Section 3.8 Related Rates

• Read and re-read the problem until you understand it.

• Draw and label a picture which gives the relevant information (if possible).

• Introduce notation. Assign a symbol to every quantity that is a function of time.

• Express the given info and the required rate in terms of derivatives and equations.

• Differentiate all variables implicitly as functions of time \( t \)

• Substitute the given into resulting equation and solve for the unknown rate.
**Example** A rocket is launched vertically upward from a point 2 miles west of an observer on the ground. What is the speed of the rocket when the angle of elevation of the observer’s line of sight to the rocket is $50^\circ$ and is increasing at $5^\circ$ per second?

If possible, always start with a graph or picture like the one below:

In the figure above the base $b$ represents the distance of the observer to the rocket’s launch pad and it is equal to 2. "$h$" represents the distance of the rocket to the ground and $\theta$ represent the angle of elevation of the observer’s line of sight. Both $h$ and $\theta$ are increasing in time. We are also given $\frac{d\theta}{dt} = 5^\circ$ per second. We are asked to find the speed of the rocket when $\theta = 50^\circ$. Speed is defined as the rate of change in distance of the rocket to the ground with respect to time i.e we are looking for $\frac{dh}{dt}$ when $\theta = 50^\circ$.

To find this we need an equation that relates $h$ to the other variable $\theta$ and the base $b = 2$. This equation we will gather using the geometry. In the given figure above we have a right angular triangle hence we will use the trigonometry to connect $h$ to others as follows

$$ \tan \theta = \frac{h}{2} $$

Now we will implicitly differentiate both sides of this equation with respect to time. Note that both variables $h$ and $\theta$ on both sides change with respect to time. This means when you do the differentiation expect $\frac{dh}{dt}$ and $\frac{d\theta}{dt}$ to pop-up every time you are differentiating an $h$ or $\theta$ term.

$$ \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{2} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = 2 \sec^2 \theta \frac{d\theta}{dt} $$

**Warning:** Calculus requires that all angles be in radians.

$$ \frac{dh}{dt} = 2 \sec^2(50^\circ \cdot \frac{\pi}{180})[5^\circ \cdot \frac{\pi}{180}] \approx 0.422 m/sec \approx 1520.7 mi/hr $$
Example a) A 13ft long ladder rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 5 ft per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 12ft from the wall.

As seen in the figure the length of the ladder is 13ft; and ”y” represents the height of the ladder on the wall at certain time t and ”x” represents distance between the feet of the ladder and the wall. Both ”y” and ”x” are changing with time as the ladder slides down. We are also given \( \frac{dx}{dt} = 5 \text{ft/sec} \). We are asked to find the ”speed” of the top of the ladder in this question as well. As in the previous example speed is the rate of change in the height (or distance to the ground) of the top of the ladder with respect to time. So we want to find \( \frac{dy}{dt} \) when \( x = 12 \text{ft} \). To calculate this derivative we again need an equation that relates ”y” to the other variable ”x” and the ladder’s length. At this point geometry becomes handy again and by using Pythagorean Theorem we write

\[
x^2 + y^2 = 13^2
\]

Now we will implicitly differentiate both sides of this equation with respect to time. Note that both variables w and f on both sides change with respect to time. This means when you do the differentiation expect \( \frac{dy}{dt} \) and \( \frac{dx}{dt} \) to pop-up every time you are differentiating an x or y term.

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}
\]

Note that we have a ”y” on the right hand side of this equality. Since we are asked to find \( \frac{dy}{dt} \) when \( x = 12 \text{ft} \), we will use the triangle and Pythagorean
Theorem to find the "w": \(12^2 + y^2 = 13^2 \Rightarrow y = 5\). Now we have everything to evaluate the derivative:

\[
\frac{dy}{dt} = -\frac{12}{5} \cdot (5) = -12 \text{ ft/sec}
\]

Again the negative sign shows that the ladder is moving down.

b) Consider the triangle formed by the ladder, wall and the floor. Find the rate at which the area of the triangle is changing when the base of the ladder is 12 ft from the wall?

The area of the triangle is \(A = \frac{1}{2}x \cdot y\) based on our figure in part (a). So for this problem we have a readily available "connection" equation and we will differentiate both sides of this equation with respect to time. Note that we need to use the Product Rule on the right hand side and since both w and f are changing with respect to time we should not forget \(\frac{dy}{dt}\) or \(\frac{dx}{dt}\) that comes because of the differentiation of a w or an f term.

\[
dA = \frac{1}{2} \cdot \frac{dy}{dt} + \frac{1}{2} \cdot \frac{dx}{dt}
\]

We know that \(x = 12\) and \(y = 5\) from the previous part. Also \(\frac{dx}{dt} = 5\) and \(\frac{dy}{dt} = -12\) from above. Putting this altogether we get \(\frac{dA}{dt} = \frac{1}{2}12(-12) + \frac{1}{2}5(5) = -\frac{119}{2} \text{ ft}^2/\text{sec}\). Once again the negative here means that the area is decreasing, which makes sense since the ladder is falling and creating a smaller triangle.
**Example** A police cruiser, approaching a right-angled intersection from the north is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 miles per hour. If the cruiser is moving at 60 miles per hour at the instant of measurement, what is the speed of the car?

Let "s" be the distance between the two cars and x be the distance of the car to the intersection and y be the distance of the cruiser’s. We have the question related info all listed on the figure. We will use the Pythagorean Theorem to find the "connection" equation again. \( x^2 + y^2 = s^2 \). Next take the derivative of both sides with respect to \( t \) to get

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2s \frac{ds}{dt}
\]

If \( x = 0.8 \) and \( y = 0.6 \), then we can find \( s \) by plugging these values into \( x^2 + y^2 = s^2 \) and solving it we will get \( s = 1 \). We also know that \( \frac{dy}{dt} = -60 \) since the car is moving south, or down as our picture states. In addition \( \frac{ds}{dt} = 20 \) since this is how much the distance between the cars is changing. We will put all of these known values in our differentiation result

\[
2(0.8) \frac{dx}{dt} + 2(0.6)(-60) = 2(1)(20) \Rightarrow \frac{dx}{dt} = 70 \text{mi/hr}
\]
**Example** Consider a conical tank. Its radius at the top is 4 feet, and its 10 feet high. Its being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is 5 feet high? Here is a rough sketch of the tank.

![Tank Sketch]

First note that we are given \( r_{\text{top}} = 4 \) and \( h_{\text{full}} = 10 \). Also \( \frac{dV}{dt} = 2 \text{ ft}^3/\text{min} \).

We are asked to find the rate of change in the height of the water so we want to find \( \frac{dh}{dt} \) when \( h = 5 \). Since we are given the derivative of the water’s volume it is a good idea to use the Conics Volume= \( \frac{1}{3} \pi r^2 h \) formula as the "connection" equation. One problem with it is that we have two variables \( r \) and \( h \) that are changing with time on the right hand side of the equation. So if we take the derivative we will have both \( \frac{dr}{dt} \) and \( \frac{dh}{dt} \) popping up and they are both unknown. We need to figure out either how to find \( \frac{dr}{dt} \) or write \( r \) in terms of \( h \) before doing the differentiation so we have only one variable to deal with. I’ll go with writing \( r \) as a function of \( h \) as the other option will require me to do this anyways. To achieve this we need to draw the two-dimensional cross-section of this tank. We will use the letters \( r \) and \( h \) to represent the variable radius and height of the water at any level. We can find the relationship between \( r \) and \( h \) from figure below using similar triangles.
So we have \( \frac{r}{h} = \frac{4}{10} \) or \( r = \frac{2}{5}h \)

Plug this expression for \( r \) back into \( V \) to get

\[
V = \frac{1}{3} \pi \left( \frac{2}{5}h \right)^2 h = \frac{4}{3(25)} \pi h^3
\]

Now let's differentiate both sides of this equation with respect to time

\[
\frac{dV}{dt} = \frac{4}{25} \pi h^2 \frac{dh}{dt}
\]

Now plug in the numbers \( \frac{dV}{dt} = 2 \) and \( h = 5 \) to get \( \frac{dh}{dt} = \frac{1}{2\pi} \).