Section 3.5 Concavity and 2nd Derivative Test

In the previous section we have used the first derivative to gather information about the graph of f. The First Derivative Test tells us where our function is increasing or decreasing not how it curves or bends. In this section we will learn to use the second derivative to gather this information. We will start our journey with geometrical explorations again. Here are couple of typical ways a curve can bend.

The first two curves together we will call ”concave up” and the last two as ”concave down” based on the observations below
Note that the tangent lines drawn to first graph are all below the graph. Also the slopes of tangent lines are increasing as \( x \) moves from left to right. Whereas in the second graph all the tangent lines are above the graph. Also the slopes of tangent lines are decreasing as \( x \) moves from left to right. So here is the definition;

**Definition** If the graph of \( f \) lies

(i) **above** all of its tangents on an interval \( I \), then it is **concave upward** on \( I \).
(ii) **below** all of its tangents on an interval \( I \), then it is **concave downward** on \( I \).

**Remarks** • By the discussion just before the definition we know that if \( f \) is concave upward then the slope’s of the tangent lines is increasing, this means \( f' \) is increasing. Hence the derivative of \( f' \) –which is \( f'' \) – is positive on the interval \( f \) is concave up.

• Similarly, if \( f \) is concave downward then the slope’s of the tangent lines is decreasing, this means \( f' \) is decreasing. Hence the derivative of \( f' \) –which is \( f'' \) – is negative on the interval \( f \) is concave down.

**Theorem (The Concavity Test)** a) If \( f''(x) > 0 \) for all \( x \) in an interval \( I \), then the graph of \( f \) is **concave up** on \( I \).

b) If \( f''(x) < 0 \) for all \( x \) in an interval \( I \), then the graph of \( f \) is **concave down** on \( I \).
**Example** Determine where the function $F(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is concave up and concave down?

Note from the Concavity Test that the sign of the second derivative determines the concavity, hence we need to first find the second derivative and then where it is changing it’s sign.

$$F'(x) = 12x^3 - 12x^2 - 24x \Rightarrow F''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2).$$

The derivative might change its sign either at a root or where it is not defined. Since our function is a polynomial it is defined everywhere. We will search for the other possibility

$$f''(x) = 0 \Rightarrow 3x^2 - 2x - 2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{7}}{3}$$

To check for the sign change we will use a sign chart again

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>$F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$ to $\frac{1 - \sqrt{7}}{3}$</td>
<td>$-$</td>
<td>$\cup$</td>
</tr>
<tr>
<td>$\frac{1 - \sqrt{7}}{3}$</td>
<td>$+$</td>
<td>$\cup$</td>
</tr>
<tr>
<td>$\frac{1 + \sqrt{7}}{3}$</td>
<td>$+$</td>
<td>$\cup$</td>
</tr>
<tr>
<td>$\frac{1 + \sqrt{7}}{3}$ to $\infty$</td>
<td>$-$</td>
<td>$\cup$</td>
</tr>
</tbody>
</table>

So $F$ is concave up on the intervals $(-\infty, \frac{1 - \sqrt{7}}{3})$ and $(\frac{1 + \sqrt{7}}{3}, \infty)$ and it is concave down on the interval $(\frac{1 - \sqrt{7}}{3}, \frac{1 + \sqrt{7}}{3})$. Note that the concavity changes at $x = \frac{1 - \sqrt{7}}{3}$ and $x = \frac{1 + \sqrt{7}}{3}$.

**Definition** A function has an inflection point at ”c” if the concavity of $f(x)$ changes at ”c”.

Warning! $f''(c) = 0$ does not immediately imply $c$ is a point of inflection. To see this consider $f(x) = x^4$. The graph of $f$ is a parabola and it is always concave up yet $f''(0) = 0$. Create the sign chart for the second derivative yourself and see that the sign of $f''(x)$ is positive on both sides of zero.
**Example** Determine where the function \( F(x) = |x^2 - 1| \) is concave up and/or concave down? Also find points of inflection. Then using two derivative tests we have given so far give a rough sketch of \( F(x) \)

To find the derivatives of \( F \) we need to re-write \( F \) as follows

\[
F(x) = |x^2 - 1| = \begin{cases} 
  x^2 - 1 & \text{if } x^2 \geq 1 \\
  -x^2 + 1 & \text{if } x^2 < 1 
\end{cases} = \begin{cases} 
  x^2 - 1 & \text{if } x \leq -1 \text{ or } x \geq 1 \\
  -x^2 + 1 & \text{if } -1 < x < 1 
\end{cases}
\]

So \( F'(x) = \begin{cases} 
  2x & \text{if } x < -1 \text{ or } x > 1 \\
  \text{dne} & \text{if } x = -1 \text{ or } x = 1 \\
  -2x & \text{if } -1 < x < 1 
\end{cases} \)

Since we will need the information from first derivative test to graph \( F \) let’s find its critical values first. Clearly, \( x = \pm 1 \) are critical points as \( F' \) does not exist there. Also \( F'(x) = 0 \) implies \( -2x = 0 \) implies \( x = 0 \) is another critical point. I will postpone the sign chart construction to later for reasons which will become soon clear. Next the second derivative

\[
F''(x) = \begin{cases} 
  2 & \text{if } x < -1 \text{ or } x > 1 \\
  \text{dne} & \text{if } x = -1 \text{ or } x = 1 \\
  -2 & \text{if } -1 < x < 1 
\end{cases}
\]

So \( F''(x) \) is never equal to zero but it fails to exist at \( x = \pm 1 \). I will next create a sign chart that gathers information from first and second derivative together. I’ll explain after the table how the choice about the shape of \( F \) is made on each interval

The red rectangle in each of the boxes indicates the shape of \( F \) on that interval. How is it determined: On the interval \((-\infty, -1)\) the first derivative is negative, so \( F \) is \( \searrow \); the second derivative on this interval is positive so
F is concave up. But the concave up shape consist of two parts as in the following picture

Note that only the left side of this shape is decreasing not the right side, so we have picked up for F only the left side of the concave up shape. Similarly, on the interval \((-1,0)\) the first derivative is positive so F is \(\nearrow\); the second derivative on the other hand is negative so F is concave down. But (again) the concave down shape consist of two parts as seen below

and only left side of this shape is increasing not the right side, so we have picked up for F the left side of the concave down shape. The rest of the red-rectangle choices are processed in the similar manner. Note that here \(x = 1\) and \(x = -1\) are point of inflections (because concavity changes there). At \(x = 0\) F has a local maximum and at \(x = \pm 1\) F has local mins. Here is a rough sketch of F following the guidelines from the sign chart:
**Second Derivative Test** For a function $f$ let $f''$ exists and be continuous on an interval $(a, b)$ and $f$ be continuous on $[a, b]$. Then if $c$ in $(a, b)$ is a critical point of $f$, then

a) If $f''(c) > 0$ then $f$ has a local minimum at $x = c$

b) If $f''(c) < 0$ then $f$ has a local maximum at $x = c$

a) If $f''(c) = 0$ then you cannot make a conclusion about $c$.

**Remarks** 1) The upside to this test is that it is very quick (provided you have the second derivative) to show whether a critical point is a max or min. The downside is that it might happen that $f''(c) = 0$. In this situation, one cannot conclude from the second derivative alone whether $f$ has a max or a min at $c$.

Why does the test work? Well, if $f''(c) > 0$ this means $f'(x) \nearrow$ on a neighborhood around $c$. Also since $c$ is a critical point $f'(c) = 0$. So the graph of $f'(x)$ around $c$ looks like roughly like the graph below.

So $f'$ changes sign from negative to positive which is by the first derivative test is where we have a local minimum.

Similarly, if $f''(c) < 0$ this means $f'(x) \searrow$ on a neighborhood around $c$. Also since $c$ is a critical point $f'(c) = 0$. So the graph of $f'(x)$ around $c$ looks like roughly like the graph below.
So $f'$ changes sign from positive to negative which is by the first derivative test is where we have a local maximum.

**Example** Use the Second derivative Test to classify the critical points of $f(x) = \frac{x}{x^2 + 4}$.

First find the critical points

$$f'(x) = \frac{1 \cdot (x^2 + 4) - x(2x)}{(x^2 + 4)^2} = \frac{-x^2 + 4}{(x^2 + 4)^2}$$

$f'(x) = 0 \Rightarrow -x^2 + 4 = 0 \Rightarrow x = -2$ and $x = 2$. $f'(x)$ dne is not possible because the denominator is never equal to zero. To classify these two critical points we need the second derivative

$$f''(x) = \frac{-2x(x^2 + 4)^2 - (-x^2 + 4)(2(x^2 + 4)2x)}{(x^2 + 4)^4}$$

$$= \frac{-2x(x^2 + 4) - (-x^2 + 4)4x}{(x^2 + 4)^3}$$

$$= \frac{-2x^3 - 8x + 4x^3 - 16x}{(x^2 + 4)^3}$$

$$= \frac{2x^3 - 24x}{(x^2 + 4)^3} = \frac{2x(x^2 - 12)}{(x^2 + 4)^3}$$

Now the second derivative test

$f''(-2) = \frac{(+)(-)}{(+) (+)} > 0$ so $x = -2$ is a local minimum

$f''(2) = \frac{(+)(-)}{(+) (+)} < 0$ so $x = 2$ is a local maximum

Note that I didn’t actually calculated the second derivatives at $x = \pm 2$ I have only checked out the signs which is all I needed.