Section 4.4 The Definite Integral

Definition For any function $f$ defined on the interval $[a, b]$, the definite integral of $f$ from $a$ to $b$ is

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i^*) \Delta x$$

whenever the limit exists and is the same for any choice of test points $c_i^*$ in the $i^{th}$ subinterval. When the limit exists, we say that $f$ is integrable on $[a, b]$.

Remarks

1) We have defined the sum on the right hand side as the Riemann Sum in Section 4.3 and mentioned that the sum approaches the area under $f$ as we let $n \to \infty$. So the definite integral we have defined above gives us the area under the curve $y = f(x)$ above the x-axis when $x$ changes from $a$ to $b$. (Note that we are assuming $f(x) \geq 0$. If the area has negative sign this means the area we are interested in is below the x-axis.—More on this later)

2) Since the definite integral is ultimately a limit if we know that $\int_{a}^{b} f(x) \, dx$ exists, then to evaluate the integral we can do so by choosing a special $\Delta x$ and special $c_i^*$.

First we will evaluate couple of integrals by using the first remark. Then we will see how we calculate an integral by using limit of a Riemann sum calculation.

Example Evaluate the integrals below by interpreting each in terms of areas.

a) Calculate $\int_{0}^{1} \sqrt{1-x^2} \, dx$

Notice first that

$$y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$$

$$x^2 + y^2 = 1$$

gives us a circle centered at the origin with radius one. But the area we are interested in is only $1/4$th of this circle because we want the part when $x$ is changing from 0 to 1 as in the figure below.
We know the area of the whole circle is $\pi r^2 = \pi 1^2 = \pi$. Since we only need 1/4th of it the area we are looking for is equal to $\pi/4$ or 
\[ \int_0^1 \sqrt{1-x^2} \, dx = \frac{\pi}{4} \]

b) Calculate $\int_3^0 |3x-5| \, dx$

We will use the same idea as above. First plot the area we are dealing with

Note that the area asked is the sum of the areas $A_1$ and $A_2$ or 
$\int_3^0 |3x-2| \, dx = A_1 + A_2$. And since both of these areas are of triangular shape we can easily calculate them

$A_1 = \frac{1}{2} \cdot 5 \cdot \frac{5}{3} = \frac{25}{6}$ and $A_2 = \frac{1}{2} \cdot 4 \cdot \frac{4}{3} = \frac{12}{6}$.

Then $\int_3^0 |3x-1| \, dx = \frac{25}{6} + \frac{12}{6} = \frac{37}{6}$
Now we will calculate the definite integrals using the definition and the tip in the second Remark, but we need a tool that will secure the existence of the integral first. And here it comes;
Theorem If \( f \) is continuous on \([a, b]\), then it is integrable on \([a, b]\).

For any curve that is continuous the area under it over a closed interval \([a, b]\) has to be finite and also term \( f(i)\Delta x \) in the Riemann Sum will always make sense i.e. defined

\[
\text{Out} = \int_a^b x \, dx
\]

Example Calculate \( \int_a^b x \, dx \) using the definition of definite integral

By the theorem above we know this integral exists because \( y = x \) is continuous everywhere hence on \([a, b]\). As we noted in Remark 2 after the definition of the definite integral since the limit exists if we choose a specific \( \Delta x \) and special \( c_i^* \)'s the limit should be equal to \( \int_a^b x \, dx \). Why? Because we can think of the Riemann Sum we got by our special choice of \( \Delta x \) and \( c_i^* \)'s as a "subsequence" of the general Riemann sum. And since the general one converges, the subsequence has to converge to the same limit, i.e the value of the integral.

We will choose \( \Delta x = \frac{b-a}{n} \) and in the below cases we will choose specific \( c_i^* \)'s. Here is how our interval looks like
Choose \( c^*_i = c_{i-1} \) Left-hand endpoint of each subinterval.

\[
\sum_{i=1}^{n} f(c^*_i) \Delta x = \sum_{i=1}^{n} c_{i-1} \cdot \frac{b-a}{n}
\]

\[
= \sum_{i=1}^{n} \left( a + (i-1) \frac{b-a}{n} \right) \cdot \frac{b-a}{n}
\]

\[
= \sum_{i=1}^{n} a \cdot \frac{b-a}{n} + \sum_{i=1}^{n} \left( \frac{b-a}{n} \right)^2 (i-1)
\]

\[
= \frac{a(b-a)}{n} n + \sum_{i=1}^{n} \left( \frac{b-a}{n} \right)^2 i - \sum_{i=1}^{n} \left( \frac{b-a}{n} \right)^2
\]

\[
= a(b-a) + \frac{(b-a)^2}{2} \frac{n(n+1)}{n} - n \left( \frac{b-a}{n} \right)^2
\]

\[
= a(b-a) + \frac{(b-a)^2}{2} \frac{n+1}{n} - \frac{(b-a)^2}{n}
\]

Take the limit \( n \to \infty \)

\[
\int_{a}^{b} x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c^*_i) \Delta x
\]

\[
= \lim_{n \to \infty} \left( a(b-a) + \frac{(b-a)^2}{2} \frac{n+1}{n} - \frac{(b-a)^2}{n} \right)
\]

\[
= a(b-a) + \frac{(b-a)^2}{2} \cdot 1 - 0
\]

\[
= (b-a) \left[ a + \frac{b-a}{2} \right] = (b-a) \left[ \frac{2a+b-a}{2} \right]
\]

\[
= (b-a) \frac{b+a}{2} = \frac{b^2-a^2}{2}
\]
Choose \( c_i^* = \frac{c_{i-1} + c_i}{2} \) Mid-point of each subinterval. Note here we will replace \( \Delta x = c_i - c_{i-1} = \frac{b-a}{n} \) which is length of each subinterval to simplify our calculations.

\[
\sum_{i=1}^{n} f\left( \frac{c_{i-1} + c_i}{2} \right) \Delta x = \sum_{i=1}^{n} \frac{c_{i-1} + c_i}{2} \cdot (c_i - c_{i-1})
\]

\[
= \sum_{i=1}^{n} \frac{c_i^2 - c_{i-1}^2}{2}
\]

\[
= \frac{1}{2} \left( (x_n^2 - x_0^2) + (x_2^2 - x_1^2) + (x_3^2 - x_2^2) + \ldots + (x_n^2 - x_{n-1}^2) \right)
\]

\[
= \frac{1}{2} (x_n^2 - x_0^2) = \frac{1}{2} (b^2 - a^2) \text{ because } x_n = b, x_0 = a
\]

Take the limit \( n \to \infty \)

\[
\int_a^b x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left( \frac{c_{i-1} + c_i}{2} \right) \Delta x
\]

\[
= \lim_{n \to \infty} \frac{1}{2} (b^2 - a^2) = \frac{b^2 - a^2}{2}
\]

Choose \( c_i^* = c_i \) Right-hand endpoint of each subinterval.

\[
\sum_{i=1}^{n} f(c_i^*) \Delta x = \sum_{i=1}^{n} c_i \cdot \frac{b-a}{n}
\]

\[
= \sum_{i=1}^{n} \left( a + \frac{b-a}{n} \right) \cdot \frac{b-a}{n}
\]

\[
= a(b-a) + \left( \frac{b-a}{n} \right)^2 \frac{n(n+1)}{2}
\]

Take the limit \( n \to \infty \)

\[
\int_a^b x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i^*) \Delta x
\]

\[
= \lim_{n \to \infty} \left( a(b-a) + \left( \frac{b-a}{n} \right)^2 \frac{n(n+1)}{2} \right)
\]

\[
= \frac{b^2 - a^2}{2}
\]

So we have showed in three different ways \( \int_a^b x \, dx = \frac{b^2 - a^2}{2} \)
Example Show that \( \int_a^b k \, dx = k(b-a) \) for any constant \( k \).

We will use the same \( \Delta x \) as before i.e \( \Delta x = \frac{b-a}{n} \) and the right hand endpoint of each subinterval as \( c_i^* = c_i \)

\[
\sum_{i=1}^{n} f(c_i^*) \Delta x = \sum_{i=1}^{n} k \cdot \frac{b-a}{n} = k \frac{b-a}{n} \cdot n = k(b-a) \rightarrow k(b-a) \text{ as } n \rightarrow \infty
\]

Example Show that \( \int_a^b x^2 \, dx = \frac{b^3 - a^3}{3} \)

This is left as an exercise for you to prove.

Cautionary Example Let \( f(x) = \begin{cases} A & \text{if } a \leq x \leq c \\ B & \text{if } c < x \leq b \end{cases} \) where \( A < B \).

Does \( \int_a^b f(x) \, dx \) exists?

Note that \( f \) is discontinuous at \( x = c \). So our theorem does not promise us any result. But looking at the graph of \( f \) below we see that \( \int_a^b f(x) \, dx \) exits and \( \int_a^b f(x) \, dx = A_1 + A_2 = A(c-a) + B(b-c) \)

Next we will talk about a little what does "negative area" mean?

If \( f \) takes both positive and negative values as in the figure below, then the Riemann Sum is the sum of the area of the rectangles that lie above the \( x \)-axis, and the negatives of the areas of the rectangles that lie below the \( x \)-axis.
When we take the limit of such Riemann Sums, we get

$$\int_{a}^{b} f(x) \, dx = A_2 - A_1$$

where $A_2$ is the area of the region above the x-axis and below the graph of $f$ and $A_1$ is the area of the region below the x-axis and above the graph of $f$.

**Side Note** If we are interested in the total area we need to calculate $\int_{-1}^{3} |2 - x| \, dx$

**Example** Evaluate the integral by interpreting each in terms of area $\int_{-1}^{3} (2 - x) \, dx$

Above we have the graph of the area we are calculating. By discussion preceding this example we know

$$\int_{-1}^{3} (2 - x) \, dx = A_1 - A_2 = \frac{3 \cdot 3}{2} - \frac{1 \cdot 1}{2} = \frac{8}{2} = 4$$
Properties of $\int_a^b f(x) \, dx$

Suppose $\int_a^b f(x) \, dx$ and $\int_a^b g(x) \, dx$ exist then

1) $\int_a^b c f(x) \, dx$ exists and $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$

2) $\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

3) $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$

4) $\int_a^a f(x) \, dx = 0$

5) If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x) \, dx \geq 0$

6) If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$

7) If $m \leq f(x) \leq M$ on $[a, b]$ then $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$

8) $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$ where $a < c < b$

We will evaluate couple of integrals next using the properties of definite integrals.

**Examples**

1) $\int_0^1 (x - 5) \, dx = \int_0^1 x \, dx - \int_0^1 5 \, dx = \frac{1^2 - 0^2}{2} - 5(1 - 0) = \frac{-9}{2}$ by Property 2

2) $\int_0^2 6x^2 = 6 \int_0^2 x^2 \, dx = 6 \frac{2^3 - 0^3}{3} = 16$ by Property 1

3) $\int_2^0 5 dx = -\int_2^0 5 \, dx = -5(2 - 0) = -10$ by Property 3

4) $\int_2^1 x^3 \sin^{-1} x \, dx = 0$ by Property 4

5) Write the given difference of integrals as a single integral of the form $\int_a^b f(x) \, dx$.

\[ \int_2^1 f(x) \, dx - \int_2^7 f(x) \, dx \]

Use Property 8 to write the first integral as $\int_2^7 f(x) \, dx = \int_2^7 f(x) \, dx + \int_7^{10} f(x) \, dx$ then

\[ \int_2^{10} f(x) \, dx - \int_2^7 f(x) \, dx = \int_2^7 f(x) \, dx + \int_7^{10} f(x) \, dx - \int_2^7 f(x) \, dx = \int_7^{10} f(x) \, dx \]

6) If $\int_0^1 f(t) \, dt = 2$, $\int_0^4 f(t) \, dt = -6$ and $\int_3^4 f(x) \, dx = 1$, find $\int_1^3 f(t) \, dt$.

First note that $\int_1^3 f(t) \, dt = \int_0^3 f(t) \, dt - \int_0^1 f(t) \, dt$

And since $\int_0^4 f(t) \, dt = \int_0^3 f(t) \, dt + \int_3^4 f(t) \, dt$ by Property 8

$\int_0^3 f(t) \, dt = -6 - 1 = -7$, $\int_1^3 f(t) \, dt = -7 - 2 = -9$
7) Use Properties of integrals to verify the inequality below without evaluating the integrals.

\[ \int_1^2 \sqrt{5-x} \, dx \geq \int_1^2 \sqrt{x+1} \, dx \]

We will use the Property 6 above to prove the inequality. So we have to show \( \sqrt{5-x} \geq \sqrt{x+1} \) on the interval \([1, 2]\). And we show this below: On this interval

\[ 2 \geq x \Rightarrow 2+2 \geq x+x \Rightarrow 5-1 = 4 \geq x+x \Rightarrow 5-x \geq x+1 \Rightarrow \sqrt{5-x} \geq \sqrt{x+1} \]

Note that we may take the square root of both sides because \( x \geq 1 \Rightarrow x + 1 \geq 2 > 0 \)

Hence by Property 6 \( \int_1^2 \sqrt{5-x} \, dx \geq \int_1^2 \sqrt{x+1} \, dx \).

8) Use Properties of integrals to verify the inequality below without evaluating the integrals.

\[ \frac{\pi}{6} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \, dx \leq \frac{\pi}{3} \]

We will use the Property 6 to verify this inequality. And we need to find the two bounds \( m \) and \( M \) first to be able to use this property. Since the integral we are trying to find bounds for is for \( \sin x \) on the interval \([\frac{\pi}{6}, \frac{\pi}{2}]\) we will plot the graph of the it on this interval.

From the graph we observe that \( \frac{1}{2} \leq \sin x \leq 1 \) on \([\frac{\pi}{6}, \frac{\pi}{2}]\). So \( m = \frac{1}{2} \) and \( M = 1 \). Then by Property 6

\[ \frac{1}{2} (\frac{\pi}{2} - \frac{\pi}{6}) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \, dx \leq 1 \cdot (\frac{\pi}{2} - \frac{\pi}{6}) \Rightarrow \frac{\pi}{6} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \, dx \leq \frac{\pi}{3} \]
9) Use Properties of integrals to verify the inequality below without evaluating the integral.

\[ \int_{0}^{\pi/2} x \sin x \, dx \leq \frac{\pi^2}{8} \]

We will use Property 6 to prove the result. Since \( \sin x \leq 1 \) for \( x \) in \( [0, \pi/2] \).

So \( x \sin x \leq x \cdot 1 = x \) on this interval. Hence

\[ \int_{0}^{\pi/2} x \sin x \, dx \leq \int_{0}^{\pi/2} x \, dx \leq \frac{\left(\frac{\pi}{2}\right)^2}{2} - 0^2 = \frac{\pi^2}{8} \]

10) Use Property 7 to evaluate the integral \( \int_{0}^{3} (x^2 + 2x) \, dx \)

To use Property 7 again we need to find the lower and upper bounds \( m \) and \( M \) for the function \( y = x^2 + 2x \). As before we could use the graph of this function to figure these bounds but instead I would like to use Calculus and what we learnt so far to show another way of finding these bounds. Finding \( m \) and \( M \) can be thought of finding the absolute maximum and minimum of \( y = x^2 + 2x \) over the interval \( [-3, 0] \). Since we have a continuous polynomial function we may use EVT and we know \( y = x^2 + 2x \) has an absolute maximum and minimum on this interval. So \( f'(x) = 2x + 2 = 0 \Rightarrow x = -1 \) then comparison gives \( f(-3) = 9 - 6 = 3 \) absolute max

\( f(-1) = 1 - 2 \) absolute min

\( f(0) = 0 \)

Hence \( -1 \leq x^2 + 2x \leq 3 \) on \([-3,0]\). Using Property 7 now we get

\[ -1(0 + 3) \leq \int_{-3}^{0} (x^2 + 2x) \, dx \leq 3(0 + 3) \Rightarrow -3 \leq \int_{-3}^{0} (x^2 + 2x) \, dx \leq 9 \]
Average Value of a Function and Mean Value Theorem for Integrals

Recall: Given a set of values \( y_1, y_2, ..., y_n \) the average of these values is defined as \( y_{\text{avg}} = \frac{y_1 + y_2 + ... + y_n}{n} \).

Consider a Riemann Sum for a definite integral

\[
\sum_{i=1}^{n} f(i) \cdot \frac{b - a}{n} = (b - a) \sum_{i=1}^{n} \frac{f(i)}{n}
\]

Note that the fraction \( \frac{f(i)}{n} \) in the sum on the right hand side is the Average of the function \( f \) at \( n \)-test points. This observation leads us to a more interesting one below

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(i) \cdot \frac{b - a}{n} = (b - a) \lim_{n \to \infty} \sum_{i=1}^{n} \frac{f(i)}{n}
\]

Divide this equation by \( b - a \)

\[
\frac{1}{b - a} \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{f(i)}{n}
\]

So the right hand side expression in the limit gives us the Average value of \( f \) on \([a, b]\). So based on this argument

**Definition** If \( f \) is integrable on the interval \([a, b]\), then the **average value of** \( f \) on \([a, b]\) is

\[
f_{\text{avg}} = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx
\]

**Mean Value Theorem for Integrals** If \( f \) is continuous on the interval \([a, b]\), then there is a \( c \) in \((a, b)\) such that

\[
f(c) = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx
\]