MATH 286 — FALL 2008 TEST II
DIFFERENTIAL EQUATIONS PLUS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

NAME: ____

SIGNATURE: ____________________

ID NUMBER: __________________________

(1) Do not open this test until you are told to begin.
(2) This exam has 7 pages including this cover and two intentionally blank pages for your use. There are 4 problems total. You have 50 minutes.
(3) No notes or books are permitted.
(4) No calculators are permitted.
(5) Please turn off all cell phones.
(6) Place your ID card on your desk for inspection.
(7) Please stay for the entirety of the test period, so as not to disturb others.
(8) Explain all your solutions as clearly as possible, citing results from class if needed.
(9) Good luck!

PROBLEM	POINTS	SCORE
1	12	
2	12	
3	8	
4	8	
TOTAL	40	

1. (a) Find the real solutions of the homogeneous differential equation

$$y^{(3)} - y^{(2)} + 4y^{(1)} - 4y = 0.$$

[8 marks]

(b) Consider the inhomogeneous linear differential equation

$$y^{(3)} - y^{(2)} + 4y^{(1)} - 4y = xe^{3x} + \cos\ 2x.$$

Find the general form of a particular solution y_p of this differential equation. However, you do NOT have to solve for the undetermined coefficients. [4 marks]

Solution: (a) The characteristic equation is

$$r^3 - r^2 + 4r - 4 = r^2(r-1) + 4(r-1) = (r-1)(r^2+4).$$

Hence the roots are $r = 1, 2i, -2i$. We need three linearly independent functions. The general solution associated to $r = 1$ is plainly $y_1 = Ae^x$, $A \in \mathbb{R}$.

As discussed in class, we only have to study what happens with one of the complex roots in the conjugate pair $\{2i, -2i\}$. Let's pick $r = 2i$. Applying Euler's formula, we get a solution of the form

$$y_2 = be^{2i} = b(\cos(2x) + i\sin(2x)).$$

However, as in class the real form of this solution is

$$y_2 = B\cos(2x) + C\sin(2x)$$

for any $B, C \in \mathbb{R}$. By superposition, all real solutions are of the form

$$y_c = y_1 + y_2 = Ae^x + B\cos(2x) + C\sin(2x), \quad A, B, C \in \mathbb{R}.$$

(b) Let $f = xe^{3x} + \cos(2x)$. The linearly independent terms of f and all of it's derivatives are (up to a scalar multiple):

$$xe^{3x}, e^{3x}, \cos(2x), \sin(2x).$$

Of these, only $\cos(2x)$ and $\sin(2x)$ appear in y_c above. Hence, we can avoid duplication by multiplying these by x. Therefore the guess for y_p is

$$y_p = Dxe^{3x} + Ee^{3x} + Fx\cos(2x) + Gx\sin(2x)$$

for undetermined coefficients D, E, F, G. □

Grading: (a) 3 points for writing down and factoring the characteristic eqation correctly, 1 point for the part of the solution associated to $r = 1$, 2 points for applying Euler to the case of $r = 2i$ (or $r = -2i$), 2 points for obtaining the real form. (b) 2 points for the linearly independent terms from all the derivatives, 2 points for the correct answer.

2. Use the eigenvalue method to find the general solution to the following system of differential equations:

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{x}$$

where $\mathbf{x} = [x_1(t)\ x_2(t)\ x_3(t)]^T$. (Hint: all the eigenvalues are integers.) [12 marks]

Solution: The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

Cofactor expanding this along the first row gives:

$$(1-\lambda)((-1-\lambda)(2-\lambda)-1) + 1(2-\lambda) + 1(-1) = 0$$

or

$$(1-\lambda)(\lambda^2-\lambda-3) + (1-\lambda) = 0.$$

Thankfully this factors easily as

$$(1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(\lambda-2)(\lambda+1) = 0.$$

Hence the (distinct) eigenvalues are $\lambda = 1, 2, -1$.

Now we find associated eigenvectors:

Case $\lambda = 1$: We end up solving the system

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{c} = \mathbf{0}.$$

By Gaussian elimination, this is equivalent to the problem

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{c} = \mathbf{0}.$$

Hence if $\mathbf{c} = [a\ b\ c]^T$ we can set $c = s \in \mathbb{R}$, $b = c = s$ and $a = 2b + c = 3s$. Hence it follows $\mathbf{v}_1 = [3\ 1\ 1]^T$ is an eigenvector.

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The other cases are similar:

Case $\lambda = 2$*:* This leads to solving

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -3 & -1 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{c} = \mathbf{0}.$$

By Gaussian elimination, this is equivalent to the problem

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{c} = \mathbf{0}.$$

So here $c = s \in \mathbb{R}$, $b = 0$ and $a = 3b + c = s$. Hence an eigenvector is $\mathbf{v}_2 = [1\ 0\ 1]^T$.

Case $\lambda = -1$*:* This leads to solving

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix} \mathbf{c} = \mathbf{0}.$$

By Gaussian elimination, this is equivalent to the problem

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{c} = \mathbf{0}.$$

So here $c = s \in \mathbb{R}$, $b = 3c = 3s$ and $a = c = s$. Hence an eigenvector is $\mathbf{v}_3 = [1\ 3\ 1]^T$.

Summing up, since the eigenvalues are all distinct, we know automatically that the found eigenvectors are all linearly independent. Hence it follows that the general solution is of the form

$$\mathbf{x} = A\mathbf{v}_1 e^t + B\mathbf{v}_2 e^{2t} + C\mathbf{v}_3 e^{-t},$$

for $A, B, C \in \mathbb{R}$. □

Grading: 3 points for computing the eigenvalues correctly (showing work). 2 points for each of the three eigenvectors to find (the student doesn't need to use Gaussian elimination, but some explanation is needed). 1 point for noting that we have distinct eigenvalues so that the eigenvectors are known to be linearly independent (or otherwise showing linear independence), 2 points for concluding with the correct general solution.

3. For which values of $a \in \mathbb{R}$ are the functions

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix} e^t, \ \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix} e^t, \ \mathbf{x}^{(3)}(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} e^t$$

linearly independent on $\mathbb{R}$? [8 marks]

Solution: To check linear independence of these functions, we can compute the Wronskian

$$W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}) = \begin{vmatrix} e^t & e^t & e^t \\ e^t & ae^t & 2e^t \\ ae^t & e^t & 3e^t \end{vmatrix} = e^{3t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 2 \\ a & 1 & 3 \end{vmatrix},$$

where in the latter equality, we've used the property of determinants (discussed in class) that says in this case that we can factor out the e^t from each column.

A theorem from class says that the given functions are linearly independent provided that this Wronskian is nonzero. Since $e^{3t} \neq 0$ for all values of $t \in \mathbb{R}$, we only need to find the values of a for which the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 2 \\ a & 1 & 3 \end{vmatrix} = 0.$$

Cofactor expanding this determinant (say along the first row) gives (after simplification)

$$(a-4)(a-1) = 0.$$

Hence the given functions are linearly independent as long as $a \neq 1, 4$. □

Grading: 2 points for writing down the correct Wronskian, 3 points for correctly computing the determinant and 1 point for factoring out e^{3t} leading to having to solve that quadratic in a. (In particular, the student must note that $e^{3t} \neq 0$.) 2 points for correctly solving the quadratic and thereby coming to the correct conclusion.

4. Prove the following result from class: Let A be an $n \times n$ matrix of real numbers. Suppose λ is an eigenvalue of A. Prove that λ is a root of its characteristic equation. [8 marks]

Proof: The assumption that λ is an eigenvalue of A means that there is a *nonzero* vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Hence from this equation we obtain

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

Thus the following $n \times n$ system of equations

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{c} = \mathbf{0}$$

has at least two solutions, the trivial zero vector solution, and $\mathbf{v}$. This happens if and only if $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ does not exist and if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

That is, λ is a root of the characteristic equation. □

Grading: 1 point each for defining what it means for λ to be an eigenvalue of A, and what the characteristic equation is in this case. 2 points for obtaining that we are solving the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{c} = \mathbf{0}$ (we'll forgive the slight inaccuracy of using $\mathbf{v}$ instead of $\mathbf{c}$). 2 points for observing that this system has a nonunique solution. 2 points for saying that therefore the determinant is zero. The grader should note that the appropriate logic of the argument is kept up, and deductions should be made for inaccurate implications.

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