1. Do the following variation on a worksheet problem. Consider a three-dimensional grid whose dimensions are 25 by 50 by 60. You are at the front lower corner of the grid and wish to get to the back upper right corner 135 “blocks” (edges) away. How many different routes are there in which you walk exactly 135 blocks?

**Solution:** The answer is \( \frac{135!}{25!50!60!} \). This is since this walk can be encoded by a sequence of 25 A-steps, 50 B-steps and 60 C-steps. [This is intended as a “gift” problem, 3 points for the expression and 3 points for some explanation.]

2. Do the following problem from homework. Use combinatorial reasoning to prove the identity (in the given form)

\[
\binom{n}{k} - \binom{n - 3}{k} = \binom{n - 1}{k - 1} + \binom{n - 2}{k - 1} + \binom{n - 3}{k - 1}
\]

**Solution:** See homework 5. [2 points for describing the set counted by the LHS, 4 points for decomposing this set “uses a”, “not a but uses b”, “not a not b but uses c” (this is disjoint).]

3. Prove the Erdős-Szekeres Theorem. That is, show that any sequence of \( n^2 + 1 \) distinct real numbers \( a_1, a_2, \ldots, a_{n^2+1} \) has either an increasing subsequence of length \( n+1 \) or a decreasing subsequence of length \( n+1 \).

**Solution:** See lecture 8. □

4. Prove Ramsey’s theorem. That is, show that the Ramsey number \( R(p, q) \) exists for integers \( p, q \geq 1 \).

**Proof:** This is in lecture 9. The proof is a double induction, with a base case that needs to be checked. The main argument is to write down the inequality \( R(p, q) \leq R(p - 1, q) + R(p, q - 1) \), and then make a carefully stated argument about why \( K_{R(p-1,q)+R(p,q-1)} \) contains a red \( K_p \) or blue \( K_q \). □

5. The UIUC “CS ÷ partying” program has 7 electives


Each student must complete precisely 4 of the electives to obtain their degree. If this program has 4130 many unemployed graduates, show that at least 100 of them must have taken exactly the same electives.
Solution: Note that \( \binom{7}{4} = 35 \). Hence there are 35 ways to obtain the degree. One can set up pigeonholes as these paths to a degree. The pigeons are the 413 students. Use the strong pigeonhole principle with quotas \( q = q_i = 100 \) for each degree path \( i \). Since
\[
4000 > n \times q - n + 1 = 35 \times 100 - 100 + 1
\]
some \( i \) has at least \( q_i = 100 \) pigeons in pigeonhole \( i \), from which the result follows.
\[\square\]