Section 3.4 Increasing and Decreasing Functions

Definition • A function \( f(x) \) is increasing if \( f(x_0) \leq f(x_1) \) whenever \( x_0 < x_1 \). It is strictly increasing if \( f(x_0) < f(x_1) \) whenever \( x_0 < x_1 \).

• A function \( f(x) \) is decreasing if \( f(x_0) \geq f(x_1) \) whenever \( x_0 < x_1 \). It is strictly decreasing if \( f(x_0) > f(x_1) \) whenever \( x_0 < x_1 \).

Note that "an increasing" function preserves the "order" between two points, whereas a decreasing one "reverses" it. So intuitively, a function is increasing if outputs don't get smaller as inputs get bigger. Similarly, a function is decreasing if outputs don't get bigger as inputs get bigger.

Geometrical Observations:

The function below is a graph of a function that is increasing. It also has the tangent lines drawn to it at several points. Note that the slopes of these tangent lines are all positive. Hence \( f' \) appears to be positive everywhere.
Next we have a graph of a function that is decreasing. It also has the tangent lines drawn to it at several points. Note that the slopes of these tangent lines are all negative. Hence $f'$ appears to be negative everywhere.

So the following theorem should come as no surprise:

**Theorem** Suppose that $f$ is a differentiable function on an interval $I$

1) If $f'(x) > 0$ for all $x$ in $I$, then $f$ is increasing $(\nearrow)$ on $I$.
2) If $f'(x) < 0$ for all $x$ in $I$, then $f$ is decreasing $(\searrow)$ on $I$.

**Proof** We have proved the first result as a corollary of Mean Value Theorem in class. Here to remind ourselves MVT we will prove the second one:

Let $f'(x) < 0$ on the interval $I$. Pick any two points $x_1$ and $x_2$ in $I$ where $x_1 < x_2$. Applying MVT on the interval $[x_1, x_2]$ (note that this small interval in contained in our bigger interval $I$) because $f$ is continuous and differentiable on $I$ hence on $[x_1, x_2]$, we know that there is a $c$ in the interval $[x_1, x_2]$ such that

$$\left(\star\right) \quad f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

By our assumption $f'(c) < 0$ also $x_2 - x_1 > 0$ because $x_1 < x_2$ implies by the $(\star)$ above that $f(x_2) - f(x_1) < 0$. Hence $f(x_2) < f(x_1)$. Since our choices of $x_1$ and $x_2$ are arbitrary this result will hold for all possible choices in $I$.

So for any $x_1 < x_2$, $f(x_1) > f(x_2)$. By definition then $f$ is decreasing on $I$.

This theorem suggests that we can gather information about $f(x)$ using $f'(x)$. The next theorem will give the full recipe of how to do exactly that. To make sure you have a better understanding of the next result we will
have couple of geometrical explorations that will pave the road to it.

Note that in each of the following cases the argument will be around a "critical point" (recall that a critical point is a point where either $f'(x) = 0$ or $f'(x)$ does not exist.)

Case 1 Assume that at the critical point $x = c$, the function $f(x)$ has a maximum. Then a typical picture you will get, will be one of the following:

Case 2 Assume that at the critical point $x = c$, the function $f(x)$ has a minimum. Then a typical picture you will get, will be one of the following:
Case 3 Assume that at the critical point $x = c$, the function has neither a minimum nor a maximum. The a typical picture you will get, will be one of the following:

or one of these
**First Derivative Test** Let $f(x)$ be continuous on the interval $[a, b]$ and $c$ a critical point of $f(x)$ on the open interval $(a, b)$, then

1) $f(c)$ is a *local maximum* if

$$f'(x) > 0 \text{ for all } x \in (a, c) \text{ and } f'(x) < 0 \text{ for all } x \in (c, b).$$

2) $f(c)$ is a *local minimum* if

$$f'(x) < 0 \text{ for all } x \in (a, c) \text{ and } f'(x) > 0 \text{ for all } x \in (c, b).$$

3) $f(c)$ is not a local extremum if $f'(x)$ has the same sign on $(a, c)$ and $(c, b)$.

**Example** Find where the function $F(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing and classify the critical points of $F$.

We have to first find the critical points. Since $F(x)$ is a polynomial the derivative does not exist case is not a possibility. So we will only explore $F'(x) = 0$ one.

$$F'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0 \Rightarrow 12x(x^2 - x - 2) = 0 \Rightarrow 12x(x+1)(x-2) = 0$$

So the critical values are $x = 0$, and $x = -1$ and $x = 2$. To classify these critical points we’ll use the First Derivative Test. It is much easier to observe the test in action on a sign chart.

![Sign chart for F'(x) and F(x)]
This chart is formed as follows: I think the first row (x-row) as the real line. On the very left we have \(-\infty\) and as we move towards right the numbers get bigger and on the very right we have \(+\infty\). In the \(F'\) row: I have ”zeros” in the middle of each vertical line emanating from each of the critical values on the top row indicating the fact that these values make the derivative equal to zero. I have determined the signs in this row \((F' \text{ row})\) by picking up test points from each interval and plugging it into the derivative and checking the sign. For example from the interval \((-\infty, -1)\) I picked up \(x = -2\), then \(F'(-2) = 12(-2)(-2 + 1)(-2 - 2) < 0\), hence I have the negative sign (-) between \(-\infty\) and -1, similarly from the interval \((-1, 0)\), I picked up \(x = -1/2\), then \(F'(-1/2) = 12(-1/2)(-1/2 + 1)(-1/2 - 2) > 0\), hence I have the positive sign (+) between -1 and 0. The rest of the derivative row signs completed in a similar manner. Then for \(F\) row I have used the information from the first theorem; when the derivative is negative function is decreasing and when it is positive it is increasing. So \(F\) is decreasing \((\searrow)\) on the intervals \((-\infty, -1)\) and \((0, 2)\) and it is increasing \((\nearrow)\) on the intervals \((-1, 0)\) and \((2, \infty)\)

Now, by using the First Derivative Test we can classify the critical points as follows: \(x = -1\) is a local minimum, \(x = 0\) is a local maximum and \(x = 2\) is a local minimum. Here is a rough sketch of \(F(x)\) based on this investigation:

**Example** Find where the function \(F(x) = \frac{(x-1)^2}{(x+1)}\) is increasing and decreasing and classify the critical points of \(F\).

The first step is the same; find the critical points first. One of the critical points is easy to recognize from the function itself because the function \(F(x)\) is not defined at \(x = -1\) hence it is not differentiable at this point. So \(F'(-1)\) does not exist, meaning \(x = -1\) is a critical point. To find the rest of the critical points I need to do the work.
\[ F'(x) = \frac{2(x-1)(x+1)-(x-1)^2}{(x+1)^2} = \frac{(x-1)(x+3)}{(x+1)^2} \]

\[ F'(x) = 0 \Rightarrow (x - 1)(x + 3) = 0 \Rightarrow x = 1 \text{ and } x = -3 \]

Also the derivative confirms our previous observation \( F'(-1) \) is not defined. So the critical values are \( x = -3, -1, 1 \). To classify these points and where the function is decreasing (\( \searrow \)) or increasing (\( \nearrow \)) I’ll use a similar chart like the one in the previous example. The only difference between this sign chart’s construction and the previous one is: I have double-vertical lines emanating from the critical value \( x = -1 \). This is to distinguish this critical value from the other two(-3 and 1), because this one makes my derivative not defined and other two make it equal to zero.

So according to sign chart: \( F \) is \( \nearrow \) on the intervals \((-\infty, -3)\) and \((1, \infty)\). \( F \) is \( \searrow \) on the intervals \((-3, -1)\) and \((-1, 1)\). By using the First Derivative Test at \( x = -3 \) we have a local maximum and at \( x = 1 \) we have a local minimum.

Here we have a rough sketch of \( F \) based on the information above.