Drift configurations

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Based on joint work with Li Li (UIUC)
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Outline

- Flag and Schubert varieties; Drift configurations
- Hilbert Samuel multiplicities
- Kazhdan-Lusztig polynomials
Let \( X = \text{Flags}(\mathbb{C}^n) = GL_n/B \) be the complete flag variety. Concretely

\[
\text{Flags}(\mathbb{C}^n) = \{ \langle 0 \rangle \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \}.
\]

\( GL_n, B, T \) acts on \( X \), so we can talk about \( T \)-fixed points:

\[
X^T = \{ e_v := \langle 0 \rangle \subset \langle \vec{e}_{v(1)} \rangle \subset \cdots \subset \langle \vec{e}_{v(1)}, \vec{e}_{v(2)}, \ldots, \vec{e}_{v(k)} \rangle \subset \cdots \subset \mathbb{C}^n \text{ for } v \in S_n \}
\]

\textit{Schubert variety:} \( X_w := \overline{Be_w} \) for \( w \in S_n \).

Locally, by \( B \)-action, can reduce study of \( X_w \) to \( e_v \) where \( v \leq w \) (in Bruhat order).
Basic problems

It has been a key problem to understand singularities of Schubert varieties.

**Problem A:** Give a combinatorial rule for the Hilbert-Samuel multiplicity $\text{mult}_{e_v}(X_w)$.

**Problem B:** Give a combinatorial rule for the Kazhdan-Lusztig polynomial $P_{v,w}(q)$.

**Problem C:** Give a relation (or relations) between these two invariants of Schubert singularities.
Our main combinatorial construction:

Figure: Pangaea(\(v, w\)) and a particular \(D \in \text{drift}(v, w)\); \(wt(D) = 10\)
Drift configurations from $v \leq w$ (covexillary)

**Goal:** Study the class of covexillary (3412-avoiding) Schubert varieties $X_w$.

Let $v = 2, 3, 4, 6, 5, 1, 7, 8, 9, 10 \leq w = 10, 9, 5, 4, 3, 7, 2, 6, 5, 1$ in Bruhat order on the symmetric group $S_n$:

![Diagram]

**Figure:** An overlay of $D(w)$ with $G(w)$ (●’s) and $G(v)$ (○’s); constructing $B(v, w)$

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Alternative, subdivide each continent into $1 \times 1$ "countries" and compute $M_{v,w}$ as the total number of drift configurations. Clearly,

$$Q_{v,w}(1) \leq M_{v,w}.$$ 

**Preview of Theorem:** [Li-Y.’10] For $w$ covexillary, $\text{mult}_{e_v}(X_w) = M_{v,w}$ and $P_{v,w}(q) = Q_{v,w}(q)$. 
Theorem: [Li-Y. '10] There is a shellable simplicial ball/sphere whose facets are drift configurations.

Figure: The 2-dimensional KL_{23465178910,10954382761}
The (Hilbert-Samuel) **multiplicity** of a point $p$ in a scheme $X$, 

$$\text{mult}_p(X) = \text{degree} \, \text{Proj}(\text{gr}_{m_p} \mathcal{O}_{p,X}) \subseteq \text{Proj}(\text{Sym}^* m_p/m_p^2).$$

Equivalently, if the Hilbert–Samuel polynomial of $\mathcal{O}_{p,X}$ is 

$$a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 \ (a_d \neq 0)$$

then $\text{mult}_p(X) = d! a_d$.

In particular, $\text{mult}_p(X) = 1$ if and only if $X$ is smooth at $p$.

Intuitively, multiplicity counts at how many points a generic hyperplane of complementary dimension meets $X$ near $p$.

Higher multiplicity means a worse singularity.
Our conjectural approach towards $\text{mult}_{e_v}(X_w)$

- A neighbourhood of $e_v \in X_w$ is given by the Kazhdan-Lusztig variety $\mathcal{N}_{v,w}$
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- Gröbner degenerations for $\prec_{v,w,\pi}$ break $N_{v,w}$, and its projectivized tangent cone, into an initial scheme $\text{init}_{\prec_{v,w,\pi}} N_{v,w} = \text{a schemey union of coordinate subspaces.}$
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This asserts multiplicity reduces to the combinatorics of counting the number of facets of a desirable simplicial complex.
Kazhdan-Lusztig ideals and varieties

\[ w = 7531462 \quad \text{and} \quad v = 5123746. \]
Kazhdan-Lusztig ideals and varieties

\[ w = 7531462 \]

\[ \begin{align*}
0 & & 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & & z_{62} & & 1 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & & z_{52} & & z_{53} & & 1 & & 0 & & 0 & & 0 & & 0 \\
0 & & z_{42} & & z_{43} & & z_{44} & & 0 & & 1 & & 0 & & 0 \\
1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
z_{21} & & z_{22} & & z_{23} & & z_{24} & & 0 & & z_{26} & & 1 & & 0 \\
z_{11} & & z_{12} & & z_{13} & & z_{14} & & 1 & & 0 & & 0 & & 0
\end{align*} \]

\[ v = 5123746. \]
Kazhdan-Lusztig ideals and varieties

The Kazhdan-Lusztig ideal is \( I_{5123746, 7531462} \)

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_{62} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & z_{52} & z_{53} & 1 & 0 & 0 & 0 & 0 \\
0 & z_{42} & z_{43} & z_{44} & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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z_{11} & z_{12} & z_{13} & z_{14} & 1 & 0 & 0 & 0
\end{pmatrix}
\]

The **Kazhdan-Lusztig ideal** is \( I_{5123746,7531462} \)

\[
= \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{11} & z_{12} \end{vmatrix}, \ldots, \begin{vmatrix} 1 & 0 & 0 \\ z_{21} & z_{22} & z_{23} \end{vmatrix}, \ldots, \begin{vmatrix} 0 & z_{42} & z_{43} & z_{44} \\ 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{11} & z_{12} & z_{13} & z_{14} \end{vmatrix}, \ldots \right\rangle.
\]

The **Kazhdan-Lusztig variety** is \( \mathcal{N}_{v,w} := \text{Spec}(\mathbb{C}[z^{(v)}]/I_{v,w}) \).
Gröbner geometry theorems

We prove our geometric multiplicity conjecture for covexillary $w$: 
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**Theorem:** [Li-Y.,’10] The essential minors form a Gröbner basis with respect to an explictly chosen term order $\prec_{v,w,\pi}$. The limit is reduced and equidimensional, with Stanley-Reisner simplicial complex homeomorphic to a shellable ball/sphere.
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**Theorem:** [Li-Y.,’10] We have a prime decomposition:

$$\text{init}_{\prec_{v,w,\pi}} l_{v,w} = \bigcap_{\mathcal{P}} \langle \tilde{z}_{ij} : (i,j) \in \mathcal{P} \rangle$$

where $\tilde{z}_{ij}$ is a change of variables depending on $\pi$. The intersection is over all drift configurations where each box is a continent.
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**Theorem:** [Li-Y.,’10] $\text{mult}_{e_v}(X_w) = M_{v,w}$ and

$$\text{mult}_{e_v}(X_w) = \det \left( \begin{pmatrix} b_i + \lambda_i - i + j - 1 \\ \lambda_i - i + j \end{pmatrix} \right)_{1 \leq i,j \leq \ell(\lambda)},$$

where $\ell(\lambda)$ is the number of nonzero parts of $\lambda$ and $b = B(v,w)$. 
Let $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. The **Hecke algebra** $\mathcal{H}$ for $S_n$ is the algebra over $A$ with basis $\{T_w : w \in S_n\}$ subject to

$$
T_{s_i} T_w = T_{s_i w} \quad \text{if } \ell(s_i w) > \ell(w)
$$

$$
T_{s_i}^2 = (q - 1) T_{s_i} + q T_{id} \quad \text{otherwise}
$$

There is the **Kazhdan-Lusztig basis** $\{C'_w : w \in S_n\}$ given by

$$
C'_w = (q^{-\frac{1}{2}})^{\ell(w)} \sum_{v \leq w} P_{v,w}(q) T_v
$$

and where $P_{v,w}(q)$ are the Kazhdan-Lusztig polynomials.
The Kazhdan-Lusztig conjecture states $P_{v,w}(q) \in \mathbb{N}[q]$ (for arbitrary Coxeter groups). For Weyl groups, and $S_n$ in particular, Kazhdan-Lusztig proved this by showing

$$P_{v,w}(q) = \sum_{j \geq 0} \dim(\mathcal{H}^{2j}(w)_v) q^j$$
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**Upshot:** Thus $P_{v,w}(q)$ is another measure of singularities of Schubert varieties.
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**Upshot:** Thus $P_{v,w}(q)$ is another measure of singularities of Schubert varieties.

$P_{v,w}(q) = 1$ if and only if $X_w$ is (rationally) smooth at $e_v$. 

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Drift configurations
[Lascoux-Schützenberger '81] proved a rule for Grassmannian Schubert varieties

[Lascoux '95] generalized his earlier work with a rule for covexillary $X_w$

[Naruse '10+] proved that for Grassmannians (and cominuscule $G/P$) that $P_{\mu,\lambda}(1) \leq \text{mult}_{e\mu}(X_{\lambda})$. 
Combinatorial rules for Kazhdan-Lusztig polynomials

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Building on these ideas, we show:

**Theorem:** [Li-Y.,’10] For $w$ covexillary,

- $P_{v,w}(q) = Q_{v,w}$;
- $P_{v,w}(1) \leq \text{mult}_{e_v}(X_w)$;
- $P_{v,w}(q) = P_{id,\Theta_{v,w}}(q)$ for an explicit, covexillary $\Theta_{v,w}$;
- defining recursions for covexillary $P_{v,w}(q)$ can be “explained” as vertex decompositions of the simplicial complex $KL_{v,w}$.
In this talk we discussed two measures of singularities of Schubert varieties

- Hilbert-Samuel multiplicities
- Kazhdan-Lusztig polynomials

By introducing
- Drift configurations

and examining their combinatorics, we obtained new, compatible combinatorial rules for these measures, in the covexillary case. We also explain how our results fit into a more general, geometric degeneration approach to compute the multiplicities $\text{mult}_{e_v}(X_w)$. 