This exam has ten problems. Answer as many problems as you can. Each question is worth 5 points (total points is 50). Show all of your work. An answer with an incomplete explanation will no receive full credit. Write your name on the top right corner of each page.

Any definitions or notation that is not defined is assumed knowledge from the course.

No external assistance permitted, including calculators of any kind. [Total time: 3 hours]

Below are some formulas you might find helpful. This list is neither guaranteed to be complete, nor do I guarantee that each formula is actually needed on the exam.

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

\[
\sum_{j \geq 0} x^j = \frac{1}{1-x}
\]

\[
\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}
\]

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, n \in \mathbb{N}\]

\[(1 - x)^{-n} = \sum_{k \geq 0} \binom{n+k-1}{k-1} x^k, n \in \mathbb{N}\]

\[e^x = \sum_{k \geq 0} \frac{x^k}{k!}\]

\[|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i_1} |A_{i_1}| - \sum_{i_1, i_2} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^n \sum_{i_1, \ldots, i_n} |A_{i_1} \cap \cdots \cap A_{i_n}|\]

\[|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = |S| - \sum_{i_1} |A_{i_1}| + \sum_{i_1, i_2} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^{n+1} \sum_{i_1, \ldots, i_n} |A_{i_1} \cap \cdots \cap A_{i_n}|\]
1. Let $F_n$ be the adjusted Fibonacci number, that counts the number of 1,2-lists of size $n$. Prove by induction that $F^2_n = F_{n-1}F_{n+1} + (-1)^n$ for $n \geq 1$.

2. How many ways are there to line up $p$ distinct people to get into $d$ dance clubs in the “Warehouse district” of Chicago? Here the order of the placement of the people in each line matters. Some clubs may have no one lining up for them. (For $p = d = 2$ the answer is 6.)

3. Find the number of integral solutions to
   \[ x_1 + x_2 + x_3 + x_4 = 45 \]
   subject to $4 \leq x_1 \leq 12$, $-5 \leq x_2 \leq 10$, $6 \leq x_3 \leq 18$ and $11 \leq x_4 \leq 17$.
   You may use inclusion-exclusion or generating series methods. (You must at least reduce the answer to an expression involving binomial coefficients.)

4. Using exponential generating series, find the number of ways to put 30 labeled (thus distinct) people into four different rooms $A$, $B$, $C$, $D$ if room $A$ must have an even number of people (possibly 0) and the other rooms must have at least one person. (Note: you must use exponential generating series to do this.)

5. Use the binomial theorem and the relation
   \[(1 + x)^{m_1}(1 + x)^{m_2} = (1 + x)^{m_1+m_2}\]
to prove the following Vandermonde convolution:
   \[ \sum_{k=0}^{n} \binom{m_1}{k} \binom{m_2}{n-k} = \binom{m_1 + m_2}{n}. \]
6. How many permutations $\pi$ of $1, 2, \ldots, 2n$ are there in which only the even integers must be deranged, i.e., $\pi(2k) \neq 2k$ for $k = 0, 1, 2, \ldots$? (Hint: use inclusion-exclusion and mimic how we did derangements without the even condition, in class. This is meant to be easy if you understand that class argument.)

7. Prove the following identity by counting the same set in two different ways.

$$\sum_k \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = \binom{2n+1}{n}.$$ 

8. Using the pigeonhole principle, or otherwise, prove that for any $N$ positive integers $a_1, a_2, \ldots, a_N$, the (nonempty) sum of some of these integers (perhaps one of the numbers itself) is divisible by $N$.

9. Let $\alpha$ and $\beta$ be partitions, as identified with their Ferrers diagrams. Let $\alpha'$ and $\beta'$ be their conjugate partitions. Interpret the RHS combinatorially. Then prove that the LHS counts the same thing in a different way.

$$\sum_{i,j} \min\{\alpha_i, \beta_j\} = \sum_k \alpha'_k \beta'_k.$$ 

10. Give, with proof, a bijection between Dyck paths from $(0, 0)$ to $(2n, 0)$ and the set of standard Young tableaux of shape $(n, n)$, i.e., fillings of the $2 \times n$ chessboard with numbers $1, 2, \ldots, 2n$ such that the rows and columns strictly increase. For example, when $n = 3$ we have five standard Young tableaux, namely:

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1 2 3 1 2 4 1 2 5 1 3 5 1 3 4
4 5 6 3 5 6 3 4 6 2 4 6 2 5 6
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