# MATH 286 — FALL 2008 TEST I
# DIFFERENTIAL EQUATIONS PLUS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

NAME: ______________________________

SIGNATURE: ________________

ID NUMBER: ______________________

(1) Do not open this test until you are told to begin.
(2) This exam has 6 pages including this cover and one intentionally blank page for your use. There are 4 problems total. You have 50 minutes.
(3) No notes or books are permitted.
(4) No calculators are permitted.
(5) Please turn off all cell phones.
(6) Place your ID card on your desk for inspection.
(7) Please stay for the entirety of the test period, so as not to disturb others.
(8) Explain all your solutions as clearly as possible, citing results from class if needed.
(9) Good luck!

| PROBLEM | POINTS | SCORE |
|---|---|---|
| 1 | 5 | |
| 2 | 10 | |
| 3 | 15 | |
| 4 | 10 | |
| TOTAL | 40 | |

1. Consider the differential equation

$$\frac{dy}{dx} = -y^4 + 5y^3 + 10.$$

Suppose your friend claims to have found a specific solution with the property that

$$\lim_{x\to\infty} y(x) = \infty.$$

Is this possible? Why or why not? [5 marks]

*Solution:* It is NOT possible. Consider the slope field for this DE. Notice that for any $x \in \mathbb{R}$ and $y$ large we have $\frac{dy}{dx} < 0$. Hence any solution is decreasing whenever $y(x)$ is large (and is in fact asymptotically equal to one of the (real) roots of $-y^4 + 5y^3 + 10 = 0$). $\square$

Grading:

- 1 mark for stating it is not possible
- 1 mark for considering the slope field rather than trying to explicitly solve the separable DE
- 2 marks for remarking that $\frac{dy}{dx} < 0$ for $y$ large
- 1 mark for any plausible explanation that the previous fact implies impossibility

It would be VERY difficult to solve this by trying to separate variables. NO points for just trying to do this without success.

2. Solve the following differential equation:

$$\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}.$$

You may leave your answer in implicit form. [10 marks]

*Solution:* This is a separable DE question:

$$\begin{aligned}
& \tfrac{dy}{dx} = \tfrac{(x-1)y^5}{x^2(2y^3-y)} \\
\Longrightarrow \quad & \tfrac{2y^3-y}{y^5} dy = \tfrac{x-1}{x^2} dx \\
\Longrightarrow \quad & \tfrac{2}{y^2} - \tfrac{1}{y^4}\ dy = \tfrac{1}{x} - \tfrac{1}{x^2}\ dx \\
\Longrightarrow \quad & \int \tfrac{2}{y^2} - \tfrac{1}{y^4}\ dy = \int \tfrac{1}{x} - \tfrac{1}{x^2}\ dx \\
\Longrightarrow \quad & -2y^{-1} + \tfrac{1}{3}y^{-3} = \ln|x| + x^{-1} + C
\end{aligned}$$

and the last line, for any $C \in \mathbb{R}$ solves the original DE. $\square$

Grading: 2 marks for each of the first three " $\Longrightarrow$ " and 4 marks for the last " $\Longrightarrow$ " (since it requires essentially 4 integrations, 1 point for each done correctly).

3. Solve the following differential equation:

$$xy' + (2x - 3)y = 4x^4$$

with initial value $y(1) = 1$ [15 marks]

*Solution:* Dividing through by $x$ gives a first order linear DE:

$$\begin{aligned} & xy' + (2x-3)y = 4x^4 \\ \Longrightarrow \qquad & y' + \tfrac{2x-3}{x}y = 4x^3 \end{aligned}$$

This latter DE is of the form $y' + P(x)y = Q(x)$, and the integrating factor is

$$\rho(x) = e^{\int P(x)dx} = e^{\int 2-3/x \ dx} = e^{2x - 3\ln|x|} = x^{-3}e^{2x},$$

where in the last equality we have used the fact that we are only interested in a solution near $x = 1$, so we may assume $x > 0$.

Multiplying through by the integrating factor gives

$$\begin{aligned} & \rho(x)\left(y' + \tfrac{2x-3}{x}y\right) = 4e^{2x} \\ \Longrightarrow \qquad & D_x(\rho(x) \cdot y) = 4e^{2x} \\ \Longrightarrow \qquad & \rho(x) \cdot y = \textstyle\int 4e^{2x}dx \\ \Longrightarrow \qquad & \rho(x) \cdot y = 2e^{2x} + C \\ \Longrightarrow \qquad & y(x) = x^3e^{-2x}(2e^{2x} + C) \end{aligned}$$

Hence

$$y(x) = 2x^3 + Cx^3/e^{2x}.$$

Since $1 = y(1) = 2 + C/e^2$ we get $C = -e^2$ and hence the final solution is

$$y(x) = 2x^3 + (-e^2)x^3/e^{2x}.$$

□

Grading:

- 2 point for dividing through by $x$ to get a first order linear DE
- 3 points for computing the integrating factor correctly
- 3 points for multiplying through by the integrating factor and using the reverse product rule
- 1 point for noting $x > 0$ allows one to set $|x| = x$
- 3 points for correctly solving for $y(x)$
- 3 points for determining $C$ from the initial condition

4. Consider a population $P(t)$ satisfying the logistic equation $P'(t) = aP - bP^2$ where $aP$ is the birth rate per month and $bP^2$ is the death rate per month. Assume that the initial population is 1000 individuals and there are 50 births and 20 deaths per month occurring at time $t = 0$.

(a) Calculate the constants $a$ and $b$ from the data. [6 marks]
(b) What is the limit population? [4 marks]

*Solution:* For (a), we know $aP(0) = a(1000) = 50$ hence $a = 50/10^3$. Similarly, $bP(0)^2 = b \times 1000^2 = 20$, so $b = 20/10^6$.

For (b), write

$$P'(t) = aP - bP^2 = bP(a/b - P),$$

which is in the form

$$P'(t) = kP(M - P).$$

We know from class that the limiting population is $M = a/b$, and so that's

$$a/b = (50/10^3)/(20/10^6) = \frac{5}{2} \times 10^3.$$

Hence the limit population is $2,500$ individuals. $\square$

Grading: Three marks for each of $a, b$ computed correctly in (a). Two marks for the right answer in (b), and two marks for citing the fact we established the limit population is $M$ in class, or otherwise proving it.

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