Math 595 “The Grassmannian”: Total positivity

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Outline

I Preliminaries about total positivity
II The totally nonnegative part of the Grassmannian
III Some (more recent) results; problems
A square matrix $A$ is **totally positive** if all of its minors are positive real numbers. (Similarly we define a **totally nonnegative** matrix.)

**Example:** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ has nine $1 \times 1$ minors

\[ \Delta_{\{1\},\{1\}} = |1|, \Delta_{\{1\},\{2\}} = |1|, \ldots, \]

nine $2 \times 2$ minors

\[ \Delta_{\{1,2\},\{1,2\}} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \Delta_{\{2,3\},\{1,3\}} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3, \ldots \]

and the $3 \times 3$ minor $\Delta_{\{1,2,3\},\{1,2,3\}} = 1$. \[\square\]
A systematic study of total positivity was initiated in the 1930’s by Gantmacher-Krein (they showed existence of $n$ distinct positive eigenvalues).

Relations/applications to:

- oscillations in mechanical systems
- stochastic processes and approximation theory
- Polya frequency sequences
- representation theory of $S_\infty$
- planar resistor networks
- quantum groups
- Somos sequences
Two basic questions:

**Question A:** How do you parametrize totally positive matrices?

**Question B:** How do you efficiently test for total positivity?

- The answers to both questions involve the theme of positivity in combinatorics.
Question A: Parametrizing totally positive matrices

The matrix

\[
\begin{bmatrix}
    d & dh & dhi \\
    bd & bdh + e & bdhi + eg + ei \\
    abd & abdh + ae + ce & abdhi + (a + c)e(g + i) + f \\
\end{bmatrix}
\]

for \(a, b, c, d, e \in \mathbb{R}_+\) is totally positive.

Fact: Every \(3 \times 3\) totally positive is of this form

Our goals for Question A:

- Give a combinatorial construction of this matrix (and the \(n \times n\) generalization).
- Give a combinatorial explanation of the positivity of minors.
Consider a **planar network** $\Gamma$ with **edge weighting** $\omega$:

- The **sources** are the extreme left nodes and the **sinks** are the extreme right nodes.
- The **weight** of a directed path in $\Gamma$ is the product of the weights of its edges.
The **weight matrix** $A(\Gamma, \omega)$ is the $n \times n$ matrix with

\[ A_{ij} = \text{sum of weights of paths from source } i \text{ to sink } j. \]

**Theorem:** $A(\Gamma, \omega)$ is totally positive.

**In class exercise:** Give a combinatorial interpretation of the determinant.
The weight matrix $A(\Gamma, \omega)$ is the $n \times n$ matrix with

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**Theorem:** $A(\Gamma, \omega)$ is totally positive.

**In class exercise:** Give a combinatorial interpretation of the determinant.

**Solution:** The determinant is

$$\det(A) = \sum_{w \in S_n} \sum_{\pi} \text{sgn}(w) \omega(\pi),$$

where

- the inner sum is over all families of paths $\pi = (\pi_1, \ldots, \pi_n)$ from the sources to the sinks
- $\omega(\pi) = \omega(\pi_1) \cdots \omega(\pi_n)$
path 1 → 2 has weight: $dh$
path 2 → 1 has weight: $bd$
path 3 → 3 has weight: $ceg$

$w = 213$ and $\text{sgn}(w) = -1$

Hence family contributes $-(dh)(bd)(ceg)$ to $\det(A)$. 
The signed expression for $\det(A)$ doesn’t manifest the claimed positivity!

**In class exercise:** How do we prove positivity of the determinant nonetheless?
Planar network IV

The signed expression for $\det(A)$ doesn’t manifest the claimed positivity!

**In class exercise:** How do we prove positivity of the determinant nonetheless?

**Solution:** We can cancel all negative contributions by a *sign reversing involution*:

- The paths intersect, so we can “path switch” at an intersection to change the $\text{sgn}$
- This gives a positive contribution to $\det(A)$ which cancels with our negative contribution.

Thus our theorem is immediate from:

**Lindström’s Lemma:** $\det(A) = \sum_\pi \omega(\pi)$ where $\pi = (\pi_1, \ldots, \pi_n)$ are nonintersecting families of paths (necessarily sending source $i$ to sink $i$).
In class exercise: Convince yourself that Lindstöm’s Lemma is true (say in our example network).
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**Solution comments:** In our example, there’s only one nonintersecting path family:

![Diagram of a planar network](image)

and so $\det(A) = fed(> 0)$.

For smaller minors of $A$, the same argument works.
**Theorem:** (Gasca-Peña) A square matrix is totally positive if and only if its **initial minors** $\Delta_{I,J}$ (where $1 \in I \cup J$ and $I$ and $J$ each form an interval) are positive. Thus one has an $n^2$ check (and this is tight).

**Goal:** Set up a combinatorial framework where this test is a special case.
Any two wires of the same color cross exactly once

There are $n^2$ chambers

Each chamber indexes a chamber minor, determined by which wires of each color are below it
**Theorem:** (Fomin-Zelevinsky) A square matrix $A$ is totally positive if and only if its chamber minors are positive.

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Solution: $\Delta_{2,1} = \frac{\Delta_{3,2}\Delta_{23,12} + \Delta_{3,1}\Delta_{13,23}}{\Delta_{13,12}}$. 

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Main conjecture: Every minor is a Laurent polynomial with positive coefficients, in chamber minors of an arbitrary double wiring diagram.

Open problem: What is a combinatorial rule for this expansion?
G. Lusztig extended the notion of total positivity (or nonnegativity) to any complex semisimple Lie group $G$ and to any $G/P$, with further work by K. Rietsch. We will discuss the particular situation of the (real) Grassmannian $Gr_k(\mathbb{R}^n)$, following work of A. Postnikov.

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The totally nonnegative part of the Grassmannian I

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**Definition:** The **totally nonnegative Grassmannian** $\text{Gr}^{TNN}_k(\mathbb{R}^n) \subset \text{Gr}_k(\mathbb{R}^n)$ is

$$\text{Gr}^{TNN}_k(\mathbb{R}^n) = GL^+_k \setminus \text{Mat}^{TNN}_{k,n}$$

where $\text{Mat}^{TNN}_{k,n}$ is the set of $k \times n$ matrices of rank $k$ whose *maximal* minors have positive determinant. (Similarly define the totally positive part of the Grassmannian.)
In class exercise: Describe $\Gr^T_{kN}(\mathbb{R}^n)$ in terms of the Plücker coordinates.
In class exercise: Describe $\text{Gr}^\text{TNN}_k(\mathbb{R}^n)$ in terms of the Plücker coordinates.

Solution: Under the Plücker embedding in $\mathbb{P}^{n\choose k} - 1$ it’s the points whose Plücker coordinates are all nonnegative.
In class exercise: Describe $\text{Gr}^T_{k}(\mathbb{R}^n)$ in terms of the Plücker coordinates.

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In class exercise: Why is the space of totally nonnegative matrices a (big cell) piece of a totally nonnegative Grassmannian?
The totally nonnegative part of the Grassmannian II

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In class exercise: Why is the space of totally nonnegative matrices a (big cell) piece of a totally nonnegative Grassmannian?

Brief solution: We use the echelon form description of the big cell in $Gr_{n}(\mathbb{R}^{2n})$, namely $[A|I]$ where $I$ is the $n \times n$ identity matrix and $A$ is arbitrary. However, if we impose that the corresponding point is in the TNN part, each $n \times n$ minor is nonnegative. Now take any TNN matrix $B$ and alternate the signs of the rows to give $A$. 

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We have two decompositions of the Grassmannian (Schubert, matroid). Either one imposes a decomposition of $\text{Gr}_k^{TNN}(\mathbb{R}^n)$. Postnikov’s work focuses on the matroid one.

**Definition:** The **TNN Grassmann cells** are $X_M^{TNN} = X_M \cap \text{Gr}_k^{TNN}(\mathbb{R}^n)$.

**Definition:** The above definition allows one to define a **TNN matroid** $M$ to be one where $X_M^{TNN} \neq \emptyset$. 
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**In class exercise:** Determine all TNN matroids for $k = 2$ and $n = 4$. 
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**Definition:** The above definition allows one to define a **TNN matroid** $M$ to be one where $X_M^{\text{TNN}} \neq \emptyset$.

**In class exercise:** Determine all TNN matroids for $k = 2$ and $n = 4$.

**Solution:** The not TNN matroids are $M = \{12, 23, 34, 13\}$, $M \cup 13$ and $M \cup 24$. This is closed under cyclic shifts of $[4]$. Why?
Interesting behavior: The $X^\text{TNN}_M$ are balls (as opposed to $X_M$ themselves).

**Theorem:** [Postnikov] Each $X^\text{TNN}_M$ is homeomorphic to an open ball. The decomposition of $\text{Gr}^\text{TNN}_k(\mathbb{R}^n)$ is a CW complex.

Note: it’s conjectured that this CW complex is even nicer (the closure of each cell is homeomorphic to a ball).

**Theorem:** [Postnikov] $X^\text{TNN}_M$ is the intersection of only $n$ permuted (TNN) Schubert cells associated to $c, c^2, c^3, \ldots, c^n$ where $c = (123 \cdots n)$ is the long cycle.

Note: One can consider the same intersection in the complex Grassmannian. This is (now) known as the Positroid. Unlike the matroid strata, they are relatively mild in singularity structure (normal, Cohen-Macaulay). See work of [Knutson-Lam-Speyer].
Some things to understand

- How to parameterize the cells (and see the gluing of cells)?
- How to index the cells (nicely)?

Answers to these questions lead to interesting combinatorial objects.
[To be continued.]