Sections 1.5 Limits involving Infinity

Limits to Infinity

Example: Consider \( f(x) = \ln(x) \)

As \( x \) approaches zero from the right hand side \((x \to 0^+)\), \( \ln(x) \) grows unboundedly in the negative direction. So if we are considering \( \lim_{x \to 0^+} \ln(x) \) what is the answer? Since there is no finite number \( L \) such that this limit is equal to; your book and some other sources claims this limit to be non-existing (or limit does not exist). But this definition is not as informative about the behavior of the function. So if and whenever possible I would like you to state the exact destinations of your limits such as in this case \( \lim_{x \to 0^+} \ln(x) = -\infty \). Having said that I do not disagree with the claim that this limit does not exist because \( +\infty \) and \( -\infty \) are not part of the real line but the curve sketching part we will cover later in the course will require us to look at the "non-existing" issue closer so why not start getting used to it now.

Definition

i) We say that \( f(x) \) approaches infinity as \( x \) approaches to \( c \) and we write \( \lim_{x \to c} f(x) = \infty \) if for every positive number \( B > 0 \) there exists a corresponding \( \delta > 0 \) such that for all \( x \) satisfying \( 0 < |x-c| < \delta \Rightarrow f(x) > B \)

ii) We say that \( f(x) \) approaches negative infinity as \( x \) approaches to \( c \) and we write \( \lim_{x \to c} f(x) = -\infty \) if for every negative number \( -B \) there exists a corresponding \( \delta > 0 \) such that for all \( x \) satisfying \( 0 < |x-c| < \delta \Rightarrow f(x) < -B \)
Informally \( \lim_{x \to c} f(x) = \infty \) says that the function can be made arbitrarily large by making \( x \) sufficiently close to \( c \). (or in the \( -\infty \) case can be made arbitrarily small by making \( x \) sufficiently close to \( c \).)

**Definition** The line \( x = c \) is called a **vertical asymptote** of the curve \( y = f(x) \) if at least one of the following is true.

\[
\lim_{x \to c^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = -\infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = -\infty
\]

**Example** Find \( \lim_{x \to 0} \frac{1}{x^2} \).

Check out the graph and the table corresponding to function \( f(x) = \frac{1}{x^2} \) when \( x \) is close to zero

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^2 )</th>
<th>( \frac{1}{x^2} )</th>
<th>( x )</th>
<th>( x^2 )</th>
<th>( \frac{1}{x^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-1} )</td>
<td>( 10^{-2} )</td>
<td>100</td>
<td>( 10^{-1} )</td>
<td>( 10^{-2} )</td>
<td>100</td>
</tr>
<tr>
<td>( 10^{-2} )</td>
<td>( 10^{-4} )</td>
<td>10,000</td>
<td>( 10^{-2} )</td>
<td>( 10^{-4} )</td>
<td>10,000</td>
</tr>
<tr>
<td>( 10^{-3} )</td>
<td>( 10^{-6} )</td>
<td>1,000,000</td>
<td>( 10^{-3} )</td>
<td>( 10^{-6} )</td>
<td>1,000,000</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( 10^{-12} )</td>
<td>( 10^{-24} )</td>
<td>( 10^{24} )</td>
<td>( 10^{-12} )</td>
<td>( 10^{-22} )</td>
<td>( 10^{24} )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

So based on both, the graph and the table the function \( \frac{1}{x^2} \) grows without a bound no matter from which direction you approach to zero (left or right), hence \( \lim_{x \to 0^+} \frac{1}{x^2} = \infty \) and \( \lim_{x \to 0^-} \frac{1}{x^2} = \infty \). So \( \lim_{x \to 0} \frac{1}{x^2} = \infty \). Hence \( x = 0 \) is vertical asymptote for \( f(x) = \frac{1}{x^2} \) by definition above.

**Example** Find \( \lim_{x \to -1} \frac{1}{|x+1|} \)

Let’s see how to get the graph of \( \frac{1}{|x+1|} \) starting with the graph of \( \frac{1}{x} \) first. After all so far graphs have been our most helpful tool to find the limit.
The red dashed line is at $x=1$. Once again as $x$ approaches from either side to $x = 1$ the function values become arbitrarily small without a bound so 
$$\lim_{x \to 1^-} \frac{-1}{|x-1|} = -\infty$$ and 
$$\lim_{x \to 1^+} \frac{-1}{|x-1|} = -\infty \text{ implies } \lim_{x \to 1^-} \frac{-1}{|x-1|} = -\infty,$$

And $x = 1$ is a vertical asymptote.

Vertical Asymptotes comes into discussion most of the time when we have to divide by zero hence especially with regards to rational functions. But you have to be careful considering the issue of vertical asymptotes when it comes to rational functions as you can observe in the Rule below:

**Locating Vertical Asymptotes of Rational Functions**

If $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where $q(c) = 0$ and $p(c) \neq 0$, then $x = c$ is a vertical asymptote of the graph of $f(x)$.

**Example** Let $f(x) = \frac{x^2 + x - 2}{x^2 - 1} = \frac{(x-1)(x+2)}{(x-1)(x+1)}$. And find the $\lim_{x \to 1} f(x)$?

Note that the rule stated above does not work for $f(x)$ because $p(1) = 1^2 + 1 - 2 = 0$ as well as $q(1) = 1^2 - 1 = 0$.
So $f(1) = \frac{0}{0}$ is the indeterminate form. This form usually says you need to do more work with your rational function such as using the factored out form of $f$ and get rid of the $x - 1$ term in the numerator and denominator and re-assess the limit again as follows:
\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x+2}{x+1} = \frac{3}{2}
\]

So \(x = 1\) is not an vertical asymptote for \(f(x) = \frac{x^2+x-2}{x^2-1}\).

**Example** Find \(\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{-1}{(x-1)^4}\).

Here at \(x = 1\) \(p(1) = -1 \neq 0\) and \(q(1) = (1 - 1)^4 = 0\) so by the rule above \(x = 1\) is a vertical asymptote for this function. To be exact \(\lim_{x \to 1} \frac{-1}{(x-1)^4} = -\infty\). Check this result creating the graph of \(f\) just like we did above for \(\frac{-1}{|x-1|}\).

### Limits at Infinity

In applications one is often interested in the asymptotic behavior of a function or in what the function does in the long run. That is \(\lim_{x \to \infty} f(x)\) or \(\lim_{x \to -\infty} f(x)\).

**Definition**

i) We say that as \(x\) approaches infinity \(f(x)\) approaches \(p\), written \(\lim_{x \to \infty} f(x) = p\), if for every \(\epsilon\) there is an \(M > 0\) such that if \(x > M\) then \(|f(x) - p| < \epsilon\)

ii) We say that as \(x\) approaches negative infinity \(f(x)\) approaches \(q\), written \(\lim_{x \to -\infty} f(x) = q\), if for every \(\epsilon\) there is a negative number \(-M\) such that if \(x < -M\) then \(|f(x) - q| < \epsilon\)
Intuitively \( \lim_{x \to \infty} f(x) = p \) means that the graph lies in a horizontal strip around \( y=p \) for all sufficiently large positive \( x \) values. And as the width of this strip decreases the graph of \( y = f(x) \) approaches the horizontal line \( y = p \) really close. You can observe this in the graphs below.

\[
\begin{array}{ccc}
\text{Example} & \text{Let } f(x) = \frac{1}{x} & \text{and consider } \lim_{x \to \infty} \frac{1}{x}.
\end{array}
\]

As \( x \to \infty \) \( f(x) \) goes to zero or \( \lim_{x \to \infty} \frac{1}{x} = 0 \) because as the denominator of the fraction gets larger the fraction itself becomes really small. Note though it is never equal to zero, but by choosing \( x \) large enough \( f(x) \) can be made as close to zero as we would like. More formally let \( \epsilon > 0 \). There is a number \( N \) such that \( \left| \frac{1}{x} - 0 \right| < \epsilon \) for \( x > N \); that is once you pass \( x = N \) marker, your function values will be always within the strip around zero just like in the graph below.

This brings us to the idea of Horizontal Asymptotes.

**Definition** The line \( y = p \) (or \( y = q \)) is called a horizontal asymptote of the curve \( y = f(x) \) if either \( \lim_{x \to \infty} f(x) = p \) (or \( \lim_{x \to -\infty} f(x) = q \)).

So in the above example \( y = 0 \) is a horizontal asymptote for \( f(x) = \frac{1}{x} \).
How do we calculate limits at infinity? Well, three things.

1) The Limit Laws and the Squeeze Theorem remain valid for limits at infinity. The usual restriction applies: For the limit of a quotient to be expressible as the quotient of the limits, the denominator needs to be nonzero. We need to add in another type of condition. I mentioned before that $\infty$ and $-\infty$ are not real numbers. They do not denote anything on the real number line. Rather they are symbols to help us express the idea that a variable, say $x$, is taking on indefinitely large (or large negative) values. Thus we should not be using it, at least at this level, for arithmetical calculations. There is nothing called $\infty/\infty$ for example. So, we will not say the limit of a quotient is the quotient of the limits in case the limit of the top or that of the bottom happens to be $\pm \infty$. This, however, does not mean that we can’t say anything about limits of the structures in which some components involve $\pm \infty$. (More on this later)

2) Besides the obvious limit $\lim_{x \to \infty} k = \lim_{x \to \infty} k = k$ for any constant $k$, we have another basic limit fact (i.e., basic building block):

$$\lim_{x \to \infty} \frac{1}{x^p} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^p} = 0$$

for all positive powers $p$ and $\frac{1}{x^p}$ allows us to talk about the limit. What does this last sentence “the domain of $\frac{1}{x^n}$ allows us to talk about the limit” mean? Well, let $p = 1/2$ and consider $\lim_{x \to -\infty} \frac{1}{x^{1/2}}$. Even discussing such a limit is absurd because for negative values of $x$, $\sqrt{x}$ is not defined so we cannot talk about a limit, because $\frac{1}{x^{1/2}}$ is not a function. So the above limit facts are true only if it makes mathematical sense to talk about it.

3) A third basic limit you will need is the polynomials: For any $k > 0$

$$\lim_{x \to \infty} x^k = \infty \quad \text{and} \quad \lim_{x \to -\infty} x^k = \begin{cases} +\infty & \text{if } k \text{ is even} \\ -\infty & \text{if } k \text{ is odd} \end{cases}$$
**Example** Find \( \lim_{x \to \infty} \frac{x+1}{x} \).

A standard technique is to divide both the denominator and the numerator by the highest power of \( x \) that appears in the denominator. In this case, this means \( x \):

\[
\lim_{x \to \infty} \frac{x+1}{x} = \lim_{x \to \infty} \frac{x+1}{x} \cdot \left( \frac{1}{x} \right) = \lim_{x \to \infty} 1 + \frac{1}{x} = 1 + 0 = 1
\]

You could also reach the same conclusion by re-writing \( f(x) = \frac{x+1}{x} = 1 + \frac{1}{x} \) and recognizing that as \( x \) goes to infinity the term \( \frac{1}{x} \) is really small compared to 1, so the long run behavior of this function is more depended on the 1 part of the function rather than the \( \frac{1}{x} \). Also note that by our definition of horizontal asymptotes \( x = 1 \) is a horizontal asymptote for \( f \).

**Example** Find \( \lim_{x \to \infty} \frac{x^2+1}{x} \).

Once again divide both the denominator and the numerator by the highest power of \( x \) that appears in the denominator. Again it is \( x \):

\[
\lim_{x \to \infty} \frac{x^2+1}{x} = \lim_{x \to \infty} \frac{x^2+1}{x} \cdot \left( \frac{1}{x} \right) = \lim_{x \to \infty} x + \frac{1}{x} = \lim_{x \to \infty} 1 + \frac{1}{x} = 1 + 0 = 1
\]

Right there we stop. Even though in an extended real line course you can write that limit directly as \( \frac{\infty+0}{\infty} = \infty \), we cannot do that in this class and we don’t need to. Think of this way at the end you are considering \( \lim_{x \to \infty} x + \frac{1}{x} \).

Just as we noted at the end of the previous example as \( x \) gets larger and larger indefinitely, so does the \( x \) term in this limit: Since \( \frac{1}{x} \) is small when \( x \) is large, it will not affect the value of \( x \) by much so \( x + \frac{1}{x} \) will become very large and hence the limit is \( \infty \).
Example Find \( \lim_{x \to \infty} \frac{x + 1}{x^2} \).

Once again divide both the denominator and the numerator by the highest power of \( x \) that appears in the denominator. This time it is \( x^2 \):

\[
\lim_{x \to \infty} \frac{x + 1}{x^2} = \lim_{x \to \infty} \frac{1/x^2}{1/x^2} \cdot \left( \frac{1/x^2}{1/x^2} \right) = \lim_{x \to \infty} \frac{1/x}{1/x} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

By comparing with the three examples above the following result for infinite limits of rational numbers should be clear:

**Theorem** Let \( R(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0} \) be a rational function then

\[
\lim_{x \to \infty} R(x) = \begin{cases} 
\infty & \text{if } m > n \\
\frac{a_m}{b_n} & \text{if } m = n \\
0 & \text{if } m < n
\end{cases}
\]

So the answers of the following example should be clear:

**Example** \( \lim_{x \to \infty} \frac{(2x - 3)^{20}(3x + 2)^{30}}{(2x + 1)^{30}} = \lim_{x \to \infty} \frac{2^{20}3^{30}}{2^{30}} = \left( \frac{3}{2} \right)^{30} \)

How about some irrational function examples:

**Example** Find \( \lim_{x \to \infty} \frac{3x - 2|x|}{x + 2} \) and \( \lim_{x \to \infty} \frac{3x - 2|x|}{x + 2} \)

Note that \( f(x) = \frac{3x - 2|x|}{x + 2} = \begin{cases} 
\frac{3x - 2x}{x + 2} = \frac{x}{x + 2} & \text{if } x > 0 \\
\frac{3x + 2x}{x + 2} = \frac{5x}{x + 2} & \text{if } x < 0
\end{cases} \)

So when discussing the infinite limits you need to consider two different functions:

\[
\lim_{x \to \infty} \frac{3x - 2|x|}{x + 2} = \lim_{x \to \infty} \frac{x}{x + 2} = 1 \text{ because } m = n = 1 \text{ and } a_1 = b_1 = 1
\]

\[
\lim_{x \to \infty} \frac{3x - 2|x|}{x + 2} = \lim_{x \to \infty} \frac{5x}{x + 2} = \lim_{x \to \infty} \frac{5}{1 + \frac{2}{x}} = \frac{5}{1} = 5.
\]
Example Find \( \lim_{x \to \infty} \frac{3x - 7}{\sqrt{5}x^2 + 4} \) and \( \lim_{x \to -\infty} \frac{3x - 7}{\sqrt{5}x^2 + 4} \).

\[
\lim_{x \to \infty} \frac{3x - 7}{\sqrt{5}x^2 + 4} = \lim_{x \to \infty} \frac{3x - 7}{x^2(5 + \frac{4}{x^2})} = \lim_{x \to \infty} \frac{3x - 7}{|x|\sqrt{5 + \frac{4}{x}}} 
\]

At this point realize that when you say \( x \) is going to infinity you are dealing with very large positive numbers. So for these \( x \) values \( |x| = x \). Hence add this extra info into your limit discussion above to get

\[
\lim_{x \to \infty} \frac{3x - 7}{\sqrt{5}x^2 + 4} = \lim_{x \to \infty} \frac{3x - 7}{x\sqrt{5 + \frac{4}{x}}} = \lim_{x \to \infty} \frac{3 - \frac{7}{x}}{\sqrt{5} + \frac{4}{x^2}} = \frac{3}{\sqrt{5}}
\]

For the other limit (negative infinity) we will have the same beginning steps up to the point where we decide the sign of the \( |x| \). This time because \( x \) goes to negative infinity means you are dealing with very "large" negative numbers \( |x| = -x \). So you will proceed in your calculations as below:

\[
\lim_{x \to -\infty} \frac{3x - 7}{\sqrt{5}x^2 + 4} = \lim_{x \to -\infty} \frac{3x - 7}{|x|\sqrt{5 + \frac{4}{x}}} = \lim_{x \to -\infty} \frac{3x - 7}{-x\sqrt{5 + \frac{4}{x}}} = -\frac{3}{\sqrt{5}}
\]

So this function has two different horizontal asymptotes one at \( y = \frac{3}{\sqrt{5}} \) and the other at \( y = -\frac{3}{\sqrt{5}} \).

Example Find \( \lim_{x \to \infty} \sqrt{x + \sqrt{x}} - \sqrt{x} \)

This would be a classic example of using the conjugate of an expression. The conjugate of \( \sqrt{x + \sqrt{x}} - \sqrt{x} \) is \( \sqrt{x + \sqrt{x}} + \sqrt{x} \). So;

\[
\lim_{x \to \infty} (\sqrt{x + \sqrt{x}} - \sqrt{x}) \cdot \frac{\sqrt{x + \sqrt{x}} + \sqrt{x}}{\sqrt{x + \sqrt{x}} + \sqrt{x}} = \lim_{x \to \infty} \frac{x + \sqrt{x} - x}{\sqrt{x + \sqrt{x}} + \sqrt{x}} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x}} + \sqrt{x}}
\]

Now you need the old trick the highest power of \( x \) in the denominator is \( \sqrt{x} \) so divide both numerator and denominator by this term:

\[
\lim_{x \to \infty} (\sqrt{x + \sqrt{x}} - \sqrt{x}) = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x}} + \sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt[4]{x^2} + 1} = \frac{1}{2}
\]