Gröbner bases and singularities of Schubert varieties

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Outline

I  Flag and Schubert varieties
II A Gröbner basis for Kazhdan-Lusztig ideals
III Projectivized tangent cones of Schubert varieties
Let $X = \text{Flags}(\mathbb{C}^n)(=GL_n/B)$ be the complete flag variety. Concretely

$$\text{Flags}(\mathbb{C}^n) = \{ \langle 0 \rangle \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \}.$$ 

$GL_n, B, T$ acts on $X$, so we can talk about $T$-fixed points:

$$X^T = \{ e_v := \langle 0 \rangle \subset \langle \vec{e}_v(1) \rangle \subset \cdots \subset \langle \vec{e}_v(1), \vec{e}_v(2), \ldots, \vec{e}_v(k) \rangle \subset \cdots \subset \mathbb{C}^n$$

for $v \in S_n\}.$

**Schubert variety:** $X_w := \overline{Be_w}$ for $w \in S_n$.

Locally, by $B$-action, can reduce study of $X_w$ to $e_v$ where $v \leq w$ (in Bruhat order).
Basic problems

Given a singularity property/local numerical invariant \( \mathcal{P} \) associated to \( X_w \), one can ask:

**Problem A:** If \( \mathcal{P} \) is a property. Which \( X_w \) satisfy \( \mathcal{P} \) at all points? At which points of \( X_w \) does \( \mathcal{P} \) hold?

**Problem B:** If \( \mathcal{P} \) is a local numerical invariant, give a combinatorial rule for computing \( \mathcal{P} \).

**Problem C:** Give a relation (or relations) between different properties/invariants \( \mathcal{P} \).
Given a singularity property/local numerical invariant $\mathcal{P}$ associated to $X_w$, one can ask:

**Problem A:** If $\mathcal{P}$ is a property. Which $X_w$ satisfy $\mathcal{P}$ at all points? At which points of $X_w$ does $\mathcal{P}$ hold?

**Problem B:** If $\mathcal{P}$ is a local numerical invariant, give a combinatorial rule for computing $\mathcal{P}$.

**Problem C:** Give a relation (or relations) between different properties/invariants $\mathcal{P}$.

**Examples:** $\mathcal{P}$ could be “smooth”, “Cohen-Macaulay”, “Gorenstein”, “normal”, “Hilbert-Samuel multiplicity”, “Kazhdan-Lusztig polynomial”, “equivariant $K$-theory localization”, ...
The opposite big cell $B_-B/B$ is an affine open neighbourhood of $e_{id} \in GL_n/B$.

Thus an affine open neighbourhood of $e_v \in X_w$ is given by

$$X_w \cap vB_-B/B \cong (X_w \cap B_-vB/B) \times \mathbb{A}^{\ell(v)}.$$ 

Define $\mathcal{N}_{v,w} := X_w \cap B_-vB/B$ be the Kazhdan-Lusztig variety.

**Goal:** Give explicit coordinates and equations for $\mathcal{N}_{v,w}$. 

Kazhdan-Lusztig varieties
The **Kazhdan-Lusztig ideal** is $I_{5123746, 7531462}$

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & z_{62} & 1 & 0 & 0 & 0 & 0 \\
0 & z_{52} & z_{53} & 1 & 0 & 0 & 0 \\
0 & z_{42} & z_{43} & z_{44} & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_{21} & z_{22} & z_{23} & z_{24} & 0 & z_{26} & 1 \\
z_{11} & z_{12} & z_{13} & z_{14} & 1 & 0 & 0
\end{pmatrix}
$$

$w = 7531462$

$v = 5123746$.
Let $\prec$ be the pure lexicographic term order on monomials in $z^v$ (=those $z_{ij}$ appearing in the generic matrix for $v$) defined by

$$z_{ij} \prec z_{k,\ell} \text{ if } j < \ell, \text{ or if } j = \ell \text{ and } i < k.$$ 

**Theorem:** [Woo-Y., '09] The defining (or essential) minors form a Gröbner basis for the Kazhdan-Lusztig ideal $I_{v,w}$. 

**Corollary:** [implicit in Fulton '92; cf. Woo-Y. '06] $N_{v,w} := \text{Spec}(\mathbb{C}[z^{(v)}])/I_{v,w}$. 

**Corollary:** [Knutson-M., '05] The essential minors of the Schubert determinantal ideal $I_w$ form a Gröbner basis with respect to any diagonal term order.
II. Applications and some comparisons to other work

1. [Woo-Y., ’06] introduced *interval pattern avoidance* and proved it could be used, in principle, to combinatorially characterize the $\mathcal{P}$-locus and which $X_w$ are $\mathcal{P}$, for any *semicontinuously stable* property $\mathcal{P}$.

2. [Knutson-Miller, ’05]’s theorem was used to give a geometric interpretation of Lascoux-Schützenberger’s Schubert/Grothendieck polynomials.


4. Our Gröbner basis explicates one of [Knutson ‘08]’s degenerations.

5. [Woo-Y., ’09] extends 2., answering a question of [Buch-Rimanyi, ’04] about specializing Grothendieck polynomials. To do this we introduce a combinatorial notion of *unspecialized Grothendieck polynomials*. 
The projectivized tangent cone and $H$-polynomials

Let $Y$ be a variety.

- Let $(\mathcal{O}_p, Y, m_p)$ be the local ring of $p \in Y$
- Associated graded ring:
  \[ \text{gr}_{m_p} \mathcal{O}_p, Y = \bigoplus_{i \geq 0} m_p^i / m_p^{i+1}. \]

- The Hilbert-series is
  \[ \text{Hilb}(\text{gr}_{m_p} \mathcal{O}_p, Y, q) = \frac{H_{p, Y}(q)}{(1 - q)^\dim Y}. \]

- $H_{p, Y}(q)$ is the $H$-polynomial
- $H_{p, Y}(1) = \text{mult}_p(Y)$ (Hilbert-Samuel multiplicity)
Main conjecture: [Li-Y., ’10] The following hold

i. $\text{gr}_{m_v} \mathcal{O}_{e_v,x_w}$ is Cohen-Macaulay

ii. $H_{e_v,x_w}(q) \in \mathbb{Z}_{\geq 0}[q]$

iii. “coefficient of $q^k$ in $H_{e_v,x_w}(q)$” is an upper-semicontinuous function

Properties (ii), (iii) are known to hold for the Kazhdan-Lusztig polynomials $P_{v,w}(q)$.

We wish to posit an analogy between $H_{v,w}(q)$ and $P_{v,w}(q)$. (More later.)
An approach towards understanding $H_{v,w}(q)$

We would like to Gröbner degenerate the KL ideal $I_{v,w}$ to its proj. tangent cone, with respect to a *local* term order.

- We study a choice of term orders $\prec_{v,w,\pi}$ that depends on $v, w$ and a *shuffling* (total ordering) of variables $\pi$.

- Gröbner degenerations for $\prec_{v,w,\pi}$ break $N_{v,w}$, and its projectivized tangent cone, into an initial scheme $\text{init}_{\prec_{v,w,\pi}}N_{v,w} = \text{a scheme} \bigcup \text{of coordinate subspaces}$.

- $H_{v,w}(q)$ is obtained from the Hilbert series of this monomial ideal

**Conjecture:** [Li-Y. ’10] There exists $\pi$ such that $\text{init}_{\prec_{v,w,\pi}}N_{v,w}$ is reduced and equidimensional with Stanley-Reisner simplicial complex is homeomorphic to a vertex-decomposable and thus shellable ball or sphere.

This asserts, e.g., multiplicity reduces to the combinatorics of counting the number of facets of a desirable simplicial complex.
We prove our geometric multiplicity conjecture for covexillary $w$.

- covexillary $w$ = “3412 avoiding”
- A. Cortez’s proof of the singular theorem conjecture, treats covexillary as a base case of her induction: other cases are “partially resolved” to this case.
- See also work of [W. Fulton ’92], [A. Lascoux ’95], [Knutson-Miller-Y., ’09].
- The KL ideals are not homogeneous wrt the standard grading.

**Theorem:** [Li-Y.,’10] The essential minors form a Gröbner basis with respect to an explicitely chosen term order $\prec_{v,w,\pi}$. The limit is reduced and equidimensional, with Stanley-Reisner simplicial complex homeomorphic to a shellable ball/sphere.
Theorem: [Li-Y.,'10] We have a prime decomposition:

\[ \text{init}_{\prec_{v,w,\pi}} l_{v,w} = \bigcap_{P} \langle \tilde{z}_{ij} : (i,j) \in P \rangle \]

where $\tilde{z}_{ij}$ is a change of variables depending on $\pi$. The intersection is over certain pipe dreams (not defined here, but cf. [Knutson-Miller-Y., '09]).

We also have explicit combinatorial formulas for $H_{v,w}(q)$ and for $\text{mult}_{e_v}(X_{w})$. Consequently:

Theorem: [Li-Y.,'10] All three parts of the the main conjecture are true for covexillary $w$ (and hence all Grassmannians).
Let $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. The Hecke algebra $\mathcal{H}$ for $S_n$ is the algebra over $A$ with basis $\{ T_w : w \in S_n \}$ subject to

$$T_{s_i} T_w = T_{s_iw} \quad \text{if } \ell(s_iw) > \ell(w)$$
$$T_{s_i}^2 = (q - 1) T_{s_i} + q T_{id} \quad \text{otherwise}$$

There is the Kazhdan-Lusztig basis $\{ C'_w : w \in S_n \}$ given by

$$C'_w = (q^{-\frac{1}{2}})^{\ell(w)} \sum_{v \leq w} P_{v,w}(q) T_v$$

and where $P_{v,w}(q)$ are the Kazhdan-Lusztig polynomials.
The Kazhdan-Lusztig conjecture states $P_{v,w}(q) \in \mathbb{N}[q]$ (for arbitrary Coxeter groups). For Weyl groups, and $S_n$ in particular, Kazhdan-Lusztig proved this by showing

$$P_{v,w}(q) = \sum_{j \geq 0} \dim(\mathcal{H}^{2j}(w)_v) q^j$$

where $\mathcal{H}^{2j}(w)_v$ is the stalk at $e_v$ of Goresky-Macpherson’s local intersection cohomology sheaf $\mathcal{H}^{2j}(w)$ of $X_w$.

**Upshot:** Thus $P_{v,w}(q)$ is another measure of singularities of Schubert varieties.

$P_{v,w}(q) = 1$ if and only if $X_w$ is (rationally) smooth at $e_v$. 
[Lascoux-Schützenberger ’81] proved a rule for Grassmannian Schubert varieties

[Lascoux ’95] generalized his earlier work with a rule for covexillary $X_w$

[Naruse ’10+] proved that for Grassmannians (and cominuscule $G/P$) that $P_{\mu,\lambda}(1) \leq \text{mult}_{e_\mu}(X_\lambda)$.

Building on these ideas:

**Theorem:** [Li-Y.,’10] For $w$ covexillary, $P_{v,w}(q) \leq H_{v,w}(q)$.

We have no geometric explanation for this result.
Summary

In this talk we discussed Kazhdan-Lusztig varieties/ideals as a local model for Schubert varieties.
By finding explicit Gröbner bases we

- Answered a question of [Buch-Rimanyi ’04] concerning combinatorics of Grothendieck polynomials and extended work of [Knutson-Miller ’05] on matrix Schubert varieties
- Studied the projectivized tangent cone and associated singularity invariants ($H$-polynomials, Hilbert-Samuel multiplicity)
- Investigated an analogy with Kazhdan-Lusztig polynomials.