Answer as many problems as you can. Each question is worth 6 points (total points is 30). HOWEVER I WILL TAKE THIS TEST OUT OF 25 POINTS. Hence a score of 30/25 is possible.

Show your work. An answer with no explanation will receive no credit. Write your name on the top right corner of each page. No external aids (notes, calculators, phones, etc) allowed.

[Total time: 50 minutes]

1. Determine the number of integral solutions to

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 2022 \]

subject to

\[ x_1 \geq 0, x_2 \geq 3, x_3 \geq 47, x_4 \geq -4, x_5 \geq -13. \]

Solution: Use the substitution \( y_1 = x_1, y_2 = x_2 - 3, y_3 = x_3 - 47, y_4 = x_4 + 4, y_5 = x_5 + 13. \) It then follows the answer is \( \binom{2022-33+5-1}{5-1} = \binom{1993}{4}. \)

2. Give a COMBINATORIAL PROOF of the following identity (i.e., double count a set). A non-combinatorial proof will receive no credit.

\[ \sum_{k=0}^{n} \binom{n}{k} 412^{n-k} = 413^n. \]

Solution: The RHS counts the number of sequences of length \( n \) using 0, 1, 2, \ldots, 412. The LHS counts the same thing by decomposing the set of all such sequences according to the number \( k \) of 412’s that appear. There are \( \binom{n}{k} \) places to put these 412’s. Then the remaining \( n-k \) positions can either be 0, 1, \ldots, 411, giving a total of \( 412^{n-k} \) possibilities, by the multiplication principle.

2. Count the number of ways to place four nonattacking rooks on a \( 8 \times 8 \) chessboard such that neither the first row nor the first column is empty.
Solution 1: Since the first row and column are chosen, there are \( \binom{7}{3}^2 \) ways to pick three other rows and columns. Then on the \( 4 \times 4 \) subgrid you have 4! options. Hence the result is \( \binom{7}{3}^2 4!(= 29400) \). \qed

Solution 2: There are two cases.

The first case is that there is a rook in (matrix position) \((1,1)\). In that case, you have seven remaining columns and rows of which you choose three. Then you place rooks in 3! ways. Thus the number is \( \binom{7}{3}^2 \times 3! \).

The second case is that the rook in row 1 is not in column 1 and hence the rook in column 1 is not in row 1. There are \( 7 \times 7 \) ways to place those rooks. For the other two rooks we need to place there are now \( \binom{6}{2}^2 \times 2! \) position them. Hence the number in this case is \( 7^2 \times \binom{6}{2}^2 \times 2! \).

The final count is therefore \( \binom{7}{3}^2 \times 3! + 7^2 \times \binom{6}{2}^2 \times 2! (= 29400) \). \qed

3. Remember this problem from class: Let \( S \) be the set of permutations of at most \( m \) A’s and at most \( n \) B’s. The claim is

\[ \#S = \left( \frac{m+n+2}{m+1} \right) - 1. \]

Precisely define a set \( S’ \) which “obviously” has cardinality \( \left( \frac{m+n+2}{m+1} \right) - 1 \) and giving a bijection \( f: S’ \to S \).

Explain why your claimed bijection is WELL-DEFINED, i.e., given \( w \in S’ \) that indeed \( f(w) \in S \). You do NOT need to show \( f \) is injective or surjective here.

**Proof:** Let \( S’ \) be the set of permutations with \( m+1 \) A’s and \( n+1 \) B’s BUT not including the word \( AA \cdots ABB \cdots B \). Since there are \( \left( \frac{m+1+n+1}{m+1} \right) \) words with \( m+1 \) A’s and \( n+1 \) B’s, this \( S’ \) has the desired cardinality.

The bijection \( f: S’ \to S \) is as follows: look at the rightmost A that appears in a word \( w \in S’ \). Then remove it and all B’s that appear to the right of it. Since \( w \neq AA \cdots ABB \cdots B \), the resulting subword must have at least one B that remains. Remove the leftmost B and all A’s the appear to the left of it. Since we started with \( m+1 \) A’s and \( m+1 \) B’s and we have removed at least one of each, that \( f(w) \in S \) (i.e., \( f \) is well-defined) follows. \qed

3. Yihong\(^1\) and Zhuo are solving the Homework 2 problem:

*In how many ways can two red and four blue rooks be placed on an \( 8 \times 8 \) chessboard so that no two rooks can attack each other?*

\(^1\)any similarities in names to students in the class is purely coincidental
Yihong suggests the following solution: Have THREE color classes: RED, BLUE, and VOID (no color).

STAGE 1: decide what colors appear in which of the eight columns of the chessboard. Since there are two RED, four BLUE, that leaves two VOID to arrange. Thus the number of ways to do this is \( \frac{8!}{2!3!2!} \).

STAGE 2: place eight rooks in \( 8! \) ways on the board. Paint the eight rooks according to the color of their column (with the VOID rooks disappearing).

Hence the answer is \( \frac{8!}{2!3!2!} \times 8! \) by the multiplication principle.

Zhuo says this count is wrong (it OVERCOUNTS by a factor of 2). Explain this overcount.

Solution: Yihong’s STAGE 1 is fine, but at STAGE 2 there are \( 6! \) to arrange to the rooks on the RED and BLUE columns. Fix such a choice. Then at the step where he places two rooks on the VOID columns there are two placements that lead to the same final outcome (i.e., phantom rooks on those columns), namely one where a rook is SW of the other, and one where it is NW of the other.

4. How many arrangements of IDENTIFICATION have no repeated I’s? For example, EIDNFITICAITON is allowed, but DIETNIIFCATON and IDENTIIIFCATON are rejected.

Solution: Consider arrangements of the form

\[ \ldots, I, \ldots, I, \ldots, I, \ldots, I, \ldots, \]

where the the far left and far right blank can be empty. The other two blanks must contain a letter. This translates to the composition problem

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 10 \]

subject to \( x_1 \geq 0, x_2 \geq 1, x_3 \geq 1, x_4 \geq 1, x_5 \geq 0 \) (and integral).

Using the transformation

\[ y_1 = x_1, y_2 = x_2 - 1, y_3 = x_3 - 1, y_4 = x_4 - 1, y_5 = x_5 \]

translates to the problem to the standard form

\[ y_1 + y_2 + y_3 + y_4 + y_5 = 7 \]

subject to \( y_i \in \mathbb{Z}_{\geq 0} \). There are \( \binom{7+5-1}{7} \) solution. Now we need to consider all \( 10!/(2!2!) \) rearrangements of the remaining letters. This gives \( 10! \times (1/2!)^2 \times \binom{7+5-1}{7} \).

4. What fraction of all of the arrangements of EFFLORESCENCE has consecutive Cs, consecutive Fs but no consecutive Es? For example, ECCLROESFFENE is one such arrangement.
Solution: Combine the two C’s into C and combine the two F’s into F. Now there are 4 E’s and exactly one of each of the other letters (L, O, R, S, N, C, F). Now consider the pattern:

_\_, E, _, E, _, E, _, E, _

The three middle _ must have at least one letter. Hence we are considering the distribution of letters governed by

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 7 \]

where \(x_2, x_3, x_4 \geq 1\) (this ensures the E’s are not consecutive). There are \(\binom{4+4}{4}\) such distributions and hence \(x = 7!\binom{8}{4}\) placements of the letters.

Now without restriction there are \(y = \frac{13!}{2!2!4!}\) many arrangements. Hence the desired probability is \(\frac{x}{y}\).  

5. A composition of \(n \geq 1\) is any vector \((x_1, x_2, \ldots, x_k) \in \mathbb{Z}_{\geq 0}^k\) with \(x_1 + x_2 + \ldots + x_k = n\) for SOME \(k \geq 1\). Let \(C_n\) be the set of all such compositions. For example, if \(n = 3\), \(C_3 = \{(3), (2, 1), (1, 2), (1, 1, 1)\}\).

Let \(P_n\) be the set of subsets of \([n-1] := \{1, 2, \ldots, n-1\}\).

Describe a BIJECTION \(f : C_n \rightarrow P_n\) that shows \(#C_n = #P_n\) (and hence both are equal to \(2^{n-1}\)).

Prove \(f\) is WELL-DEFINED, i.e., if \(c \in C_n\) then \(f(c)\) is indeed in \(P_n\). You do NOT need to prove it is injective nor surjective.

Solution: We define a map \(f : C_n \rightarrow P_n\) as follows. Suppose \((x_1, x_2, \ldots, x_k) \in C_n\).

Discard \(x_k\) leaving \((x_1, x_2, \ldots, x_{k-1})\). Now we build a set \(S\). If \(k = 1\) then \(S = \emptyset\). Otherwise the first element of \(S\) is \(x_1\). The second element is \(x_1 + x_2\), etc. Since \(x_k > 0\), it follows that each \(1 \leq x_1 + x_2 + \ldots + x_i \leq n - 1\) for each \(i\). Hence the map is well-defined.  

5. Give a COMBINATORIAL proof of the following identity:

\[
\binom{2}{2} \binom{n}{2} + \binom{3}{2} \binom{n-1}{2} + \binom{4}{2} \binom{n-2}{2} + \ldots + \binom{n}{2} \binom{2}{2} = \binom{n+3}{5}.
\]

Hint: think about the MIDDLE, not the last, nor the first.

Proof: The RHS counts selections of five numbers from \(\{1, 2, \ldots, n+3\}\). Consider the middle number of this selection. The least it could be is 3, then there are \(\binom{2}{2}\) numbers to select to the left and \(\binom{n}{2}\) to the right. In general if the middle number is \(i\), there are \(\binom{i-1}{2}\) options to the left and \(\binom{n+3-i}{2}\) options to the right. This decomposes all selections \(S\) into \(S_2 \cup S_3 \cup \cdots \cup S_i \cup \cdots S_{n+1}\) (disjoint union) where \(#S_i = \binom{i-1}{2} \times \binom{n+3-i}{2}\), and the result follows by the addition principle.  

□