Sections 1.3 Computation of Limits

We will shortly introduce the "limit laws." Limit laws allows us to evaluate the limit of more complicated functions using the limit of simpler ones. 

**Theorem** Suppose that "c" is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists. Then;

1) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

2) $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

3) $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

4) If $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

5) $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ n is a positive integer

Indeed the same rules hold for directional limits, so that (for instance) if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^-} g(x)$ exists, then

$\lim_{x \to a^-} [f(x) + g(x)] = \lim_{x \to a^-} f(x) + \lim_{x \to a^-} g(x)$

**Corollary** If f(x) is a polynomial, then $\lim_{x \to a} f(x) = f(a)$.

**Proof:** If f(x) is a polynomial, we know that $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$. Then we have
\[
\lim_{x \to a} f(x) = \lim_{x \to a} a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \\
= \lim_{x \to a} [a_n x^n] + \lim_{x \to a} [a_{n-1} x^{n-1}] + \ldots + \lim_{x \to a} [a_1 x] + \lim_{x \to a} [a_0] \\
= a_n \lim_{x \to a} [x^n] + a_{n-1} \lim_{x \to a} [x^{n-1}] + \ldots + a_1 \lim_{x \to a} [x] + \lim_{x \to a} [a_0] \\
= a_n (\lim_{x \to a} x)^n + a_{n-1} (\lim_{x \to a} x)^{n-1} + \ldots + a_1 (\lim_{x \to a} x) + a_0 \\
= a_n a^n + a_{n-1} a^{n-1} + \ldots + a_1 a + a_0 = f(a)
\]

**Example** Find the \(\lim_{x \to 5} (2x^2 - 3x + 4)\).

We will use the same steps as we did in the proof of the above Corollary.

\[
\lim_{x \to 5} (2x^2 - 3x + 4) = \lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4 \text{ by Law 1} \\
= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 \text{ by Law 2} \\
= 2[\lim_{x \to 5} x]^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 \text{ by Law 5} \\
= 2(5)^2 - 3(5) + 4 = 39
\]

**Example** Find the \(\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}\).

First note that \(\lim_{x \to -2} 5 - 3x = 5 - 3 \lim_{x \to -2} x = 11 \neq 0\). So we may use the Law 4 for quotients.

\[
\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} x^3 + 2x^2 - 1}{\lim_{x \to -2} 5 - 3x} \\
= \frac{(-2)^3 + 2(-2)^2 - 1}{11} = \frac{-1}{11}
\]

**Theorem** If \(f\) is a rational function and "a" is in the domain of \(f\), then \(\lim_{x \to a} f(x) = f(a)\).

After the Example above the proof of this theorem should be clear. Let’s see what happens if ”a” is not in the domain of our rational function.
Example Find \( \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} \).

We cannot simply substitute \( x = 0 \) since \( f(0) \) is not defined. Hence we cannot apply the Law 4 for quotients (denominator is zero). However we can rationalize the numerator as follows:

\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} = \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} \cdot \frac{\sqrt{x^2 + 9} + 3}{\sqrt{x^2 + 9} + 3} = \lim_{x \to 0} \frac{(x^2 + 9) - 9}{x^2(\sqrt{x^2 + 9} + 3)} = \lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2 + 9} + 3)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{3 + 3} = \frac{1}{6}
\]

Example Find \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \)

Note that again we cannot use the Law 4 for quotients because the denominator is zero at \( x = 2 \). So we need another algebra trick:

\[
\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4
\]

In the above example note that \( f(x) \neq x + 2 \). The rational function is very much like \( x + 2 \) except it is not defined at \( x = 2 \). So we are using the power of the limit.

Theorem Let \( \lim_{x \to a} f(x) = L \) and \( n \) is any positive integer, then

\[
\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}
\]

If \( n \) is even, we need \( \lim_{x \to a} f(x) > 0 \).
**Example** Find \( \lim_{x \to -2} \sqrt{x^4 + 3x + 6} \).

\[
\lim_{x \to -2} \sqrt{x^4 + 3x + 6} = \sqrt{\lim_{x \to -2} (x^4 + 3x + 6)} = \sqrt{(-2)^4 + 3(-2) + 6} = 4
\]

**Example** Find \( \lim_{x \to -2} \sqrt{x - 2} \).

Your first reaction to this problem should be that limit does not exits. Because any open interval around \( x=2 \) contains values of \( x \) not in the domain of \( \sqrt{x - 2} \). But if we correct the question and discuss the limit from the right we have an answer:

\( \lim_{x \to -2^+} \sqrt{x - 2} = 0 \). If you consider \( \lim_{x \to -2^-} \sqrt{x - 2} \) does not exist. By a fact we have seen before you can also conclude that \( \lim_{x \to -2} \sqrt{x - 2} \) does not exist.

**Some Cautionary Examples:** Note that I have underlined the verb "exists" in the statement of the limit laws. You have to be cautious of this fact when you do the limit calculations. If either one of the limits do not exists, then you cannot automatically assume that you are able to use these rules. The Limit Laws guarantee results only if both results exits. So natural question might come to your mind such as:

**Question** Does \( \lim_{x \to a} [f(x)+g(x)] \) exits even though \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) does not exists?

**Answer** It depends on \( f(x) \) and \( g(x) \). Check out the following:

Let \( f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \)

and \( g(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases} \)

Both \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^+} g(x) \) do not exist because both of their one-sided limits are not equal. \( \lim_{x \to 0^+} f(x) = 1 \neq \lim_{x \to 0^-} f(x) = 0 \) and \( \lim_{x \to 0^+} g(x) = 0 \neq \lim_{x \to 0^-} g(x) = 1 \)

But \( f(x) + g(x) = 1 \) for all \( x \). So \( \lim_{x \to 0} [f(x) + g(x)] = 1 \) hence the limit of the sum exists.
Question Does \( \lim_{x \to a} f(x)g(x) \) exist even though \( \lim_{x \to a} f(x) \) exists but \( \lim_{x \to a} g(x) \) does not.

Answer It again depends on \( f(x) \) and \( g(x) \). Take your \( f(x) = x \) and \( g(x) = \frac{1}{x} \). \( \lim_{x \to 0} x = 0 \) and \( \lim_{x \to 0} 1/x \) does not exist. But \( \lim_{x \to 0} x \cdot 1/x = 1 \) and exists. (Note that \( x \cdot 1/x \) is not defined at \( x = 0 \) because \( 1/x \) is not.)

Question What is \( \lim_{x \to 1}[3f(x) + 2g(x)] \) where \( f(x) \) is the blue one and \( g(x) \) is the red.

\[
\text{Note again we cannot use the limit laws because } \lim_{x \to 1} g(x) \text{ does not exist. So instead note that the directional limits exist and hence you can use the limit laws for the directional limits and compare them for the existence of } \lim_{x \to 1}[3f(x) + 2g(x)].
\]

(Recall: \( \lim_{x \to 1}[3f(x) + 2g(x)] \) exists iff \( \lim_{x \to 1^-}[3f(x) + 2g(x)] \) and \( \lim_{x \to 1^+}[3f(x) + 2g(x)] \) exist and they are equal.) Now:

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} f(x) = 1, \quad \lim_{x \to 1^-} g(x) = -2 \quad \text{and} \quad \lim_{x \to 1^+} g(x) = -1
\]

Using our limit laws for directional limits, we have

\[
\lim_{x \to 1^-} [3f(x) + 2g(x)] = \lim_{x \to 1^-} [3f(x)] + \lim_{x \to 1^-} [2g(x)] = 3 \cdot 1 + 2 \cdot (-2) = -1
\]

and

\[
\lim_{x \to 1^+} [3f(x) + 2g(x)] = \lim_{x \to 1^+} [3f(x)] + \lim_{x \to 1^+} [2g(x)] = 3 \cdot 1 + 2 \cdot (-1) = 1
\]

Since these two directional limits do not agree, we conclude that \( \lim_{x \to 1}[3f(x) + 2g(x)] \) does not exist.

So that you have more fun while solving the hw problems and in discussion sessions I’ll state the following result from your book without proof.
Theorem For any number a, we have

<table>
<thead>
<tr>
<th>$\lim_{x \to a} \sin(x)$</th>
<th>$= \sin(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lim_{x \to a} \cos(x)$</td>
<td>$= \cos(a)$</td>
</tr>
<tr>
<td>$\lim_{x \to a} e^x$</td>
<td>$= e^a$</td>
</tr>
<tr>
<td>$\lim_{x \to a} \ln(x)$</td>
<td>$= \ln(a)$, for $a &gt; 0$</td>
</tr>
</tbody>
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If $p(x)$ is a polynomial, and $\lim_{x \to a} f(x) = L$ then $\lim_{x \to a} p(f(x)) = L$

We will re-visit the results of this theorem when we deal with "continuity" in the next section. Now let’s learn another tool that will help us to evaluate some important limits such as $\lim_{x \to 0} \frac{\sin(x)}{x}$

Theorem If $f(x) \leq g(x)$ when $x$ is near "a" (except possibly at a) and the limits of $f$ and $g$ exists as $x$ approaches $a$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Squeeze Theorem/Sandwich Theorem If $f(x) \leq g(x) \leq h(x)$ when $x$ is near "a" (except for possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then,

$$\lim_{x \to a} g(x) = L$$

Example Show that $\lim_{x \to 0} x^2 \sin(\frac{1}{x}) = 0$.

First note that you cannot use the limit law 3 for products of functions because $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist. However, since

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \text{ for all } x \neq 0$$

we have

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \text{ since } x^2 \geq 0 \text{ always.}$$
Also observe this relation in the graph below:

![Graph Image]

We know that \( \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0 \). Taking \( f(x) = -x^2 \), \( g(x) = x^2 \sin\left(\frac{1}{x}\right) \) and \( h(x) = x^2 \) in the Squeeze Theorem, we obtain
\[ \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \]

**Little Exercise** Try yourself to show \( \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 \). Here you will need your directional limit abilities along with the Squeeze Theorem.