It also applies more generally to orbit closures indexed by non-crossing clans.

**Theorem 3.2.** *Suppose  $\gamma$  is non-crossing. Then  $M_\gamma$  is scheme-theoretically cut out by  $I_\gamma$ . Also:*

- (I) The defining equations of  $I_\gamma$  form a Gröbner basis with squarefree lead terms, with respect to the term order  $\prec_{p,q}$ .
- (II) The Gröbner limit  $\text{init}_{\prec_{p,q}}(I_\gamma)$  has a prime decomposition whose components are naturally indexed by pairs of semistandard tableaux  $(T, U)$  where
- $T$  is a flagged tableaux of shape  $\lambda(\gamma)$  with flagging  $\vec{f}(\lambda(\gamma))$ ; and
  - $U$  is a flagged tableaux of shape  $\mu(\gamma)$  with flagging  $\vec{f}(\mu(\gamma))$ .
- (III)  $\text{multidegree}_{\mathbb{Z}^{2n}}(\mathbb{C}[Z]/I_\gamma) = \Upsilon_\gamma(X; Y)$  and  $\mathcal{K}_{\mathbb{Z}^{2n}}(\mathbb{C}[Z]/I_\gamma) = \Upsilon_\gamma^K(X; Y)$ .

*Example 3.3.* Continuing our previous example, one checks that

$$\begin{aligned} \text{init}_{\prec_{2,2}}(I \bigwedge_{+-}) &= \langle z_{42}z_{33}, z_{41}z_{33}, z_{41}z_{32}, z_{11}z_{22} \rangle \\ &= \langle z_{11}, z_{41}, z_{42} \rangle \cap \langle z_{11}, z_{41}, z_{33} \rangle \cap \langle z_{11}, z_{32}, z_{33} \rangle \cap \langle z_{22}, z_{41}, z_{42} \rangle \cap \\ &\quad \langle z_{22}, z_{41}, z_{33} \rangle \cap \langle z_{22}, z_{32}, z_{33} \rangle. \end{aligned}$$

Now consider the pipe diagrams associated to each prime component of  $\text{init}_{\prec_{2,2}}(I \bigwedge_{+-})$ : we define them to be obtained by placing a “+” in position  $(i, j)$  if  $z_{ij}$  appears in the component. These are respectively:

$$\begin{array}{cccc} \left[ \begin{array}{cccc} + & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot \end{array} \right], & \left[ \begin{array}{cccc} + & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & + & \cdot \\ + & \cdot & \cdot & \cdot \end{array} \right], & \left[ \begin{array}{cccc} + & \cdot & \cdot & \cdot \\ \hline \cdot & + & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right], & \left[ \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & + & \cdot & \cdot \\ + & + & \cdot & \cdot \end{array} \right], \\ & \left[ \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{array} \right], & \left[ \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & + & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right]. \end{array}$$

To compute the  $\mathbb{Z}^{2n}$  multidegree one uses additive grading that assigns  $z_{i,j}$  the weight  $x_j - y_i$ . Then

$$\begin{aligned} \text{multidegree}_{\mathbb{Z}^{2n}}(\mathbb{C}[Z]/I_\gamma) &= (x_1 - y_4)(x_2 - y_4) \cdot (x_1 - y_1) + (x_1 - y_4)(x_3 - y_3) \cdot (x_1 - y_1) \\ &\quad + (x_3 - y_2)(x_3 - y_3) \cdot (x_1 - y_1) + (x_1 - y_4)(x_1 - y_3) \cdot (x_2 - y_2) \\ &\quad + (x_1 - y_4)(x_3 - y_3) \cdot (x_2 - y_2) + (x_2 - y_3)(x_3 - y_3) \cdot (x_2 - y_2) \end{aligned}$$

In each term, we use “.” to separate the factors coming from +’s below and above the horizontal line of the corresponding pipe diagram. Factoring gives

$$\begin{aligned} &[(x_1 - y_4)(x_2 - y_4) + (x_1 - y_4)(x_2 - y_3) + (x_2 - y_3)(x_3 - y_3)] \cdot [(x_1 - y_1) + (x_2 - y_2)] \\ &= s_{(1,1),(2,3)}(x_1, x_2, x_3, x_4; y_4, y_3, y_2, y_1) s_{(1,0),(2,4)}(x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4) \end{aligned}$$

in agreement with the theorem. One can also similarly verify the  $K$ -polynomial claim by computing the  $K$ -polynomial of the simplicial complex associated to  $\text{init}_{\prec_{p,q}} I_\gamma$ .  $\square$

*Proof of Theorem 3.2:* We recall [Wy13a, Theorem 2.5]: Let

$$E_p = \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p\} \text{ and } E^q = \text{span}\{\vec{e}_{p+1}, \vec{e}_{p+2}, \dots, \vec{e}_n\},$$

where  $\vec{e}_i$  is the  $i$ -th standard basis vector of  $\mathbb{C}^n$  and  $\rho : \mathbb{C}^n \rightarrow E_p$  is the natural projection map.

**Theorem 3.4.**  $Y_\gamma$  is the set of flags  $F_\bullet$  such that the following three conditions hold:

- (1)  $\dim(F_i \cap E_p) \geq \gamma(i; +)$  for all  $i$ ;



- (2)  $\dim(F_i \cap E^q) \geq \gamma(i; -)$  for all  $i$ ;  
(3)  $\dim(\rho(F_i) + F_j) \leq j + \gamma(i; j)$  for all  $i < j$ .

Recall that  $\pi : GL_n \rightarrow GL_n/B$  is the natural map. Consider the following diagram:

$$\begin{array}{ccc} \pi^{-1}(Y_\gamma) & \subset & GL_n \subset \text{Mat}_n \supset \overline{\pi^{-1}(Y_\gamma)} := M_\gamma \subseteq V(I_\gamma) \\ & & \downarrow \pi \\ Y_\gamma & \subset & GL_n/B \end{array}$$

**Lemma 3.5.** *Suppose  $g \in GL_n$ . Then  $g \in \pi^{-1}(Y_\gamma)$  if and only if  $g$  vanishes on all generators (i), (ii) and (iii) of  $I_\gamma$ .*

*Proof.* We make the usual identification of  $gB \in GL_n/B$  with the flag

$$F_\bullet : \langle \vec{0} \rangle \subset F_1 \subset F_2 \subset \dots \subset F_{n-1} \subset \mathbb{C}^n,$$

where  $F_i$  is spanned by the leftmost  $i$  columns of  $g$ .

Fix  $F_\bullet \in Y_\gamma$ . By Theorem 3.4, the conditions (1), (2) and (3) of that theorem hold. We examine their implications on  $g$ :

(1) and (2): Consider the map  $\phi : F_i \rightarrow E^q$  obtained by the projection of  $\vec{v} \in F_i \subset \mathbb{C}^n$  onto  $\mathbb{C}^n/E_p \cong E^q$ . Since  $\ker \phi = F_i \cap E_p$ , by the rank-nullity theorem, (1) is equivalent to

$$\text{rank } \phi = \dim F_i - \dim \ker \phi \leq i - \gamma(i, +).$$

Equivalently, the  $g$  associated to  $F_\bullet$  vanishes on the minors (i). Similarly,  $F_\bullet$  satisfies (2) if and only if  $g$  vanishes on the minors (ii).

(3):  $\rho(F_i) + F_j$  is isomorphic to the column space of the  $n \times (i + j)$  matrix whose first  $i$  columns coincide with the first  $i$  columns of  $g$ , but with the lower-left  $q \times i$  submatrix zeroed out, and whose next  $j$  columns coincide with the first  $j$  columns of  $g$  (unaltered). Thus  $g$  vanishes on the generators (iii) if and only if  $F_\bullet$  satisfies (3).  $\square$

Since  $I_\gamma$  vanishes on  $\pi^{-1}(Y_\gamma)$  we must have  $M_\gamma := \overline{\pi^{-1}(Y_\gamma)} \subseteq V(I_\gamma)$ . We would know  $M_\gamma = V(I_\gamma)$  (as sets) if the latter is shown to be irreducible.

Let  $\tilde{I}_\gamma$  be generated by the generators (i) and (ii). We will need to recall the following well-known and easy fact about Gröbner bases, stated in the specific form we need:

**Lemma 3.6.** *Let  $A$  and  $B$  be disjoint collections of commuting variables. Suppose  $f_1, \dots, f_n$  is a Gröbner basis of  $\mathbb{k}[A]$  with respect to a pure lexicographic term order  $\prec_A$ , and that  $g_1, \dots, g_m$  is a Gröbner basis of  $\mathbb{k}[B]$  with respect to a pure lexicographic term order  $\prec_B$ . Let  $\prec_{A;B}$  be the pure lexicographic term order on  $\mathbb{k}[A, B]$  extending  $\prec_A$  and  $\prec_B$  that favors  $A$  over  $B$ . Then  $G = \{f_1, \dots, f_n, g_1, \dots, g_m\}$  is a Gröbner basis with respect to  $\prec_{A,B}$ .*

*Proof.* Indeed, if  $S(f_i, g_j)$  is the  $S$ -polynomial then

$$S(f_i, g_j) := \text{LT}(g_j)f_i - \text{LT}(f_i)g_j = -(g_j - \text{LT}(g_j))f_i + (f_i - \text{LT}(f_i))g_j.$$

Thus, using the multivariate division algorithm, dividing  $S(f_i, g_j)$  by  $G$  (listed in the order  $f_i, g_j, \dots$ ) gives remainder 0. Now apply Buchberger's criterion [Ei95, Section 15.4].  $\square$

**Claim 3.7.**  $\tilde{I}_\gamma$  is a prime ideal that scheme-theoretically cuts out  $\overline{\pi^{-1}(X_{u(\gamma)}^{v(\gamma)})}$ . The generators form a Gröbner basis (with squarefree lead terms) with respect to  $\prec_{p,q}$ .

*Proof.* By definition,  $\tilde{I}_\gamma$  is the ideal sum of a Schubert determinantal ideal associated to  $v(\gamma)$  living in the first  $p$  rows with a Schubert determinantal ideal associated to  $u(\gamma)$  living in the bottom  $q$  rows. The generators for each of these is individually Gröbner (with squarefree lead terms) for the term order given [KnMiYo09, Theorem 3.8]. Now the Gröbner assertion holds by Lemma 3.6.

Since  $\pi^{-1}(X_{u(\gamma)}^{v(\gamma)})$  clearly vanishes on  $\tilde{I}_\gamma$  we have  $\overline{\pi^{-1}(X_{u(\gamma)}^{v(\gamma)})} \subseteq V(\tilde{I}_\gamma)$  (and both zero sets are of the same dimension).

Since its generators are squarefree and Gröbner, by semicontinuity,  $\tilde{I}_\gamma$  is a radical ideal. On the other hand,  $V(\tilde{I}_\gamma)$  is clearly irreducible since it is the Cartesian product of two (irreducible) matrix Schubert varieties. Hence by the Nullstellensatz,  $\tilde{I}_\gamma$  is prime and so  $\overline{\pi^{-1}(X_{u(\gamma)}^{v(\gamma)})} = V(\tilde{I}_\gamma)$  (scheme-theoretic equality).  $\square$

Now we have

$$\overline{\pi^{-1}(X_{u(\gamma)}^{v(\gamma)})} = V(\tilde{I}_\gamma) \supseteq V(I_\gamma) \supseteq M_\gamma.$$

However, by [Wy12] we know  $Y_\gamma = X_{u(\gamma)}^{v(\gamma)}$  so  $M_\gamma = \overline{\pi^{-1}(X_{u(\gamma)}^{v(\gamma)})}$  and hence  $V(\tilde{I}_\gamma) = V(I_\gamma)$ . Furthermore, by the Nullstellensatz,  $I_\gamma \subseteq \tilde{I}_\gamma (= \sqrt{\tilde{I}_\gamma})$ . However, by definition  $I_\gamma \supseteq \tilde{I}_\gamma$  and hence  $I_\gamma = \tilde{I}_\gamma$ . Thus (I) now follows by Claim 3.7 since the additional generators (with squarefree lead terms) that are in  $I_\gamma$  but not  $\tilde{I}_\gamma$  do not affect Gröbnerness of the latter's generators, for general reasons.

Since  $M_\gamma \subseteq V(I_\gamma)$  and now  $I_\gamma = \tilde{I}_\gamma$  is prime, we must have  $M_\gamma = V(I_\gamma)$  and the first sentence of the theorem holds.

In view of the equality  $I_\gamma = \tilde{I}_\gamma$ , (II) is now easy from [KnMiYo09, Section 4] since the latter is the ideal sum of two vexillary Schubert determinantal ideals. (Specifically note that our grading of  $z_{ij}$  is transpose to the convention used in *loc. cit.*)

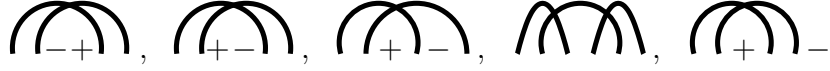
Given (II), (III) follows from Proposition 2.5 and the conclusion of our proof of the main theorems of Section 2. Alternatively, by the same line of reasoning as [KnMi05, Corollary 2.3.1],  $\text{multidegree}_{\mathbb{Z}^{2n}}(\mathbb{C}[Z]/I_\gamma)$  represents  $[Y_\gamma]_T$ . However, *a priori* this representative is not the same as  $\Upsilon_\gamma(X; Y)$ . That these are in fact equal follows from (II), Proposition 2.6 and Proposition 2.11. The authors of *loc. cit.* in fact explain how their argument works in ordinary  $K$ -theory; cf. [KnMi05, Remarks 2.3.3 and 2.3.4].  $\square$

**3.2. Conjectures.** Some of the assertions of Theorem 3.2 seem to hold generally.

**Conjecture 3.8.** *The generators of  $I_\gamma$  are a Gröbner basis with respect to some lexicographic ordering. In particular,  $I_\gamma$  is a radical ideal.*

We emphasize that the term order needed generally depends on  $\gamma$ . Conjecture 3.8 has been verified exhaustively for  $p + q \leq 6$  as well as in enough cases for larger  $p + q$  for us to be convinced.

*Example 3.9.* When  $(p, q) = (1, 2), (2, 1)$ , all  $\gamma$  are non-crossing. When  $(p, q) = (2, 2), (3, 2)$ ,  $\prec_{p,q}$  succeeds in making the defining generators of  $I_\gamma$  Gröbner. This term order also succeeds for  $(p, q) = (2, 3)$  if one add some generators obtained by column operations on the  $P_{i,j}$  matrices. The first interesting examples seem to be at  $(p, q) = (3, 3)$  where



are the instances where the defining generators (or the modification alluded to above) are not Gröbner with respect to  $\prec_{p,q}$ .  $\square$

**Conjecture 3.10.** *Theorem 3.2(III) holds for all  $\gamma$ .*

Equivalently, Conjecture 3.10 claims that  $\text{multidegree}_{\mathbb{Z}^{2n}}(\mathbb{C}[Z]/I_\gamma)$  and  $\mathcal{K}_{\mathbb{Z}^{2n}}(\mathbb{C}[Z]/I_\gamma)$  satisfy the divided difference and isobaric divided difference recurrences. This has been verified by exhaustive computer checks through  $p + q = 7$ .

We are not yet confident to assert that  $I_\gamma$  is prime, although further discussion may appear elsewhere.

These algebraic problems are closely related to two combinatorial questions:

**Question 1.** *Give a manifestly nonnegative combinatorial rule for the expansion of  $\Upsilon_\gamma^{(\beta)}(X; Y)$  into monomials in  $x_i - y_j + \beta x_i y_j$ .*

**Question 2.** *Give a manifestly nonnegative combinatorial rule for the expansion of  $\Upsilon_\gamma^{(\beta)}(X; Y)$  into  $\mathfrak{S}_\gamma^{(\beta)}(X; Y)$ .*

Brion's formula [Br01] states:

$$[Y_\gamma] = \sum_{w \in \mathcal{S}_n} c_{\gamma,w} [X_w] \in H^*(GL_n/B),$$

for explicit, combinatorially defined coefficients  $c_{\gamma,w} \in \{0, 1\}$ . In view of Proposition 2.6, this formula implies a solution to Question 2 when  $\beta = 0$  and each  $y_i = 0$ , by using any monomial expansion formula (e.g., [FoKi96, BeBi92]) for  $\mathfrak{S}_w(X)$ .

A result of A. Knutson [Kn09, Theorem 3] shows how to obtain the  $K$ -theoretic expansion of a multiplicity-free subvariety (such as  $Y_\gamma$ ) given the cohomology expansion. This provides answers to Questions 1 and 2 in ordinary  $K$ -theory.

However, we are not aware of any formula (in ordinary cohomology or  $K$ -theory) that is geometrically natural from the perspective of Gröbner degenerations of  $I_\gamma$ .

Question 2 in the case of  $\Upsilon_\gamma(X; Y)$  for matchless  $\gamma$  is equivalent to certain (yet unsolved) Schubert calculus problems. Once the matchless case is solved, a formula for the general case can be obtained by applying the operators  $\partial_i$ . These expansions involve coefficients in  $\mathbb{Z}_{\geq 0}[y_2 - y_1, \dots, y_n - y_{n-1}]$ .

#### 4. SINGULARITIES OF THE ORBIT CLOSURES

We use a modification of  $I_\gamma$  to compute measures of the singularities of  $p \in Y_\gamma$ .

**4.1. Representative points.** We pick representative points of each  $\mathcal{O}_\gamma$  to work with. Call a permutation  $\sigma$   $\gamma$ -**shuffled** if it is an assignment of

- $1, 2, \dots, p$  (in any order) to the vertices of  $\gamma$  that have a “+” or are the left end of an arc; and
- $p + 1, p + 2, \dots, n$  (in any order) to the remaining positions.

Now let  $F_{\bullet}^{\gamma, \sigma} = \langle \vec{v}_1, \dots, \vec{v}_n \rangle$  be the flag given by

$$\vec{v}_i = \begin{cases} e_{\sigma(i)} & \text{if vertex } i \text{ is a sign or the right end of an arc} \\ e_{\sigma(i)} + e_{\sigma(j)} & \text{if } i \text{ and } j \text{ form an arc and } i < j. \end{cases}$$

We recall the following easy facts for convenience:

**Lemma 4.1.** *Let  $\gamma \in \text{Clan}_{p,q}$  be given.*

- (I)  $F_{\bullet}^{\gamma, \sigma} \in \mathcal{O}_{\gamma}$  for any  $\gamma$ -shuffled  $\sigma$ .
- (II)  $v(\gamma)$  is  $\gamma$ -shuffled.
- (III) If  $\gamma$  is matchless then the  $T$ -fixed points  $\mathcal{O}_{\gamma}^T = \{F_{\bullet}^{\gamma, \sigma} \mid \sigma \text{ is } \gamma\text{-shuffled}\}$ .
- (IV) If  $\gamma$  is not matchless then  $\mathcal{O}_{\gamma}$  contains no  $T$ -fixed points.
- (V) Every point of  $Y_{\gamma}$  is locally isomorphic to some  $F_{\bullet}^{\beta, \sigma}$  for some  $\beta \prec \gamma$  and  $\beta$ -shuffled  $\sigma$ .
- (VI) Let  $\mathcal{P}$  be any upper-semicontinuous property of points of  $Y_{\gamma}$ . Then  $Y_{\gamma}$  globally has property  $\mathcal{P}$  if and only if some  $T$ -fixed point  $F_{\bullet}^{\tau, \sigma}$  has property  $\mathcal{P}$  for every matchless  $\tau \prec \gamma$ .

*Proof.* (I) follows easily from a theorem of T. Matsuki-T. Oshima [MaOs90] and A. Yamamoto [Ya97] that  $\mathcal{O}_{\gamma}$  is precisely the set of flags  $F_{\bullet}$  such that

- $\dim(F_i \cap E_p) = \gamma(i; +)$ ;
- $\dim(F_i \cap E^q) = \gamma(i; -)$ ;
- $\dim(\pi(F_i) + F_j) = j + \gamma(i; j)$ .

(II) is immediate from the definitions. For (III) clearly the “ $\supset$ ” inclusion is obvious. On the other hand, the set of  $\gamma$ -shuffled permutations is clearly a left coset in  $S_p \times S_q \setminus S_n$ , and so has order  $|S_p \times S_q| = p!q!$ . This is precisely the number of  $T$ -fixed points contained in any closed  $K$ -orbit, as each is isomorphic to the flag variety for the group  $K$ . Thus the inclusion is an equality. For (IV), simply note that there are  $\binom{n}{p}$  closed orbits, each containing  $p!q!$   $T$ -fixed points (as just noted), for a total of  $\binom{n}{p} \cdot p!q! = n!$   $T$ -fixed points contained in the closed orbits. This means that no non-closed orbit can contain a  $T$ -fixed point. (Alternatively, (IV) follows directly from [Sp85, Corollary 6.6].) For (V), the elements of  $GL_p \times GL_q$  provide the isomorphisms. Finally, for (VI), the matchless clans are the minimal elements of the closure order.  $\square$

Part (IV) contrasts with Schubert varieties where every point is locally isomorphic to a  $T$ -fixed point. However, in view of (VI) these points of  $Y_{\gamma}$  are still of special interest.

**4.2. The patch ideal.** Given a permutation  $\sigma$ , let  $M_{n, \sigma}$  be the specialization of the generic matrix  $M_n$  obtained by setting  $z_{ij} = 1$  if  $i = \sigma(j)$  and  $z_{ij} = 0$  if  $j > \sigma^{-1}(i)$ . For example, if  $\sigma = 1324$  then (now writing the permutation matrix for  $\pi$  with a 1 in position  $(\pi(i), i)$ ):

$$M_{4,1324} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{2,1} & z_{2,2} & 1 & 0 \\ z_{3,1} & 1 & 0 & 0 \\ z_{4,1} & z_{4,2} & z_{4,3} & 1 \end{pmatrix}.$$

For a clan  $\beta$ , let  $v = v(\beta)$  and let  $L_{\beta}$  be the lower triangular unipotent matrix defined by having 1's in positions  $(v(j), v(i))$  whenever  $i < j$  is matched in  $\beta$ . Now define  $M_{n, \beta} =$

$L_\beta M_{n,v(\beta)}$ . So if for example  $\beta = \wedge+-$  then  $v(\beta) = 1324$ ,

$$L_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{n,\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{2,1} & z_{2,2} & 1 & 0 \\ z_{31} + 1 & 1 & 0 & 0 \\ z_{4,1} & z_{4,2} & z_{4,3} & 1 \end{pmatrix}.$$

Finally, define the **patch ideal**  $I_{\gamma,\beta}$  of  $Y_\gamma$  at  $\beta$  to be generated by the same polynomials as the  $K$ -orbit determinantal ideal except that  $M_n$  is replaced by  $M_{n,\beta}$  in the definition.

*Example 4.2.* Suppose  $\gamma = \bigcap$  and we continue with  $\beta = \wedge+-$ . Then the reader can check that  $I_{\gamma,\beta}$  is generated by the determinant of

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ z_{2,1} & z_{2,1} & z_{2,2} & 1 \\ 0 & z_{31} & 1 & 0 \\ 0 & z_{4,1} & z_{4,2} & z_{4,3} \end{pmatrix}.$$

□

The following is standard; see the discussion of [InYo12]:

**Proposition 4.3.**  $\text{Spec}(\text{Fun}(M_{n,\beta})/I_{\gamma,\beta})$  is set-theoretically equal to a local neighbourhood of  $Y_\gamma$  around the point  $F_\bullet^{\beta,v(\beta)}$ . (The point  $\mathbf{0}$  corresponds to  $F_\bullet^{\beta,v(\beta)}$ .)

*Proof.* Let  $g = L_\beta v(\beta)$ . Then  $gB_-B/B \cap Y_\gamma$  is an affine open neighbourhood of  $Y_\gamma$  around  $gB$ . Coordinates for  $gB_-B/B$  are given by  $M_{n,\beta}$ . In view of Theorem 3.4, any matrix of  $M_{n,\beta}$  representing a flag in  $Y_\gamma$  must vanish on generators of  $I_{\gamma,\beta}$  and conversely, by Lemma 3.5. □

**Conjecture 4.4.**  $I_{\gamma,\beta}$  is a radical ideal.

Conjecture 4.4 has been verified for all patch ideals  $I_{\gamma,\beta}$  with  $\gamma \geq \beta$  through  $p + q = 6$ . Additionally, it has been verified exhaustively for patch ideals  $I_{\gamma,\tau}$  with  $\tau$  a *matchless* clan for  $n = 7$ , as well as for the cases  $(p, q) = (2, 6)$  and  $(3, 5)$ . Numerous other successful checks of  $I_{\gamma,\tau}$  with  $\tau$  matchless in the case  $(p, q) = (4, 4)$  have also been performed.

**4.3.  $H$ -polynomials and Kazhdan-Lusztig-Vogan polynomials.** We propose an analogy between two families of polynomials, one of which are the Kazhdan-Lusztig-Vogan (KLV) polynomials.

Standard references on KLV polynomials are [Vo83, LuVo83]. In their most general form, these polynomials are indexed by pairs  $(Q, \mathcal{L})$  and  $(Q', \mathcal{L}')$ , where  $Q, Q'$  are  $K$ -orbits on  $G/B$ , and  $\mathcal{L}, \mathcal{L}'$  are  $K$ -equivariant local systems on  $Q, Q'$ , respectively. For the associated polynomials to be nonzero, the pairs  $(Q, \mathcal{L})$  and  $(Q', \mathcal{L}')$  must be related in  $\mathcal{G}$ -Bruhat order, defined in [Vo83]. Since all  $K$ -equivariant local systems on all orbits are trivial in the example we are considering, for us the KLV polynomials will be indexed simply by pairs of orbits (or rather, by the corresponding pairs of clans)  $\beta, \gamma$  such that  $\mathcal{O}_\beta \subseteq \overline{\mathcal{O}_\gamma}$ . Furthermore, the coefficient of  $q^i$  in the polynomial  $P_{\beta,\gamma}(q)$  measures the dimension of the  $2i$ -th intersection homology group of  $\overline{\mathcal{O}_\gamma}$  in a neighborhood of a point of  $\mathcal{O}_\beta$ , as follows from [LuVo83, Theorem 1.12]. This mirrors the relationship between Schubert varieties and ordinary Kazhdan-Lusztig polynomials.

Consider the  $\mathbb{Z}$ -graded Hilbert series of  $\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Z}$ , the associated graded ring of the local ring  $\mathcal{O}_{p,Z}$  of a variety  $Z$ . This is denoted by  $\text{Hilb}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Z}, q)$ . The  $H$ -polynomial  $H_{p,Z}(q)$  is defined by

$$\text{Hilb}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Z}, q) = \frac{H_{p,Z}(q)}{(1-q)^{\dim Z}},$$

and  $H_{p,Z}(1)$  is the **Hilbert-Samuel multiplicity**  $\text{mult}_{p,Z}$ .

**Conjecture 4.5.** (i)  $\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Y_\gamma}$  is Cohen-Macaulay.

(ii)  $H_{p,Y_\gamma}(q) \in \mathbb{Z}_{\geq 0}[q]$ .

(iii)  $H_{p,Y_\gamma}(q) \in \mathbb{Z}_{\geq 0}[q]$  is upper-semicontinuous.

In fact (i) implies (ii), by standard facts from commutative algebra. However, (i) and (ii) seem to be logically independent of (iii).

Properties (ii) and (iii) are true for the KLV polynomial  $P_{\beta,\gamma}(q)$ . Property (ii) follows from [LuVo83, Theorem 1.12], while property (iii) holds due to recent work of W.M. McGovern [McGo13]. Thus the above conjecture is our rationale for drawing an analogy between  $H_{\beta,\gamma}(q)$  and  $P_{\beta,\gamma}(q)$ . (Here,  $H_{\beta,\gamma}(q)$  is the  $H$ -polynomial  $H_{p,Y_\gamma}(q)$  where  $p$  is any point of  $\mathcal{O}_\beta \subseteq Y_\gamma$ .) An analogous analogy and conjecture was proposed in the Schubert variety setting in [LiYo11].

**Theorem 4.6.** If  $\gamma$  is non-crossing then  $H_{\beta,\gamma}(q) \in \mathbb{Z}_{\geq 0}[q]$  and  $P_{\beta,\gamma}(q) \leq H_{\beta,\gamma}(q)$  (coefficient-wise inequality).

*Proof.* When  $\gamma$  is non-crossing  $Y_\gamma = X_{v(\gamma)}^{u(\gamma)}$ . The KLV polynomial is the  $IH$ -Poincaré polynomial at a point of  $X_{v(\gamma)}^{u(\gamma)}$ . By [KnWoYo12], this is therefore the product of Kazhdan-Lusztig polynomials for  $X_{v(\gamma)}$  and for  $X^{u(\gamma)}$ . The same is true for the  $H$ -polynomial. However,  $v(\gamma)$  and  $u(\gamma)$  are vexillary. It is a theorem of [LiYo11] that for the Schubert varieties involved, the  $H$ -polynomials have nonnegative coefficients and bound the Kazhdan-Lusztig polynomials. Nonnegativity and this bound on polynomials is preserved by multiplication.  $\square$

*Example 4.7.* The inequality of Theorem 4.6 does not always hold when  $\gamma$  is not non-crossing. For example, if  $\gamma = \overline{(\uparrow)}$  then  $P_{-++++,\gamma}(q) = 1 + q^2$ , as one can verify using ATLAS (<http://www.liegroups.org>). However, we have  $H_{-++++,\gamma}(q) = 1 + q$ .  $\square$

A. Woo and the first author have found an explicit combinatorial rule for  $P_{\beta,\gamma}(q)$  when  $\gamma$  is non-crossing.

The following also seems true:

**Conjecture 4.8.**  $\text{Spec}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Y_\gamma})$  is reduced.

Using the patch equations one can exhaustively check Conjectures 4.5 and 4.8 for all  $(p, q)$  where  $p + q \leq 7$ . We have also done checks for some larger cases.

## APPENDIX

Below we give the polynomials  $\Upsilon_\gamma(X; Y)$  for all  $\gamma \in \text{Clans}_{2,2}$ .

$\gamma$	$\Upsilon_\gamma(X; Y)$
--++	$(x_2 - y_2)(x_2 - y_1)(x_1 - y_2)(x_1 - y_1)$
-+-+	$(x_1 - y_2)(x_1 - y_1)(x_1 - y_4 - y_3 + x_2)(x_2 - y_1 + x_3 - y_2)$
-++-	$(x_1 - y_2)(x_1 - y_1)(-x_1y_3 + y_4y_3 + y_3^2 - x_2y_3 + x_1x_3 - y_4x_3 - x_3y_3 + x_2x_3 + x_2x_1 - y_4x_2 - y_4x_1 + y_4^2)$
+---+	$(x_1 - y_4)(x_1 - y_3)(-x_1y_2 + y_1y_2 + y_2^2 - x_2y_2 + x_1x_3 - x_3y_1 - x_3y_2 + x_2x_3 + x_2x_1 - x_2y_1 - y_1x_1 + y_1^2)$
+--+	$(x_1 - y_4)(x_1 - y_3)(x_1 - y_1 - y_2 + x_2)(x_3 - y_3 - y_4 + x_2)$
+-+-	$(x_2 - y_4)(x_2 - y_3)(x_1 - y_4)(x_1 - y_3)$
- $\frown$ +	$(x_1 - y_2)(x_1 - y_1)(x_2 - y_1 + x_3 - y_2)$
-+ $\frown$	$(x_1 - y_2)(x_1 - y_1)(x_1 - y_4 - y_3 + x_2)$
$\frown$ -+	$(x_1 - y_4 - y_3 + x_2)(-x_1y_2 + y_1y_2 + y_2^2 - x_2y_2 + x_1x_3 - x_3y_1 - x_3y_2 + x_2x_3 + x_2x_1 - x_2y_1 - y_1x_1 + y_1^2)$
$\frown$ +-	$(x_1 - y_1 - y_2 + x_2)(-x_1y_3 + y_4y_3 + y_3^2 - x_2y_3 + x_1x_3 - y_4x_3 - x_3y_3 + x_2x_3 + x_2x_1 - y_4x_2 - y_4x_1 + y_4^2)$
+ $\frown$	$(x_1 - y_4)(x_1 - y_3)(x_1 - y_1 - y_2 + x_2)$
+ $\frown$ -	$(x_1 - y_4)(x_1 - y_3)(x_3 - y_3 - y_4 + x_2)$
- $\frown$ +	$(x_1 - y_2)(x_1 - y_1)$
$\frown$ +	$-x_1y_2 + y_1y_2 + y_2^2 - x_2y_2 + x_1x_3 - x_3y_1 - x_3y_2 + x_2x_3 + x_2x_1 - x_2y_1 - y_1x_1 + y_1^2$
$\frown$ $\frown$	$(x_1 - y_4 - y_3 + x_2)(x_1 - y_1 - y_2 + x_2)$
$\frown$ -	$-x_1y_3 + y_4y_3 + y_3^2 - x_2y_3 + x_1x_3 - y_4x_3 - x_3y_3 + x_2x_3 + x_2x_1 - y_4x_2 - y_4x_1 + y_4^2$
+ $\frown$ -	$(x_1 - y_4)(x_1 - y_3)$
$\frown$ +	$x_1 - y_1 - y_2 + x_2$
$\frown$ -	$x_2 - y_1 - y_3 + 2x_1 - y_4 - y_2 + x_3$
+ $\frown$ -	$x_1 - y_4 - y_3 + x_2$
$\frown$	1

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