Section 4.1 Anti Differentiation

**Definition:** If \( \frac{dF(x)}{dx} = f(x) \) then we say that \( F(x) \) is an anti-derivative of \( f \).

**Example**

1) \( F(x) = \frac{x^3}{3} \) is an anti-derivative of \( f(x) = x^2 \) because 
\[
\frac{dF}{dx} = \frac{3x^2}{3} = x^2 = f(x)
\]

2) \( F(x) = x \ln x - x \) is an anti-derivative of \( f(x) = \ln x \). Show this yourself by taking derivative of \( F(x) \)

3) \( F(x) = \frac{x^2}{2} \) is an anti-derivative of \( f(x) = x \). Show this yourself by taking derivative of \( F(x) \)

4) \( F(x) = \frac{x^2}{2} + 1 \) is an anti-derivative of \( f(x) = x \)

4) \( F(x) = \frac{x^2}{2} + c \) is an anti-derivative of \( f(x) = x \)

**Theorem** If \( F'(x) = f(x) \) and \( G'(x) = f(x) \) then \( F(x) = G(x) + c \) for some constant factor \( c \).

**Proof** \( F'(x) = f(x) \) and \( G'(x) = f(x) \) implies that \( (G-F)' = f - f = 0 \). Recall that we proved as a corollary of the Mean Value Theorem that if a function has a derivative zero then it is constant. Hence \( G(x) - F(x) = c \) (for some constant \( c \)). That is, \( G(x) = F(x) + c \)

**Definition** All anti-derivatives of a continuous function form a family \( F(x) + c \) where \( c \) is an arbitrary constant. Thus the family \( F(x) + c \) is called the Indefinite Integral of \( f \) and we write

\[
\int f(x) \, dx = F(x) + c
\]

**Remark** "\( x \)" in the sign \( \int f(x) \, dx \) is not a dummy variable.
Here are some basic antiderivative formulas:

1) \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \) where \( n \neq -1 \)
2) \( \int 0 \, dx = C \)
3) \( \int k \, dx = kx + C \) where \( k \) is a constant
4) \( \int \cos x \, dx = \sin x + C \)
5) \( \int \sin x \, dx = -\cos x + C \)
6) \( \int \sec^2 x \, dx = \tan x + C \)
7) \( \int \sec x \tan x \, dx = \sec x + C \)
8) \( \int \csc^2 x \, dx = -\cot x + C \)
9) \( \int \csc x \cot x \, dx = -\csc x + C \)
10) \( \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \)
11) \( \int -\frac{dx}{\sqrt{1-x^2}} = \arccos x + C \)
12) \( \int \frac{dx}{1+x^2} = \arctan x + C \)
13) \( \int e^x \, dx = e^x + C \)
14) \( \int \frac{dx}{x} = \ln |x| + C \)

I’ll only give here the proof of the last formula you could easily verify the others in the similar way.

**Proof of (14)** Since \( |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \) we will also divide the differentiation of \( \ln |x| \) into two cases.
If \( x < 0 \) then \( \ln |x| = \ln(-x) \Rightarrow \frac{d(\ln(-x))}{dx} = \frac{1}{-x} (-1) = \frac{1}{x} \)
If \( x > 0 \) then \( \ln |x| = \ln(x) \Rightarrow \frac{d(\ln(x))}{dx} = \frac{1}{x} \)

**Example** Find the indefinite integral: \( \int \frac{1}{x^5} \, dx \)
\( \int \frac{1}{x^5} \, dx = \frac{x^{-4+1}}{-5+1} + C = \frac{x^{-4}}{-4} + C = \frac{-1}{4x^4} + C \)
**Theorem** Suppose that \( f(x) \) and \( g(x) \) have antiderivatives and let \( a \) and \( b \) be any constants. Then, 
\[
a \int f(x) \, dx + b \int g(x) \, dx
\]
is an antiderivative of 
\[
\int [af(x) + bg(x)] \, dx .
\]
That is,
\[
\int [af(x) + bg(x)] \, dx = a \int f(x) \, dx + b \int g(x) \, dx + C
\]

**Proof** Since both \( f(x) \) and \( g(x) \) have antiderivatives we have 
\[
d\frac{dx}{dx} \int f(x) \, dx = f(x) \text{ and } d\frac{dx}{dx} \int g(x) \, dx = g(x).
\]
Let \( H(x) = a \int f(x) \, dx + b \int g(x) \, dx \). Then
\[
\frac{d}{dx} (H(x)) = \frac{d}{dx} (a \int f(x) \, dx + b \int g(x) \, dx)
\]
\[
= \frac{d}{dx} (a \int f(x) \, dx) + \frac{d}{dx} (b \int g(x) \, dx)
\]
\[
= a(\frac{d}{dx} \int f(x) \, dx) + b(\frac{d}{dx} \int g(x) \, dx)
\]
\[
= af(x) + bg(x)
\]
Therefore, \( H(x) \) is an antiderivative of \( \int [af(x) + bg(x)] \, dx \)

**Theorem** Suppose \( F(x) \) is an antiderivative of \( f(x) \), then 
\( kF(x) \) is an antiderivative of \( kf(x) \). In other words;
\[
\int kf(x) \, dx = k \int f(x) \, dx
\]

**Proof** Since 
\[
\frac{d}{dx} F(x) = f(x)
\]
by derivative rules we know 
\[
\frac{d}{dx} kF(x) = k \frac{d}{dx} F(x) = kf(x).
\]

**Example** Find the indefinite integral: 
\[
\int (4x^3 + 6x^2 - 3) \, dx
\]
\[
\int (4x^3 + 6x^2 - 3) \, dx = \int 4x^3 \, dx + \int 6x^2 \, dx - \int 3 \, dx = 4 \int x^3 \, dx + 6 \int x^2 \, dx - 3 \int 1 \, dx
\]
\[
= \frac{4x^4}{4} + \frac{6x^3}{3} - 3x + C
\]
\[
= x^4 + 2x^3 - 3x + C
\]
Note that here even though each of the antiderivatives in the above example had an integral constant "\( c \)", I have bundled them all up in one big constant \( C \) so that only one such constant appears and provides notation wise less complicated answer.
Example Find the indefinite integral: \( \int \sqrt{x} + \frac{1}{\sqrt{x}} \)

\[
\int \sqrt{x} + \frac{1}{\sqrt{x}} \, dx = \int \sqrt{x} \, dx + \int \frac{1}{\sqrt{x}} \, dx
= \int x^{1/2} \, dx + \int x^{-1/2} \, dx
= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C
= \frac{2}{3} x^{3/2} + 2x^{1/2} + C
\]

Example Find the indefinite integral: \( \int (\theta^2 + \sec^2 \theta) \, d\theta \)

\[
\int (\theta^2 + \sec^2 \theta) \, d\theta = \int \theta^2 \, d\theta + \int \sec^2 \theta \, d\theta
= \frac{\theta^3}{3} + \tan \theta + C
\]

Example Find the indefinite integral: \( \int \frac{\cos \theta}{1 - \cos^2 \theta} \, d\theta \)

Since the only rules we have seen so far are only the formulas provided above and the two properties we got from the last theorem our current knowledge is not enough to calculate the integral as it is. So we need to re-write the integrand by using trig identities in a different way that will hopefully help us. Recall \( 1 - \cos^2 \theta = \sin^2 \theta \) so plug this in the denominator

\[
\int \frac{\cos \theta}{1 - \cos^2 \theta} \, d\theta = \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta
= \int \frac{1}{\sin \theta \sin \theta} \, d\theta
= \int \csc \theta \cot \theta \, d\theta
= - \csc \theta + C
\]
We have learnt in this section so far that the antiderivatives of a function form a family of functions. As you will encounter in your later courses (such as Differential Equation class) you will be interested in finding one particular antiderivative function among these infinitely many. To be able to obtain this particular antiderivative we need more information about the original function $f(x)$ that just its derivative. Next two examples will give you an idea how to find such a particular antiderivative and what you might need to find it.

**Example** Find $y(x)$ if $\frac{dy}{dx} = 6x^2$ and $y(0) = -1$

The definition of antiderivative tells us $\int \frac{dy}{dx} \, dx = y(x)$. So this means $6x^2 \, dx = y(x)$. By using our experience with the antiderivatives so far we have $y(x) = 6\frac{x^3}{3} + C = 2x^3 + C$. In order to find the value for $C$ to find the particular antiderivative we need to use the information they gave us: $y(0) = -1$. This means we put in a 0 for $x$. You will get: $y(0) = 2(0)^3 + C = 0 + C$. We know that $y(0) = -1$ so $-1 = 0 + C \Rightarrow C = -1$. So the particular antiderivative we are after is: $y(x) = 2x^3 - 1$

**Example** Find $f(x)$ if $f''(x) = \sin x$ and $f'(0) = 1$ and $f(0) = 6$

This is a two step problem. First you will need to find $f'(x)$ and then $f(x)$. Do not forget to figure out the constants along the way. This is left as an exercise to you. Ask your TA to solve it in class if you need help.