Next meeting time: 10am on Thursday at Noyes 1

1 Schur’s Lemma

Theorem 1.1. Let $f_1 : G \to GL(V_1)$ and $f_2 : G \to GL(V_2)$ be two irreducible representations of $G$, let $f$ be linear map of $V_1$ into $V_2$ such that it intertwines with two representations, i.e. $f_2(s) \circ f = f \circ f_1(s)$, $\forall s \in G$, then

(i) If $f_1$ and $f_2$ are not isomorphic, then $f=0$

(ii) If $f_1$ and $f_2$ are isomorphic, there exists basis such that $f = \lambda I$, a homothety.

Proof. (i) If $f_1$ and $f_2$ are not isomorphic, note that the ker($f$) is $G$-stable. Since we have, $\forall v \in ker(f)$, $\forall s \in G$, $f(f_1(s)v) = f_2(s)f(v) = 0$, thus $f_1(s)v \in ker(f)$. Suppose $f \neq 0$, then since ker($f$) is a $G$-module and $V_1$ is irreducible, we must have ker($f$) = 0. Then $f(V_1)$ is a isomorphic copy of $V_1$ in $V_2$. Then, $V_1 \cong_G f(V_1) \cong_G V_2$. So suppose we are in the second case,

(ii) View $V_1 = V_2$ and $f_1 = f_2$, we see the eigenspace of $f$ is also $G$-stable. Suppose $v \in V_\lambda$, $\forall s \in G$, $f(f_1(s)v) = f_1(s)f(v) = \lambda f_1(s)v$, thus $f_1(s)v \in V_\lambda$. Since we are in an algebraically closed field, we must have one eigenvalue $\lambda$ whose eigenspace $V_\lambda$ is nontrivial. Then, since it is $G$-stable and $V_1$ is irreducible, $V_\lambda = V_1$. Thus, $f = \lambda I$.

2 Class Functions and Characters

Definition 2.1. For a group $G$ and complex numbers $\mathbb{C}$, Class($G$) is the set of all functions $f : G \to \mathbb{C}$ such that $\forall t, s \in G$, $f(tst^{-1}) = f(s)$.

Lemma 2.2. Class($G$) is a inner product vector space with pointwise addition and scalar multiplication.

Proof. Since functions $f : G \to \mathbb{C}$ is a vector space over $\mathbb{C}$, it suffices to verify that Class($G$) is closed under operations. For $\alpha, \beta \in \mathbb{C}$, for $f, g \in \text{Class}(G)$, $\forall t, s \in G$,

$$\alpha f(tst^{-1}) + \beta g(tst^{-1}) = \alpha f(s) + \beta g(s)$$
Thus, $\alpha f(s) + \beta g(s) \in \text{Class}(G)$. Class($G$) is a vector space.

For the inner product, define

$$ (f | g) := \frac{1}{|G|} \sum_{t \in G} f(t) \overline{g(t)} $$

This is the standard inner product for complex vector space but with a factor $\frac{1}{|G|}$, which makes it still a inner product.

**Theorem 2.3** (Serre).

(i) If $\chi$ is a character of irreducible representation, then $(\chi | \chi) = 1$

(ii) If $\chi$ and $\chi'$ are character of non-isomorphic irreducible representations, then $(\chi | \chi') = 0$

The proof relies on the Schur’s lemma and a nice averaging trick to give relations between the entries of the matrix, and introduces a new symmetric bilinear form that is easy to work with and is the same as the inner product when apply to character.

**Lemma 2.4.** For a linear map $h$ from $V_1$ to $V_2$ and respectively, their representations $\phi_1$ and $\phi_2$, define

$$ h' = \frac{1}{g} \sum_{t \in G} (f_2(t))^{-1} h f_1(t) $$

Then,

(i) If $f_1$ and $f_2$ are not isomorphic, we have $h' = 0$

(ii) If they are isomorphic, we have basis such that $h'$ is a homothety of ratio $(1/n) \text{Tr}(h)$, where $n$ is the dimension of $V_1 \cong_G V_2$.

**Proof.** The averaging gives us an intertwiner $h'$, i.e. $f_2(t)^{-1} h' f_1(t) = h'$, $\forall t \in G$. Thus, (i) is clear from Schur’s lemma. For (ii), we need to check the number on the diagonal. By a straightforward computation from the definition, we have

$$ \text{Tr}(h') = \frac{1}{|G|} \sum_{t \in G} \text{Tr}((f_1(t))^{-1} h f_1(t)) = \text{Tr}(h) $$

Thus, since for $h' = \lambda I$, $\text{Tr}(h') = n \lambda = \text{Tr}(h)$, $h' = (1/n) \text{Tr}(h) * I$. □

**Definition 2.5.** Define a symmetric bilinear form for scalar functions on $G$

$$ \langle \phi, \psi \rangle = \sum_{t \in G} \phi(t) \psi(t^{-1}) $$

It is worth noting that when we plug in characters, the form is the same as the inner product.

Now, we rewrite $h'$, $f_1$ and $f_2$ elementwise $f_1(t) = (r_{j_1 i_1}(t))$, $f_2(t) = (r_{i_2 j_2}(t))$, $h = (h_{ij})$, and $h' = (h'_{ij})$.

A direct computations gives us the relation,

$$ h'_{i_2 j_1} = \frac{1}{|G|} \sum_{t, j_1, j_2} r_{i_2 j_2} (t^{-1}) h_{j_2 j_1} r_{j_1 i_1}(t) \quad (1) $$

$$ = \frac{1}{|G|} \sum_{j_1, j_2} \left( \sum_t r_{i_2 j_2} (t^{-1}) r_{j_1 i_1}(t) \right) h_{j_2 j_1} \quad (2) $$

$$ = \sum_{j_1, j_2} \langle r_{j_1 i_1}, r_{i_2 j_2} \rangle h_{j_2 j_1} \quad (3) $$
Corollary 2.6. In case (i), we have
\[ \langle r_{j_1i_1}, r_{i_2j_2} \rangle = 0 \]
for any \( i_1, i_2, j_1, j_2 \).

**Proof.** Observe that (3) is a linear form with respect to \( h_{j_2j_1} \) and vanishes for all \( h_{j_2j_1} \). Thus, the coefficients are 0.

Corollary 2.7. In case (ii), we have
\[ \langle r_{j_1i_1}, r_{i_2j_2} \rangle = \frac{1}{n} \delta_{j_2i_1} \delta_{j_2j_1} = \begin{cases} 
\frac{1}{n} & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\
0 & \text{otherwise} 
\end{cases} \]
for any \( i_1, i_2, j_1, j_2 \).

**Proof.** We split the argument into two cases:
1. \( i_1 \neq i_2 \): we have (3) to be a zero linear form, thus the coefficients are 0.
2. \( i_1 = i_2 \): Note that, in this case we have
   \[ h'_{ii} = \frac{1}{n} \text{Tr}(h) = \frac{1}{|G|} \sum_{j_1, j_2} \frac{1}{n} \delta_{j_2j_1} h_{j_2j_1} \]
   Thus, the coefficients of this linear form and the coefficients of (3) must be equal since they are equal for all inputs.

Proof of Orthogonality Theorem 2.3. We have
\[ (\chi | \chi') = \langle \chi, \chi' \rangle = \langle \text{Tr}(f), \text{Tr}(f') \rangle = \langle \sum_{i=1}^{n} r_{ii}, \sum_{j=1}^{n} r'_{jj} \rangle \]

If \( \chi \) and \( \chi' \) are non-isomorphic, by Corollary 2.6, all \( \langle r_{ii}, r'_{jj} \rangle = 0 \). Thus, \( (\chi | \chi') = 0 \)

If \( \chi \) and \( \chi' \) are isomorphic, by Corollary 2.7, \( (\chi | \chi') = \langle \sum_{i=1}^{n} r_{ii}, \sum_{j=1}^{n} r_{jj} \rangle = \sum_{i=1}^{n} \langle r_{ii}, r_{ii} \rangle = n * \frac{1}{n} = 1 \)

3 Consequences of Orthogonality of Characters

**Theorem 3.1.** For a linear representation \( V \) with character \( \phi \), suppose \( V \) decomposes into irreducible representations,
\[ V = W_1 \oplus W_2 \oplus \ldots \oplus W_k. \]
Then, if \( W \) is an irreducible representation with character \( \chi \), the number of \( W_i \) isomorphic to \( W \) is equal to \( (\phi | \chi) \).
Proof. Suppose $W_i$ has character $\chi_i$, we know that $\phi = \chi_1 + \ldots + \chi_k$. Thus, $(\phi|\chi) = \sum_{i=1}^{k}(\chi_i|\chi) = \sum_{i:W_i \text{ isomorphic to } W}(\chi_i|\chi) = \sum_{i:W_i \text{ isomorphic to } W} 1 = \# \text{ of } W_i \text{ that is isomorphic to } W$. 

Corollary 3.2. The number of $W_i$ isomorphic to $W$ does not depend on the decomposition. 

Since $(\phi|\chi)$ does not depend on the decomposition.

Corollary 3.3. Two representations with the same character are isomorphic.

We use every irreducible representations to "test" the number of copies of it appeared in two representations. Since they have the same character, we get two equal decompositions up to isomorphism.

Theorem 3.4. If $\chi$ is a character of representation $V$, $(\chi|\chi)$ is a positive integer and if $(\chi|\chi) = 1$, $V$ is irreducible.

Proof. First, we group together the isomorphic components in the decomposition, that is $V = \oplus_{i=1}^{l} n_i W_i$ and $W_i \not\cong W_j$ if $i \neq j$. Easy to see $n_i$ is a positive integer. Then, $(\phi|\phi) = \sum_{i=1}^{l} n_i^2 \in \mathbb{Z}^+$. Also, if $(\phi|\phi) = 1$, we must have one $n_i = 1$ and 0 for all other $n_j$. Thus, $V$ is irreducible. 

4 Characters Tables

A character table is a compact description of the irreducible characters of a group. The columns are the conjugacy classes of the group, and the rows are the irreducible characters.

Example 4.1. Expanded Character table for $S_3$

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(12)</th>
<th>(23)</th>
<th>(13)</th>
<th>(123)</th>
<th>(132)</th>
</tr>
</thead>
<tbody>
<tr>
<td>trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>sign</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>standard</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Definition 4.2. The standard representation of $S_n$ is a subrepresentation of the permutation representation. Let $V$ be a $n$ dimensional vector space, and $\beta = \{v_1, \ldots, v_n\}$ be a basis. Then $S_n$ acts naturally on $V$ by permuting the basis vectors. The $n - 1$ dimensional subspace of $V$, $\{(x_1, \ldots, x_n)|\sum_{i=1}^{n} x_i = 0\}$ is the standard representation.

How does one compute such a character table? Say we only knew two of the irreducible representations, how could we find the character values of the third? Let's call this mystery representation $p$ with character $\chi$. $a = \chi(1) = 2$ can be deduced from first two well-known representations by the fact that $\sum_{i=1}^{3} \deg(W_i)^2 = |G| = 6$ deduced from regular representation. Then using the fact that irreducible characters are orthogonal, we can produce a system of linear equations

$$\pi - 3 \bar{b} + 2 \bar{c} = 0$$
\[ \pi + 3\delta + 2\tau = 0 \]

This implies \( b = 0 \) and \( c = -1 \) We can now compress this table by aggregating members of the same conjugacy class. For \( S_n \) it is well known that the conjugacy classes are cycles with same cycle type.

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</table>

To calculate character tables of general finite groups, there is the Burnside-Dixon-Schneider algorithm which reduces the problem to computing eigenvalues. For the symmetric group, there is the Murnaghan-Nakayama Rule.

There are a plethora of open problems, even about the character table of \( S_n \).

**Fact 1.** The sum across any row of a compressed character table of a finite group is a non negative integer.

**Proof.** Let \( f \) be a representation of \( G \) acting on itself by conjugation. i.e \( g \cdot h = ghg^{-1} \). It is well known that the multiplicity of \( \chi \) in \( \chi_f \) is \( (\chi_f|\chi) = \sum_K \chi(K) \), where \( K \) is a conjugacy class of \( G \) (Exercise 7.71, [4]), but that is simply the row sum of \( \chi \).

The problem is to now give a combinatorial interpretation of the row sums, re-explaining why they are non negative (Problem 11, [5]). For certain rows [1] gives an explanation.

**Fact 2.** The density of 0 in the extended character table of \( S_n \) tends towards 1 as \( n \to \infty \).

This is shown in [2].

**References**


