Section 3.1 Linear Approximations

Since the beginning of this course I have been talking about the "tangent problem". Now that we have solved the tangent problem, meaning we know how to write the equation of tangent line to any curve at any given point (*some restriction applies) I would like to get a use out of those tangent lines. First let's observe the relationship between the tangent line and the curve in the applet below

For the applet go to the web page : www.calculusapplets.com (and search for linear approximation).

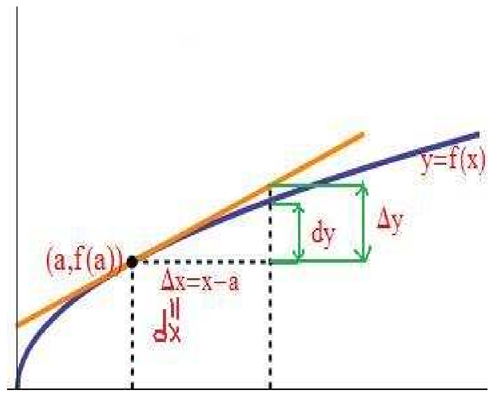
Now the important observation: If we zoom in "enough" around the point $(a, f(a))$ to the curve, the curve and the tangent line becomes nearly inseperable from each other. This is not only true for the point $(a, f(a))$ but also for all nearby points. So the tangent to the graph of a function $f(x)$ at a point $(a, f(a))$ is a decent approximation to $f(x)$ at points near a.

This has a lot of great applications, but one of the most straightforward is that we can use the tangent line to $f(x)$ at "a" to approximate the values of $f(x)$ at points close to a. Since its frequently hard to evaluate functions at random points without a calculator, this will give us a technique to approximate certain quantities with only a little calculus know-how.

There are plenty of functions that we can evaluate easily. For instance, evaluating polynomials is not very difficult, since they only involve operations like addition, subtraction, and multiplication. For the same reason, computing the value of rational functions is also relatively easy. But is it easy to compute the value of functions like $\sqrt{x}$ or $\ln(x)$. It is not. We will use the tool we will introduce in this section to calculate them approximately.

Definition The linear or tangent line approximation of $f(x)$ at "a" is

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$



Remarks: 1) This equation is mathematical way of writing what we have observed from the applet: The tangent line approximates f(x). Note that the equation of the tangent line(T) drawn to $y = f(x)$ at the point $(a, f(a))$ is

$$y_T(x) - f(a) = f'(a)(x-a) \Rightarrow y_T(x) = f(a) + f'(a)(x-a)$$

The tangent line approximation formula says, the function value at x i.e. $f(x)$ is very close to the value of x on the tangent line i.e. $y_T(x)$
2) The formula gives a good approximation near the tangent point "a". As you move away from "a"however, the approximation grows less accurate.
3) This approximation can be written alternatively in the following form;

$$\Delta y \approx dy = f'(a)\Delta x = f'(a)dx$$

where $\Delta y = f(x) - f(a)$ and $\Delta x = dx = x - a$. The term $dy = f'(a)\Delta x$ plays a special role in applied calculus and it is called the *differential of y*.

Example Find the linear approximation to $f(x) = \ln(x)$ at $a = 1$.

$f(1) = \ln(1) = 0$ also $f'(1) = \frac{1}{x}|_{x=1} = 1$ .So

$$\ln(x) \approx f(1) + f'(1)(x-1) = 0 + 1 \cdot (x-1) = x - 1$$

The approximation says for $x = 1$ and all nearby points you can think of $\ln(x)$ like the linear function $x - 1$. Here you may call the point a as the base point and you may change it as follows: Let $x = 1 + u \Rightarrow u = x - 1$

then $\ln(1+u) \approx u$ with the new base point $u_0 = a - 1 = 0$.

Basic List of Linear Approximations

The following list of approximations is generated using $a = 0$ as the base point and assume that $|x| \ll 1$ (much smaller than 1).

1) $\sin x \approx x($ if $x \approx 0)$
2) $\cos x \approx 1($ if $x \approx 0)$
3) $e^x \approx 1 + x($ if $x \approx 0)$
4) $\ln(1+x) \approx x($ if $x \approx 0)$
5) $(1+x)^r \approx 1 + rx($ if $x \approx 0)$

Proofs Proof of 1) Take $f(x) = \sin(x)$ then $f'(x) = \cos(x)$, $f(0) = 0$ and $f'(0) = 1$

$$\sin(x) \approx f(0) + f'(0)(x - 0) = 0 + 1 \cdot (x) = x$$

The proofs of 2-3 are similar to the one above. We have proved 4th one in our first example.

Proof of 5) Let $f(x) = (1+x)^r$, then $f(0) = 1$ and $f'(x) = r(1+x)^{r-1} \Rightarrow f'(0) = r$. So ;

$$(1+x)^r \approx f(0) + f'(0)(x - 0) = 1 + rx$$

Example Find the linear approximation of $f(x) = \frac{e^{-2x}}{\sqrt{1+x}}$ near $a = 0$.

One way to go about is to do the approximation by finding $f(0)$ and $f'(0)$ and using the formula. I'll show you here another way by using the list we generated above:

$$e^{-2x} \approx 1 + (-2x)(e^u \approx 1 + u \text{ where } u = -2x)$$

by (3) above in the list.Then by (5)

$$\sqrt{1+x} = (1+x)^{1/2} \approx 1 + \frac{1}{2}x$$

Put these two approximation together:

$$\frac{e^{-2x}}{\sqrt{1+x}} \approx \frac{1-2x}{1+\frac{1}{2}x} = (1-2x)(1+\frac{x}{2})^{-1}$$

Moreover $(1+\frac{1}{2}x)^{-1} \approx 1-\frac{1}{2}x$, (using $(1+u)^{-1} \approx 1-u$ with $u = x/2$). Hence;

$$\frac{e^{-2x}}{\sqrt{1+x}} \approx (1-2x)(1-\frac{1}{2}x) = 1-2x-\frac{1}{2}x+x^2$$

Now, we discard that last x^2 term,because we have already thrown out a number of other x^2 (and higher order) terms in making these approximations. Remember, were assuming that $|x| \ll 1$. This means that x^2 very small, x^3 even smaller, etc. We can ignore these higher-order terms, because they are very, very small. This yields

$$\frac{e^{-2x}}{\sqrt{1+x}} \approx 1-2x-\frac{1}{2}x = 1-\frac{5}{2}x$$

You can now quickly read from this approximation $f(0) = 1$ and $f'(0) = \frac{-5}{2}$

Example Find the limit $\lim\limits_{x\to 0} \frac{(1+2x)^{10}-1}{x}$ by approximating the quotient.

$$(1+2x)^{10} \approx 1+10(2x)(\text{ Use } (1+u)^r \approx 1+ru \text{ where } u=2x \text{ and } r=10)$$

So

$$\frac{(1+2x)^{10}-1}{x} = \frac{1+20x-1}{x} = 20$$

Example Approximate the value of $e^{0.5}$.
We will use the list we have created above and use $e^x \approx 1+x$. So
$e^{0.5} \approx 1+0.5 = 1.5$
Example Approximate the value of $\sqrt{4.1}$

Let $f(x) = \sqrt{x}$ so that were attempting to approximate $f(4.1) = \sqrt{4.1}$.Notice that I know the value of $f(x)$ and $f'(x)$ at $x = 4$, and so I can solve for the equation of the line tangent to the graph of $f(x)$ at $x = 4$. Specifically, since $f(4) = 2$ and $f'(4) = \frac{1}{4}$, the equation of the line tangent to $f(x)$ at $x = 4$ is

$$y-2 = \frac{1}{4}(x-4) \text{ or } y = \frac{1}{4}(x-4)+2$$

Now we use linearization: this tangent line is supposed to be a good approximation to $f(x)$ near $x = 4$, and so we can estimate $\sqrt{4.1}$ by plugging 4.1 into the equation of the tangent line. This gives

$$\sqrt{4.1} \approx \frac{1}{4}(4.1-4)+2 = 2\frac{1}{40}$$

Example Lets say we are on Planet Q, and that a satellite is whizzing overhead with a velocity v. We want to find the time dilation that the clock onboard the satellite experiences relative to my wristwatch. We borrow the following equation from special relativity

$$T_m = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here, T_m is the time I measure on my wristwatch, and T is the time measured onboard the satellite.

$$T_m = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} \approx T + \frac{1}{2}\left(\frac{v^2}{c^2}\right)$$

Above approximation is using Number (5) from the list of approximations we have generated before with
$(1+u)^r \approx 1 + ru$ where $u = -\frac{v^2}{c^2}$, and $r = -\frac{1}{2}$

If $v = 4$ km/s, and the speed of light (c) is 3×10^5 km/s, $\frac{v^2}{c^2} \approx 10^{-10}$. There is hardly any difference between the times measured on the ground and in the satellite. Nevertheless, engineers used this very approximation (along with several other such approximations) to calibrate the radio transmitters on GPS satellites. (The satellites transmit at a slightly offset frequency.)

Example During the medical procedure, the size of a roughly spherical tumor is estimated by measuring its diameter and using the formula $V = \frac{4}{3}\pi R^3$ to compute its volume. If the diameter is measured as 2.5cm with a maximum error of 2%, how accurate is the volume measurement?
A sphere of radius R and diameter $x = 2R$ has volume

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(\frac{x}{2}\right)^3 = \frac{1}{6}\pi x^3$$

so the volume using the estimated diameter $x = 2.5$cm is
$V = \frac{1}{6}\pi(2.5)^3 \approx 8.181$ cm^3. The error made in computing this volume using the diameter 2.5 when the actual diameter is $2.5 + \Delta x$ is

$$\Delta V = V(2.5 + \Delta x) - V(2.5) \approx V'(2.5)\Delta x$$

The measurement of the diameter can be off by as much as 2%, that is, by as much as $0.02(2.5) = 0.05$cm in either direction. Hence, the maximum error in the measurement of the diameter is $\Delta x = \pm 0.05$cm, and the corresponding maximum error in the calculation of volume is

$$\text{Maximum error in volume } = \Delta V \approx (V'(2.5))(\pm 0.05)$$

Since

$$V'(x) = \frac{1}{6}\pi(3x^2) = \frac{1}{2}\pi x^2 \text{ and } V'(2.5) \approx 9.817$$

it follows that

$$\text{Maximum error in volume } = (9.817)(\pm 0.05) \approx \pm 0.491$$

Thus, at worst, the calculation of the volume as 8.181 cm^3 is off by 0.491 cm^3, so the actual volume V must satisfy

$$7.690 \leq V \leq 8.672$$