MATH 286 — FALL 2008 TEST III
DIFFERENTIAL EQUATIONS PLUS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

NAME: ____

SIGNATURE: ____________________

ID NUMBER: __________________________

(1) Do not open this test until you are told to begin.
(2) This exam has 8 pages including this cover and four intentionally blank pages for your use. There are 3 problems total. You have 50 minutes.
(3) No notes or books are permitted.
(4) No calculators are permitted.
(5) Please turn off all cell phones.
(6) Place your ID card on your desk for inspection.
(7) Please stay for the entirety of the test period, so as not to disturb others.
(8) Explain all your solutions as clearly as possible, citing results from class if needed.
(9) Good luck!

PROBLEM	POINTS	SCORE
1	10	
2	10	
3	20	
TOTAL	40	

1. Consider an $n \times n$ matrix which consists of 1's on the diagonal, 1's right below the diagonal and 0's everywhere else. For example, if $n = 2, 3, 4, 5$ we get

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) For $n = 2, 3$ show that the characteristic polynomial is $(1 - \lambda)^n$. [2 marks]

(b) For $n = 2, 3$ show that the defect of the eigenvalue $\lambda = 1$ is $n - 1$, i.e., there is only one eigenvector up to scalar multiples. [4 marks]

(c) Now prove (a) and (b) for all $n \geq 2$. [4 marks]

Solution: For (a) we compute the characteristic polynomials

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

In fact, the determinant of any lower triangular matrix is just the product of the diagonal entries. Otherwise we can just expand along the first row.

Notice that for the $n = 3$ calculation, we have

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \times \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix}$$

shows that the computation is recursive.

For (b) we need to solve the system

$$(\mathbf{A} - \mathbf{I})\mathbf{v} = \mathbf{0}$$

where $\mathbf{A}$ is the given coefficient matrix. In these cases

$$\mathbf{A} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}.$$

An easy computation with Gaussian elimination shows that the solutions are

$$(0, s)^T \quad \text{and} \quad (0, 0, s)^T$$

respectively, where $s \in \mathbb{R}$.

Hence there is only one eigenvector (up to scalar multiples) in each case.

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(c) For the generalization of (a), just induct on $n \geq 2$, observing that the cofactor expansion along the first row reduces the determinant to the $n-1$ case (times $1-\lambda$).

For the generalization of (b), we have to solve the system

$$(\mathbf{A} - \mathbf{I})\mathbf{v} = \mathbf{0}$$

where in general $\mathbf{A}$ is a matrix with 1's along the first off diagonal; again by Gaussian elimination, it follows that the general solution is $(0, 0, \ldots, s)$, for $s \in \mathbb{R}$, and so one eigenvector, up to scalar multiples. $\square$

[Grading: for (a), 1 point each for each correct computation. For (b), 2 points for each Gaussian elimination done to conclude the result. For (c), give two marks for each argument. The argument for eigenvalues should include an induction (if only implicitly), expanding along the first row. The argument about eigenvectors should just follow the same style as (b).]

2. (a) State the convergence theorem for Fourier series. [3 marks]

(b) Prove that the Fourier series of an even function $f(t)$ (of period $P = 2L$) only involves cosine terms. [7 marks]

Solution:

(a) If a periodic function f is piecewise smooth. Then it's Fourier series converges

- to $f(t)$ at each point t where f is continuous; and
- to $\frac{1}{2}[f(t+) + f(t-)]$ otherwise.

[Grading for (a): one point for the assumption and one point for each of the conclusions.]

(b) The assumption that a function f is even is the statement that

$$f(-t) = f(t)$$

for each $t \in \mathbb{R}$.

The problem is asking us to check that the sine Fourier coefficient

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = 0.$$

Now we have

$$\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = \frac{1}{L}\int_{-L}^{0} f(t)\sin\frac{n\pi t}{L}\,dt + \frac{1}{L}\int_{0}^{L} f(t)\sin\frac{n\pi t}{L}\,dt.$$

By a change of variables, it follows that

$$\frac{1}{L}\int_{-L}^{0} f(t)\sin\frac{n\pi t}{L}\,dt = \frac{1}{L}\int_{0}^{L} f(-t)\sin\frac{n\pi(-t)}{L}\,dt.$$

Finally since $f(-t) = f(t)$ and $\sin\frac{n\pi(-t)}{L} = -\sin\frac{n\pi t}{L}$ we conclude

$$\frac{1}{L}\int_{-L}^{0} f(t)\sin\frac{n\pi t}{L}\,dt = -\frac{1}{L}\int_{0}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = -\frac{1}{L}.$$

Thus substituting this into the sum above gives the result. □

[Grading for (b): 1 points for stating knowing what it means for f to be even; 1 point for writing down the correct Fourier coefficient to check is equal to 0; 2 points for splitting the integral; 2 points for the change of variables; 1 point for showing the two summands sum to 0.]

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3. By the method of separation of variables, find a formal solution to the following boundary problem for a one-dimensional heat equation with zero endpoint temperatures (*be sure to justify each step of the procedure to ensure full credit*):

$$
\begin{aligned}
&(1) \qquad & u_t &= u_{xx} \\
&(2) \qquad & u(0,t) &= u(L,t) = 0 \\
&(3) \qquad & u(x,0) &= x \quad \text{for} \quad 0 < x < L
\end{aligned}
$$

[20 marks]

Hint: You will eventually want to know that

$$\int u \sin u du = -u \cos u + \sin u + C.$$

Solution: Assume that the solution separates as

$$u(x,t) = X(x)T(t).$$

[1 mark] Substituting this assumption into the partial differential equation (1) gives

$$XT' = X''T$$

[1 marks] Dividing gives

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

for some constant λ (this follows from the fact that X''/X depends only on x and T'/T only depends on t). [2 marks – must provide an argument]

From (2) we have

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0.$$

Without loss of generality, we may assume $T(t) \not\equiv 0$ and so

$$X(0) = X(L) = 0.$$

This gives us the endpoint problem

$$X'' + \lambda X = 0; \quad X(0) = X(L) = 0.$$

[2 points for the endpoint problem]

The eigenvalues and eigenfunctions of this endpoint problem are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin\frac{n\pi x}{L}.$$

[2 points for the eigenvalues, 2 points for the eigenfunctions– in each 1 is attributed to explanation]

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A brief justification of this can be given as follows: we have to consider the cases that $\lambda = 0, +\alpha^2, -\alpha^2$, corresponding to the possibility of zero, positive and negative eigenvalues. Checking this, one concludes that only positive eigenvalues are possible. In fact the characteristic equation

$$r^2 + \alpha^2 r = 0$$

leads to the general solution

$$X = A \cos \alpha x + B \sin \alpha x$$

from $X(0) = 0$ implies $A = 0$ and $X(L) = 0$ implies $\alpha = n\pi x/L$.

Using the above eigenvalues we find that $T = T_n$ satisfies the first order differential equation:

$$T' - \frac{n^2\pi^2 k}{L} T = 0$$

and so

$$T_n(t) = \exp(-n^2\pi^2 kt/L^2),$$

for $n = 1, 2, 3, \ldots$.. [2 points]

Hence, let

$$u_n(x,t) = X_n T_n = \sin \frac{n\pi x}{L} \exp(-n^2\pi^2 kt/L).$$

[1 mark]

Thus we build the formal solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t),$$

where the coefficients c_n are to be determined. [1 mark]

To do this, we use the condition $u(x,0) = f(x) := x$.

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} = f(x)$$

[2 points]

By taking the odd extension of $f(x)$ and thus looking at the Fourier sine expansion of $f(x)$ we see

$$c_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} dx.$$

Thus it remains to compute this integral.

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When $n = 0$ we get

$$c_0 = \frac{2}{L}\int_0^L x dx = L.$$

when $n > 0$ we get

$$c_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L} dx = \frac{2L}{n^2\pi^2}\int_0^{n\pi} u \sin u du$$

where the last equality was using a change of variables $u = n\pi x/L$.

Using the giving integration formula, we get

$$= \frac{2L}{n^2\pi^2}[-u\cos u + \sin u]_0^{n\pi} = \frac{2L}{n\pi}(-1)^{n+1}.$$

[4 points for a complete argument computing the coefficient]

Substituting these choices for c_n gives our formal solution. □