Sections 1.1/1.2 Intro to Calculus and Limit

The distance one travels at a constant speed had long been modeled as a linear function. If we are traveling at a constant velocity $m$, then the total distance traveled from time 0 to time $x$ is $f(x) = mx$.

If you pick any two arbitrary points from the line and calculate the difference quotient that gives you the slope; it will give you the same answer regardless of the choice of points you go with. And the slope of the line is equal to the constant speed.

Question: Why wouldn’t the same geometry apply to a variable speed problem?

The answer hinges on the definition of the slope of a curve at a point first. For instance, suppose we wanted to find the slope of the curve $y = f(x)$ at the point $P = (x_0, f(x_0))$. This corresponds to finding the speed of the object at time $x = x_0$ with the displacement function $f(x)$ btw– how is the slope defined? Like everything else in math start with what we know and let’s see how can we extend this into a new definition. To see what is going on let’s pick up a function, say $f(x) = x^2 - 6x + 4$ and a point of interest, say $P = (x_0, y_0) = (2, -4)$. The big idea is to consider secant lines. Specifically, pick a point $Q = (x_1, f(x_1))$ on the graph of $f(x)$ and find the slope of the line through $P$ and $Q$ (you can do this since 2 points determine a line). This is the so called secant line through $P$ and $Q$. 

Figure 1: Secant line through $P = (2, -4)$ and $Q = (3, -5)$

Now if you calculate the slopes of several secant lines what will you see? The slope of the secant line for the $f(x)$ we picked is:

$$m_{sec} = \frac{f(x_1) - (-4)}{x_1 - 2} = \frac{f(x_1) + 4}{x - 2}$$

And here is a table with $x_1$ values greater than 2 for $f(x) = x^2 - 6x + 4$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$f(x_1)$</th>
<th>$m_{sec}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-5</td>
<td>$\frac{-5 - 4}{2} = -\frac{9}{2}$ = -4.5</td>
</tr>
<tr>
<td>2.5</td>
<td>-4.75</td>
<td>$\frac{-4.75 - 4}{2.5 - 2} = -1.5$</td>
</tr>
<tr>
<td>2.1</td>
<td>-4.19</td>
<td>$\frac{-4.19 - 4}{2.1 - 2} = -1.9$</td>
</tr>
<tr>
<td>2.01</td>
<td>-4.01995</td>
<td>$\frac{-4.01995 - 4}{2.01 - 2} = -1.99$</td>
</tr>
</tbody>
</table>

So the slopes of these lines are not constant as we have seen in the constant velocity case. But if you zoom in tight on the point $(2, -4)$ you will get the graph below.

The graph looks very much like a straight line. In fact, the more you zoom in, the straighter it gets and the less it matters which two points are used to compute a slope. So here is the strategy: Let $Q$ approach the point $P$, the secant lines will begin to approach the tangent line just as we see in the
So by this graphical observation we can conclude that the slope of the secant lines will start to approach the slope of the tangent line. Hence, if we can see what happens to the quantity

\[ m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]

as \( Q \) approaches \( P \), or equivalently as \( x_1 \) approaches \( x_0 \), we will have the slope of the tangent line as desired. Soon we will be writing “what happens to the quantity \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \) as \( x_1 \) approaches \( x_0 \) as:

\[
\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}
\]

We will call this quantity the derivative of \( f(x) \) at \( x_0 \), which we will abbreviate as \( f'(x_0) \).

To see a rigorous example of this let’s go back to our parabola \( x^2 - 6x + 4 \) and check the table values below (Note that this time I have used \( x_1 \) values less than 2)

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( f(x_1) )</th>
<th>( m_{sec} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>-2.75</td>
<td>( \frac{-2.75+3}{1.5-2} = -2.5 )</td>
</tr>
<tr>
<td>1.9</td>
<td>-3.79</td>
<td>( \frac{-3.79+4}{1.9-2} = -2.1 )</td>
</tr>
<tr>
<td>1.99</td>
<td>-3.9799</td>
<td>( \frac{-3.9799+4}{1.99-2} = -2.01 )</td>
</tr>
</tbody>
</table>

By comparing the two tables we observe even though the function \( m_{sec}(x) = \frac{f(x)+4}{x-2} \) is not defined at \( x = 2 \) it appears that as we approach \( x=2 \) from right or left (on the x-axis) \( m_{sec}(x) \) is going to the value -2. So the slope of the tangent line at \((2, -4)\) is -2.
Example On Halloween let’s say you decided to drop pumpkins from the roof of Altgeld Hall – totally illegal for the reasons it will become clear soon. And assume Altgeld Hall is 80 meters high.

The equation of motion for objects near the earth surface implies that the height above ground "h" of the pumpkin is:

\[ h = 80 - 5t^2 \text{ in meters} \]

where "t" measures time in seconds. The average speed of the pumpkin (difference quotient)= \( \frac{\Delta h}{\Delta t} = \frac{\text{distance traveled}}{\text{time elapsed}} \)

When the pumpkin hits the ground, \( h=0, \) \( 80 - 5t^2 = 0. \) Solve this to find \( t = 4. \) Thus it takes 4 seconds for the pumpkin to reach the ground

\[
\text{Average Speed} = \frac{0 - 80}{4 - 0} = -20 \text{ m/sec}
\]

A spectator is probably more interested in how fast the pumpkin is going when it slams into the ground. To find the instantaneous velocity at \( t=4, \) we need to evaluate:

\[
\lim_{t \to 4} \frac{h(t) - h(4)}{t - 4} = -40 \text{ m/sec about 90 mi/hr}
\]

Here the result is negative because the pumpkin’s y-coordinate is decreasing: it is moving downward.

We will discuss the notion of "limit" next to understand the ideas explained above better.
**LIMITS**

**Intuitive Definition** For a function $f(x)$, the *limit of $f$ as $x$ approaches $a$*, written $\lim_{x\to a} f(x)$ is the quantity that outputs are approaching as inputs approach $a$.

In other words, $\lim_{x\to a} f(x)$ is where it looks like a graph of $f(x)$ is heading as $x$ approaches $a$.

**Example** Consider the graph of $f(x)$ below. What is the $\lim_{x\to 2} f(x)$?

As input approach 2, outputs are approaching 9. Hence $\lim_{x\to 2} f(x) = 9$

**Example** Consider the function $g(x)$ below. What is $\lim_{x\to 2} g(x)$?

The function $g(x)$ is the same as $f(x)$ except at $x = 2$, where $g(x) = 3$. Notice, however, that in evaluating the limit we only care about where the function seems to be going as $x \to 2$, and not about the value of the function at 2. Hence we still have $\lim_{x\to 2} g(x) = 9$
Example Consider the graph of \( h(x) \) below. What is \( \lim_{x \to 2} h(x) \)?

The function \( h(x) \) is the same as \( f(x) \) and \( g(x) \) except at \( x=2 \), where \( h(x) \) isn’t even defined. But again, computing the limit only requires that we look where the function is headed as \( x \to 2 \) and not that we know anything about \( h(2) \) (which in this case does not exist!!), and so we have \( \lim_{x \to 2} h(x) = 9 \).

The important lesson to take away from these examples is that the values of the limit \( \lim_{x \to a} f(x) \) has (in general) nothing to do with the value of \( f(x) \).

Example Consider the function \( f(x) \) depicted below. What is \( \lim_{x \to 0} f(x) \)?

For this function, as inputs approach 0 from the left, outputs approach -1. But as inputs approach 0 from the right, outputs approach 1. Since there is not a single number that outputs as inputs approach 0, we say

\[
\lim_{x \to 0} f(x) \text{ does not exist}
\]

Because of this, it is convenient to talk about ”directional limits.”
**Intuitive Definition** For a function \( f(x) \), the limit of \( f \) as \( x \) approaches \( a \) from the left, written \( \lim_{x \to a^-} f(x) \) is the quantity that outputs are approaching as inputs approach \( a \) from the left. The limit of \( f \) as \( x \) approaches \( a \) from the right, written \( \lim_{x \to a^+} f(x) \), is the quantity that outputs are approaching as inputs approach \( a \) from the right.

**Example** Consider the function \( f(x) \) from the previous example. What is \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^+} f(x) \)? We said above that \( f(x) \) approaches 1 and -1 as inputs approach 0 from the right and left (respectively). Hence we have,

\[
\lim_{x \to 0^-} f(x) = -1 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 1
\]

**Fact:** \( \lim_{x \to a} f(x) \) exists if and only if \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) exist and are equal.

This fact comes in handy for evaluating limits of functions that look pretty tricky. We'll see an example of this soon.

**Example** The function \( y = \sin\left(\frac{1}{x}\right) \) is another example of a ”limit does not exist case”.

Consider the limit \( \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \). This function does not even have a directional limit at 0!. You can see from the graph that as \( x \to 0^+ \), the function is not approaching a single value. Indeed, for any value \( c \) in the interval \([-1, 1]\) that you like, there is a sequence of points approaching 0 whose outputs are approaching to \( c \). In this sense, then the function fails to have directional limit because it is approaching ”too many” values as \( x \to 0^+ \).