

AN ESTIMATION METHOD FOR GAME COMPLEXITY

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We looked at a method for estimating the complexity measure of *game tree size*. It seems effective for a number of children’s games such as *Tic-Tac-Toe*, *Connect Four*, and *Othello*.

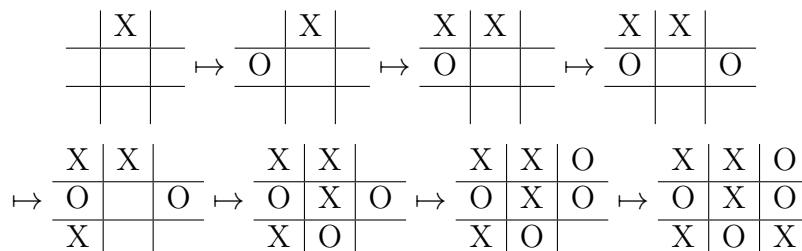
G. H. Hardy [4, pg. 17] estimated the game tree size (number of legal games) of Chess to be $10^{10^{50}}$ or “in any case a second order exponential”, but gave no reasoning. Claude Shannon wrote a seminal paper [9] on computer Chess. Based on master games collected by the psychologist Adriaan de Groot, he estimated $\approx 10^3$ options per (white, black) move pair, and that an average game is ≈ 80 *plies* (half-moves). Thus, he surmised that the game tree size (and game tree complexity) is *at least* $\approx 10^{120}$. This *Shannon number* is often compared to the number of atoms in the observable universe $\approx 10^{80}$.

We take another avenue. Given a game \mathcal{G} of two players P_1 and P_2 , generate a game g by uniformly at random selecting each move from legal possibilities. If $c_j = c_j(g)$ is the number of options at ply j , and g has N plies, let $X(g) = \prod_{j=1}^N c_j$. Independently repeat this process, producing random games g_1, g_2, \dots, g_n . The proposed estimate for the game tree size is

$$(1) \quad \text{gts}(\mathcal{G}) \approx \frac{1}{n} \sum_{i=1}^n X(g_i).$$

As justified below, (1) is an equality in the $n \rightarrow \infty$ limit. Our thesis is that it gives *fairly precise* estimates for many games of pure strategy. This estimation method is straightforward to implement, parallelizable, and space efficient. It requires no sophistication, such as use of databases (e.g., of human play, nor of endgame positions).

To illustrate, if $\mathcal{G} = \text{Tic-Tac-Toe}$, $\text{gts}(\mathcal{G}) = 255168$ (www.se16.info/hgb/tictactoe.htm is our resource). One random game we generated is



Here, $N = 9$ and $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9) = (9, 8, 7, 6, 5, 4, 3, 2, 1)$ and $X = 9! = 362880$. Estimating, using $n = 2000$ and repeating for a total of 10 trials, gives

255051, 260562, 252352, 256586, 250916, 256457, 257380, 251800, 257448, 248988.

We “guess” that $\text{gts}(\text{Tic-Tac-Toe}) \approx 2.55(\pm 0.04) \times 10^5$. The “ (± 0.04) ” refers to the usual *standard error of the mean*. It is comforting that this agrees with the known value.

The technique is an instance of *sequential importance sampling*. This Monte Carlo method can be traced back to Herman Kahn and Theodore Harris’ article [5], who credit John von

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Neumann. Let S be a finite set. Assign $s \in S$ probability $p_s \in (0, 1]$. Define a random variable $X : S \rightarrow \mathbb{R}$ by $X(s) = 1/p_s$. From the definition of expectation,

$$\mathbb{E}[X] = \sum_{s \in S} p_s X(s) = \sum_{s \in S} p_s \times (1/p_s) = \sum_{s \in S} 1 = \#S.$$

By the law of large numbers, $\frac{1}{n} \sum_{i=1}^n X(s_i) \rightarrow \#S$. The application to enumeration of S was popularized by the article [7] of Donald Knuth. He used it to estimate the number of self-avoiding walks in a grid. It has been applied, e.g., by Lars Rasmussen [8] to estimate permanents, and by Joseph Blitzstein and Persi Diaconis [2] to estimate the number of graphs with a given degree sequence. In our use, assign $p_g = 1/\prod_{j=1}^N c_j$ to each leaf node of depth N in the game tree of \mathcal{G} . Now, while we found no literature on the efficacy of (1) exactly for game tree size, there is plenty close in point, and it may have been observed without flourish.¹ At any rate, we hope this letter contributes some useful experience.

It's simple to modify the above to estimate all sorts of other statistics such as $\text{agl}(\mathcal{G})$, the average game length of \mathcal{G} . If we define $Y(g) = N \cdot \prod_{j=1}^N c_j$ then $\text{agl}(\mathcal{G}) = \mathbb{E}[Y]/\mathbb{E}[X]$. Thus, we can Monte Carlo estimate $\text{agl}(\mathcal{G})$. One knows $\text{agl}(\text{Tic-Tac-Toe}) = (5 \times 1440 + 6 \times 5328 + 7 \times 47952 + 8 \times 72576 + 9 \times 127872)/255168 \approx 8.25$. This is accurately estimated using 2000 trials. In addition, we can also estimate the percentage of wins (by either player) and draws. For draws, use the random variable $Z(g) = \delta_g \times \prod_{j=1}^N c_j$ where $\delta_g = 1$ if g is draw, and $\delta_g = 0$ otherwise. Then estimate $\mathbb{E}[Z]/\mathbb{E}[X]$. For Tic-Tac-Toe, the draw (that is, "cat's game") rate is $46080/255168 \approx 18.1\%$. Our simulations agree closely.²

Consider $\mathcal{G} = \text{Connect 4}$. Though commercialized by Milton Bradley (now Hasbro) in 1974, it has a longer history. Among its alternate names is *Captain's Mistress*, stemming from folklore that the game absorbed Captain James Cook (1728–1779) during his historic travels. The game is played on a vertical board with seven columns of height six. P_1 uses \bullet while P_2 uses \circ . P_1 moves first and chooses a column to drop their first disk into. The players alternate. At each ply, any non-full column may be chosen. The game terminates when there are four consecutive disks of the same color in a row, column or diagonal.

We encode an entire game with a *tableau* by recording the ply at which a disk was placed. A randomly sampled game g is below; it has $X(g) = 5.59 \times 10^{17}$:

			●			
	●	●	●	●		
	●	●	●	●		
	●	●	●	●	●	
●	●	●	●	●	●	●

		20				
	21	15	17	16		
	19	10	6	14		
	11	8	2	13	18	
3	9	4	1	5	7	12

Thus, \bullet and \circ correspond to odd and even labels, respectively. Since each column is increasing from bottom to top, every game of N plies can be viewed as a distribution

¹See [6]. Also, a different Monte Carlo technique was applied in [1] to estimate the number of legal positions in games. There one uses the idea of enumerating a superset U of S and sampling from U uniformly at random to estimate the probability a point is in S . In contrast, the probability distribution we use is far from uniform. Finally, AlphaGo and AlphaZero use Monte Carlo methods in Go and Chess move selection [10]. Indeed what is described partly forms the rudiments of an AI: simulating many games for each choice of move and pick the one that produces the highest estimated winning percentage.

²Code available at <https://github.com/ICLUE/Gametreesize>

of $1, 2, \dots, N$ into 7 distinguishable rooms, where each room can have at most 6 occupants. For a fixed choice of occupancy (o_1, \dots, o_7) , the number of such arrangements is the multinomial coefficient $\binom{N}{o_1 o_2 \dots o_7}$. Thus if T_N is the total of such arrangements, then rephrasing in terms of exponential generating series,

$$T_N = \text{coefficient of } x^N \text{ in } N! \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \right)^7.$$

Thus, $\#\text{gts}(\mathcal{G}) \leq \sum_{N=7}^{42} T_N = 40645234186579304685384521259174 \approx 4.06 \times 10^{31}$, as may be determined quickly using a computer algebra system.

Shannon's number is an estimated lower bound for Chess' *game tree complexity* (gtc). This is the number of leaves of the smallest full width (all nodes of each depth) decision tree determining the game-theoretic value of the initial position. Similarly, in [1, Section 6.3.2], the average game length of Connect Four *in practice* is estimated to be 36 ply with an average of 4 legal moves/ply, whence $\text{gtc}(\text{Connect 4}) \approx 4^{36} \approx 4.72 \times 10^{21}$. We applied (1) with 12 trials of the method using $n = 10^8$. Based on this, the game tree size appears not so far from the upper bound:

$$\text{gts}(\text{Connect 4}) \approx 8.34(\pm 0.05) \times 10^{28}.$$

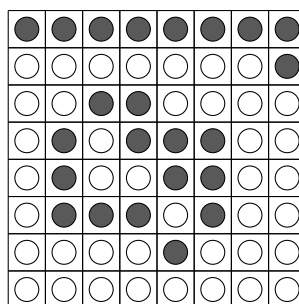
Also, $\text{agl}(\text{Connect 4}) \approx 41.03(\pm 0.01)$ plies. While P_1 wins with *perfect play* (see [1] and tromp.github.io/c4/c4.html), there is a caution: it is likely that P_1 wins less overall, at $\approx 27.71(\pm 0.21)\%$ than P_2 at $\approx 32.13(\pm 0.20)\%$ (with draws at $\approx 40.16(\pm 0.30)\%$).

Finally, let $\mathcal{G} = \text{Othello}$ (introduced into the United States in 1975 by Gabriel Industries, it is the modern version of *Reversi*). This game is played on an 8×8 board with disks \bullet and \circ , played by P_1 and P_2 , respectively. The rule is that P_1 places \bullet in a square if and only if there is another \bullet in the same row, column or diagonal and \circ 's are contiguously between them. If the placement is valid, each of these \circ 's flip to \bullet 's. The same rule applies to placing \circ (except with the colors switched). A player may pass only if they do not have a move. The game ends when neither player has a legal move. The winner is the one with the most disks. Finally, in Othello, the central squares start as



Naïvely, $\text{gts}(\text{Othello}) \leq 60! \approx 8.32 \times 10^{81}$ (by filling all 60 initially open squares in all possible ways). The gtc estimate of [1] is 10^{58} , based on an "in practice" average game length of 58 ply and an average of 10 options/ply. Elsewhere, 10^{54} is estimated for gts, but without explanation/citation (see en.wikipedia.org/wiki/Computer_Othello).³

One randomly generated Othello game g ended with \circ winning:



³A priori, this is in contradiction with Allis' estimate, since by definition, $\text{gtc}(\mathcal{G}) \leq \text{gts}(\mathcal{G})$. However, this can be reconciled as it looks like [1] does not start with the four center squares filled.

This gives $X(g) = 2.49 \times 10^{54}$. With $n = 2 \times 10^6$ (24 trials), (1) gives

$$\text{gts}(\text{Othello}) \approx 6.47(\pm 0.19) \times 10^{54}.$$

Also $\text{agl}(\text{Othello}) \approx 60.00(\pm 0.0004)$ ply, the draw rate is $\approx 4.95(\pm 0.30)\%$, P_1 's win rate is $\approx 43.36(\pm 1.56)\%$, and P_2 has $\approx 51.69(\pm 1.51)\%$ of wins. (Othello is currently unsolved, but the 4×4 and 6×6 versions have a forced win for P_2 ; see www.tothello.com.)

Estimates for statistics of other games can be similarly attempted. Candidates include *Checkers*, *Dots and Boxes*, *Go*, and *Hex*. The second named author has studied a simplified version of Chess, towards the understanding the difficulty of applying (1) to the full game. In addition, choosing each move *uniformly* at random is inessential; one might wish to modify the probabilities, with the aim of reducing variance of the estimates. Finally, while antithetical to the crude approach espoused, if one does add evaluation and database information, (1) may be reinterpreted to estimate game tree *complexity* and the number of "sensible games". Treatment of these topics will appear elsewhere.

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