Sections 1.4 Continuity

**Intuitive Definition or Pencil Test** If you can trace the graph of a function without lifting your pencil, the function is continuous.

For motivation we will consider graphically 4 fundamental discontinuities:

**Case 1:** $f(a)$ is not defined – a hole in the graph case.

Consider the graph of the function $f(x) = \frac{x^2 - 1}{x - 1}$

Note that $\lim_{x \to 1} f(x)$ exists but $f(1)$ is not defined.

**Case 2** $f(a)$ is defined but still a hole in the graph.

Consider the graph of $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ -1 & \text{if } x = 1 \end{cases}$

Note that $\lim_{x \to 1} f(x)$ exists and $f(1)$ is defined but $\lim_{x \to 1} f(x) \neq f(1)$
Case 3 Jump Discontinuity

Consider the graph of $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ -x^2 & \text{if } x < 1 \end{cases}$

$f(1)$ exists but $\lim_{x \to 1} f(x)$ does not exist.

Case 4 Vertical Asymptotes

Consider $f(x) = \frac{1}{(x-1)^2}$

Here $f(x)$ grows unbounded as $x \to 1^+$ or $x \to 1^-$. Hence $f(1)$ does not exist and $\lim_{x \to 1} f(x)$ does not exist.

So based on our observations above the below definition should not come as a surprise.
Definition A function $f$ is continuous at $x = a$ if

1) $\lim_{x \to a} f(x)$ exists

2) $f(a)$ is defined

3) $\lim_{x \to a} f(x) = f(a)$

Remark: If $f(x)$ is not continuous at $x = a$, it is discontinuous at $x = a$.

There are a few important things to notice about continuous functions:

1) evaluating limits of continuous functions is easy, because it's the same as evaluating the function;

2) a function can only be continuous on its domain, since the definition involves evaluating $f$ at $a$ (in particular if $f(a)$ is not defined, then $f(x)$ can't be continuous at $a$);

3) a continuous function has a limit at $a$ (in particular, if $\lim_{x \to a} f(x)$ does not exist, $f$ can't be continuous).

Types of discontinuity A function can fail to be continuous in a few different ways. The two big ways we see for a function to fail to be continuous at a point are jump discontinuities and removable discontinuities.

- A jump discontinuity is a point $a$ so that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but are not equal.

- A removable discontinuity is a point $a$ so that $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$.

Jump discontinuities are irredeemable, in the sense that unless we give a major make-over to the graph of $f(x)$ we cannot get rid of them. However, removable discontinuities needs only fixing at only one point, namely at "a". We can remove this type of discontinuity and get a continuous function. Here is how such an operation is done on an example:
Example Recall we have showed using the Squeeze Theorem
\[
\lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0.
\]
Clearly \(f(0)\) is not defined. So this function is a discontinuous function. But since the limit exists at \(x = 0\), we can "fill in the hole" by defining a new function \(h(x)\) as follows:
\[
h(x) = \begin{cases} 
  x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}
\]
\(h(x)\) now is continuous at \(x = 0\) \(\lim_{x \to 0} h(x) = h(0)\) so we have removed the discontinuity.

Definition A function is continuous from the right at \(x = a\) if
\[
\lim_{x \to a^+} f(x) = f(a)
\]

Example \(f(x) = \sqrt{x - 2}\). Applying the Pencil Test to the graph of \(f(x)\) we see that \(f\) is continuous at every point in \((2, \infty)\)

\[
\text{Out}[3]=
\]

Also we have previously showed that \(\lim_{x \to 2^+} f(x) = 0\). We say that \(f\) is continuous on its domain \([2, \infty)\). Closed bracket at 2 implies that the function is continuous from right.

Definition A function \(f\) is continuous on an interval \([a, b]\) if it is continuous at every point in the interval. At the end points of the interval "a" and "b" by continuity we understand one sided-continuity. At "a" we require continuity from right and at "b" we require continuity from left.

Remark If a function is continuous on all of \(\mathbb{R}\) it is referred to simply as continuous.

Example Let \(p(x)\) be any polynomial. In Section 1.3 we have showed that \(\lim_{x \to a} p(x) = p(a)\) at all \(a\) in \(\mathbb{R}\). So by definition above \(p(x)\) is continuous.

In fact, almost all the functions that we have talked about in our library are continuous: polynomials, rational functions (on their domain –so in particular where the denominator is not 0), trig functions (on their domain),
exponential functions, log functions (on their domain), and the absolute value function are all continuous.
We can also put together two continuous functions and get continuous functions: the sum, difference, product, scaling, quotient, and composition of two continuous functions are all continuous on the domain of the resultant function!

**Example** Find \( \lim_{x \to e^{7/\pi}} \sin(x) \)?

Since \( \sin(x) \) is continuous by definition of continuity, \( \lim_{x \to e^{7/\pi}} \sin(x) = \sin(e^{7/\pi}) \)

Since continuity is defined as a limit, the limit laws have a direct correlation to continuous functions.

**Theorem** If \( f \) and \( g \) are continuous at \( x=a \) and \( c \) is a constant, then the following are also continuous at \( x=a \)

1) \( f \pm g \)

2) \( cf \)

3) \( (f \cdot g) \)

4) If \( g(a) \neq 0 \) \( \frac{f}{g} \)

**Example** Where is \( f(x) = \frac{\ln x + \tan^{-1}(x)}{x^2 - 1} \) continuous?

Divide and conquer: \( \ln(x) \) is continuous on its domain \((0, \infty)\) and \( \tan^{-1}(x) \) is continuous on all of \( \mathbb{R} \). Like wise \( x^2 - 1 \). By the property (1) above the numerator \( \ln(x) + \tan^{-1}(x) \) continuous on \((0, \infty)\). By Property (4) \( f(x) \) is continuous on \((0, \infty)\) where \( x^2 - 1 \neq 0 \). Hence \( f(x) \) is continuous on \((0, 1) \cup (1, \infty)\)

**Theorem** If \( f \) is continuous at \( b \) and \( \lim_{x \to a} g(x) = b \), then \( \lim_{x \to a} f(g(x)) = g(b) \). In other words;

\[
\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))
\]
Example Find \( \lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) \)

Since \( \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) \) is not defined at \( x = 1 \) we can’t simply push \( x = 1 \) into the function. But, if \( \lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = b \) and \( \arcsin(x) \) is continuous at \( b \), we can use the last theorem.

\[
\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \lim_{x \to 1} \frac{1 - \sqrt{x} \cdot (1 + \sqrt{x})}{1 - x} = \lim_{x \to 1} \frac{1 + \sqrt{x} - \sqrt{x} - x}{1 - x} = \lim_{x \to 1} \frac{1 - x}{(1 - x)(1 + \sqrt{x})} = \lim_{x \to 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}
\]

Recall that the domain of \( \arcsin(x) \) is \([−1, 1]\). Since \( 1/2 \) is in this domain \( \arcsin(x) \) is continuous at \( x = 1 \). By last theorem,

\[
\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) = \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}
\]

**Theorem** If \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \), then the composite function \( f \circ g(x) = f(g(x)) \) is continuous at \( a \).

**Example** Where is \( F(x) = \ln(1 + \cos(x)) \) continuous?

Note that \( F(x) = f(g(x)) \) where \( f(x) = \ln(x) \) and \( g(x) = 1 + \cos(x) \). \( F(x) \) is only defined when \( 1 + \cos(x) > 0 \). So \( F(x) \) is undefined when \( \cos(x) = -1 \) or \( x = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \). So \( F \) is continuous on the intervals between these values.
The intermediate Value Theorem (IVT)

**Theorem** Suppose that $f$ is continuous on the closed interval $[a, b]$ and either;
- $f(a) < N$ and $f(b) > N$ or
- $f(a) > N$ and $f(b) < N$

Then there is a number $c$ in the interval $(a, b)$ for which $f(c) = N$.

**Remarks:**
1) In geometric terms IVT says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ then graph of $f$ cannot jump over the line. It must intersect $y = N$ somewhere.

2) The IVT states that there is such a number $c$ it neither says what this $c$ is nor claims $c$ to be unique. It is shown in the figures below that the value $N$ can be taken on once or more that once.

3) It is important that the function $f$ in the theorem be continuous, IVT is not true in general for discontinuous functions as can be seen in the example below.

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Note that there are no $c$-values on $[-1, 1]$ where $-1 < f(c) = 1/2 < 1$ or any other $N$ where $-1 < N < 1$ for that matter.
IVT has many useful applications. We will see two of them.

**Example** Show that the equation $x^{2008} + 2008x - 1$ has a root.

A root of the equation $f(x) = x^{2008} + 2008x - 1$ is a solution to the equation $f(x) = 0$, so we have to prove that $f(x) = 0$ has a solution.

First, we notice that $f(x)$ is a continuous function, since $f(x)$ is a polynomial, and polynomials are continuous. Further we can evaluate

\[
f(0) = 0^{2008} + 2008 \cdot 0 - 1 = -1 < 0 = N
\]
\[
f(1) = 1^{2008} + 2008 \cdot 1 - 1 = 1 + 2008 - 1 = 2008 > 0 = N
\]

Since $f(x)$ is a continuous function on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$, we may apply the IVT and conclude that there exists a number $c$ in the interval $(0, 1)$ with $f(c) = 0$ or $c^{2008} + 2008 \cdot c - 1 = 0$, which is exactly what we wanted to show.

**Side Note** Indeed, the conclusion of the intermediate value theorem is somewhat mysterious in that it states the existence of a number $c$ which has a nice property $f(c) = N$ but does not actually tell you what that number is. This is about the only rule we have to show that a certain number exists without actually producing it, so if someone asks you to show a certain number exists but does not make you actually find it, you should start thinking intermediate value theorem.

Here’s a trickier example that is a good example of this principle

**Example** Show that there exists a number $x$ in the interval $[0, \pi/2]$ with $x = \cos(x)$.

This problem asks us to show there exists a number $x$ which has a nice property (this time, nice property means $x = \cos(x)$) but does not ask us to say exactly what it is, so we are thinking that we should use the intermediate value theorem.

The first thing we need to do is translate this problem into a problem that we can use the intermediate value theorem to solve. In particular, given a continuous function $g(x)$ and some hypotheses, the intermediate value theorem lets us conclude the existence of a number $c$ with $g(c) = N$, where $N$ is some real number. So let’s translate our problem into the solution of an equation.
Specifically, we notice that finding a number $x$ with $x = \cos(x)$ is the same as finding a solution to the equation $f(x) = 0$, where $f(x) = \cos(x) - x$. Why is this true? A solution to $f(x) = 0$ is a number so that $\cos(x) - x = 0$, or equivalently, so that $\cos(x) = x$. This is exactly the condition we want!

Great, so let’s try to show that for the function $f(x) = \cos(x) - x$ there exists a solution to $f(x) = 0$. Since we know we have to use the intermediate value theorem, we start by observing that $f(x)$ is the difference of two continuous functions, and is therefore itself continuous. Furthermore we can see that

\[
f(0) = \cos(0) - 0 = 1 - 0 = 1 > 0
\]
\[
f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2} < 0
\]

Therefore, since $f(x)$ is a continuous function on $[0, \frac{\pi}{2}]$ with $f(0) > 0$ and $f\left(\frac{\pi}{2}\right) < 0$, we may apply the IVT and conclude that there exists a number $c$ in the interval $(0, \frac{\pi}{2})$ with $f(c) = 0$. This magical value of $c$ therefore satisfies $\cos(c) - c = 0$, and so $\cos(c) = c$. That’s just what we want.