Math 595 “The Grassmannian”: Total positivity

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Outline

I Preliminaries about total positivity

II The totally nonnegative part of the Grassmannian

III Some (more recent) results; problems

Preliminaries about total positivity I

A square matrix A is **totally positive** if all of its minors are positive real numbers. (Similarly we define a **totally nonnegative** matrix.)

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ has nine 1×1 minors

$$\Delta_{\{1\},\{1\}} = |1|, \Delta_{\{1\},\{2\}} = |1|, \ldots,$$

nine 2×2 minors

$$\Delta_{\{1,2\},\{1,2\}} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \Delta_{\{2,3\},\{1,3\}} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3, \ldots$$

and the 3×3 minor $\Delta_{\{1,2,3\},\{1,2,3\}} = 1$. □

Preliminaries about total positivity II

A systematic study of total positivity was initiated in the 1930's by Gantmacher-Krein (they showed existence of n distinct positive eigenvalues).

Relations/applications to:

- oscillations in mechanical systems
- stochastic processes and approximation theory
- Polya frequency sequences
- representation theory of S_∞
- planar resistor networks
- quantum groups
- Somos sequences

Preliminaries about total positivity III

Two basic questions:

Question A: How do you parametrize totally positive matrices?

Question B: How do you efficiently test for total positivity?

- The answers to both questions involve the theme of positivity in combinatorics.

Question A: Parametrizing totally positive matrices

The matrix

$$\begin{bmatrix} d & dh & dhi \\ bd & bdh+e & bdhi+eg+ei \\ abd & abdh+ae+ce & abdhi+(a+c)e(g+i)+f \end{bmatrix}$$

for $a, b, c, d, e \in \mathbb{R}_+$ is totally positive.

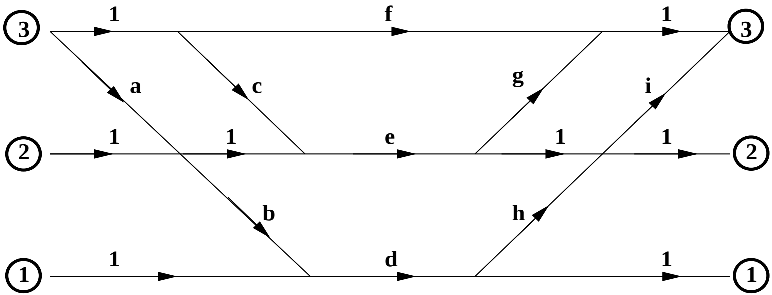
Fact: Every 3×3 totally positive is of this form

Our goals for Question A:

- Give a combinatorial construction of this matrix (and the $n \times n$ generalization).
- Give a combinatorial explanation of the positivity of minors.

Planar networks I

Consider a **planar network** Γ with **edge weighting** ω:



- The **sources** are the extreme left nodes and the **sinks** are the extreme right nodes.
- The **weight** of a directed path in Γ is the product of the weights of its edges.

Planar networks II

- The **weight matrix** $A(\Gamma, \omega)$ is the $n \times n$ matrix with

$$A_{ij} = \text{ sum of weights of paths from source } i \text{ to sink } j.$$

Theorem: $A(\Gamma, \omega)$ is totally positive.

In class exercise: Give a combinatorial interpretation of the determinant.

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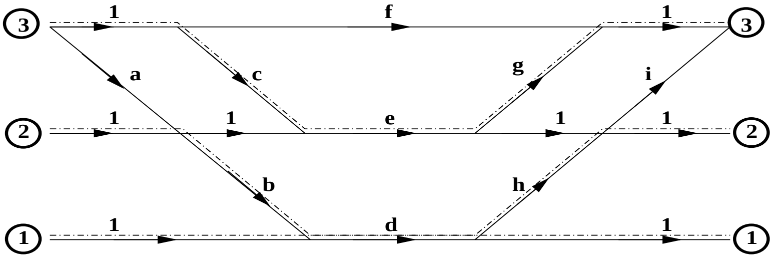
Solution: The determinant is

$$\det(A) = \sum_{w \in S_n} \sum_{\pi} \operatorname{sgn}(w)\omega(\pi),$$

where

- the inner sum is over all **families** of paths $\pi = (\pi_1, \ldots, \pi_n)$ from the sources to the sinks
- $\omega(\pi) = \omega(\pi_1) \cdots \omega(\pi_n)$

Planar network III



- path $1 \to 2$ has weight: dh
- path $2 \to 1$ has weight: bd
- path $3 \to 3$ has weight: ceg
- $w = 213$ and $\mathrm{sgn}(w) = -1$

Hence family contributes $-(dh)(bd)(ceg)$ to $\det(A)$.

Planar network IV

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Solution: We can cancel all negative contributions by a *sign reversing involution*:

- The paths intersect, so we can "path switch" at an intersection to change the sgn
- This gives a positive contribution to $\det(A)$ which cancels with our negative contribution.

Thus our theorem is immediate from:

Lindström's Lemma: $\det(A) = \sum_{\pi} \omega(\pi)$ where $\pi = (\pi_1, \ldots, \pi_n)$ are nonintersecting families of paths (necessarily sending source i to sink i).

Planar network V

In class exercise: Convince yourself that Lindstöm's Lemma is true (say in our example network).

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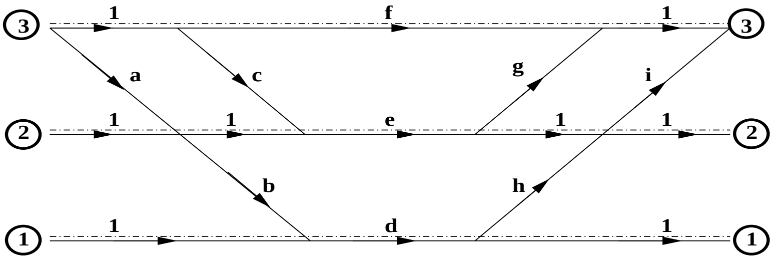
Solution comments: In our example, there's only one nonintersecting path family:



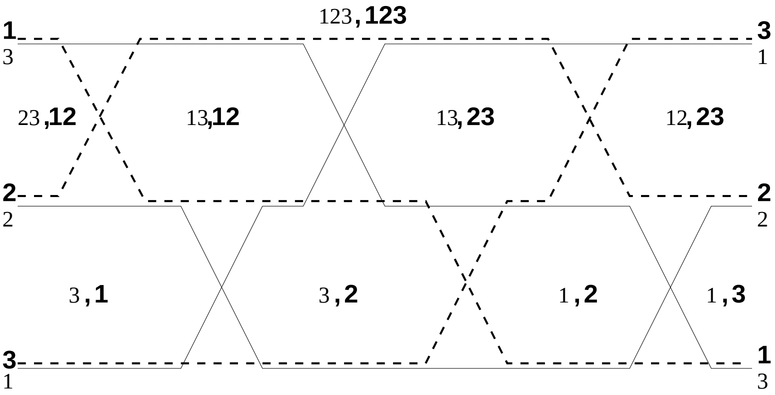
and so $\det(A) = fed (> 0)$.
For smaller minors of A, the same argument works. □

Question B: efficient tests for total positivity

Theorem: (Gasca-Peña) A square matrix is totally positive if and only if its **initial minors** $\Delta_{I,J}$ (where $1 \in I \cup J$ and I and J each form an interval) are positive. Thus one has an n^2 check (and this is tight).

Goal: Set up a combinatorial framework where this test is a special case.

Double wiring diagrams



- Any two wires of the same color cross exactly once
- There are n^2 chambers
- Each chamber indexes a **chamber minor**, determined by which wires of each color are below it

Chamber minors and positivity

Theorem: (Fomin-Zelevinsky) A square matrix A is totally positive if and only if its chamber minors are positive.

The proof is immediate from
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Solution: $\Delta_{2,1} = \frac{\Delta_{3,2}\Delta_{23,12}+\Delta_{3,1}\Delta_{13,23}}{\Delta_{13,12}}$. □

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Main conjecture: Every minor is a Laurent polynomial with positive coefficients, in chamber minors of an arbitrary double wiring diagram.

Open problem: What is a combinatorial rule for this expansion?

The totally nonnegative part of the Grassmannian I

G. Lusztig extended the notion of total positivity (or nonnegativity) to any complex semisimple Lie group G and to any G/P, with further work by K. Rietsch. We will discuss the particular situation of the (real) Grassmannian $Gr_k(\mathbb{R}^n)$, following work of A. Postnikov.

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Definition: The **totally nonnegative Grassmannian** $\mathrm{Gr}_k^{TNN}(\mathbb{R}^n) \subset \mathrm{Gr}_k(\mathbb{R}^n)$ is

$$\mathrm{Gr}_k^{TNN}(\mathbb{R}^n) = GL_k^+ \backslash Mat_{k,n}^{TNN}$$

where $Mat_{k,n}^{TNN}$ is the set of $k \times n$ matrices of rank k whose *maximal* minors have positive determinant. (Similarly define the totally positive part of the Grassmannian.)

In class exercise: Describe $\mathrm{Gr}_k^{TNN}(\mathbb{R}^n)$ in terms of the Plücker coordinates.

The totally nonnegative part of the Grassmannian II

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Solution: Under the Plücker embedding in $\mathbb{P}^{\binom{n}{k}-1}$ it's the points whose Plücker coordinates are all nonnegative. □

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Brief solution: We use the echelon form description of the big cell in $Gr_n(\mathbb{R}^{2n})$, namely $[A|I]$ where I is the $n \times n$ identity matrix and A is arbitrary. However, if we impose that the corresponding point is in the TNN part, each $n \times n$ minor is nonnegative. Now take any TNN matrix B and alternate the signs of the rows to give A. □

Decomposing $\mathrm{Gr}_k^{TNN}(\mathbb{R}^n)$ I

We have two decompositions of the Grassmannian (Schubert, matroid). Either one imposes a decomposition of $\mathrm{Gr}_k^{TNN}(\mathbb{R}^n)$. Postnikov's work focuses on the matroid one.

Definition: The **TNN Grassmann cells** are $X_M^{\mathrm{TNN}} = X_M \cap \mathrm{Gr}_k^{TNN}(\mathbb{R}^n)$.

Definition: The above definition allows one to define a **TNN matroid** M to be one where $X_M^{\mathrm{TNN}} \neq \emptyset$.

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In class exercise: Determine all TNN matroids for $k = 2$ and $n = 4$.

Solution: The not TNN matroids are $M = \{12, 23, 34, 13\}$, $M \cup 13$ and $M \cup 24$. This is closed under cyclic shifts of $[4]$. Why? □

Decomposing $\mathrm{Gr}_k^{TNN}(\mathbb{R}^n)$ II

Interesting behavior: The X_M^{TNN} are balls (as opposed to X_M themselves).

Theorem: [Postnikov] Each X_M^{TNN} is homeomorphic to an open ball. The decomposition of $\mathrm{Gr}_k^{TNN}(\mathbb{R}^n)$ is a CW complex.

Note: it's conjectured that this CW complex is even nicer (the closure of each cell is homeomorphic to a ball).

Theorem: [Postnikov] X_M^{TNN} is the intersection of only n permuted (TNN) Schubert cells associated to $c, c^2, c^3, \ldots, c^n$ where $c = (123 \cdots n)$ is the long cycle.

Note: One can consider the same intersection in the *complex* Grassmannian. This is (now) known as the Positroid. Unlike the matroid strata, they are relatively mild in singularity structure (normal, Cohen-Macaulay). See work of [Knutson-Lam-Speyer].

Some things to understand

- How to parameterize the cells (and see the gluing of cells)?
- How to index the cells (nicely)?

Answers to these questions lead to interesting combinatorial objects.
[To be continued.]

References

A. Postnikov, *Total positivity, Grassmannians, and networks*, preprint. arxiv:math/0609764

S. Fomin and A. Zelevinsky, *Total positivity: tests and parametrizations*, The Mathematical Intelligencer **22**(2000), 23–33.