SOLUTIONS Section 9.1

1. \( A = \begin{bmatrix}
1 & -3/2 & 0 \\
-3/2 & 3 & -2 \\
0 & -2 & 8 \\
\end{bmatrix} \) and \( q = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} \)

2. \( q = 2x^2 + 6y^2 + 9z^2 + 6xy + 8xz + 14yz \)

3. \[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3/2 & 0 \\
0 & 0 & 0 & 0 & 3/2 \\
\end{bmatrix}
\]

4. \( x_1^2 + 5x_2^2 + 6x_4^2 + 4x_1x_2 + 6x_1x_3 + 8x_1x_4 + 6x_3x_4 \)

5. \( P = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \)

(a) old matrix \( A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \), new matrix = \( P^T A P = \begin{bmatrix} 24 & 1 \\ 1 & -1 \end{bmatrix} \)

\( q = 24x^2 + 2xy - y^2 \) in the new coord system

(b) \[
\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} X \\ Y \end{bmatrix}
\]

\( x = 3X + 2Y, \ y = X - Y \)

\( q = (3X + 2Y)^2 + 4(3X + 2Y)(X - Y) + 3(X - Y)^2 = 24X^2 + 2XY - Y^2 \)

(c) Using \( X,Y \) coords. Substitute \( X = 1, Y = 2 \) into the new \( q \) formula to get \( q = 24 + 4 - 4 = 24 \).

Using \( x,y \) coords. First get the \( x,y \) coords.

\( P \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} \) so the old coords of \( B \) are \( x = 7, \ y = -1 \).

Substitute \( x = 7, y = -1 \) into the old \( q \) formula to get \( q = 49 - 28 + 3 = 24 \) again.

6. (a) Solve for \( x \) and \( y \) to get \( x = \frac{1}{2}(X + Y), \ y = \frac{1}{2}(Y - X) \). Then substitute:

\( q = \frac{1}{4} (X + Y)^2 + 3 \frac{1}{4}(X + Y)(Y - X) - 5 \frac{1}{4}(Y - X)^2 = - \frac{7}{4}X^2 + 3XY - \frac{1}{4}Y^2 \)

(b) Let \( P \) be the usual basis changing matrix (which converts from new to old coords). We know that

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

• this is \( P^{-1} \)

Take the inverse of \( P^{-1} \) to get \( P = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \).

The new basis vectors are \( u = (1/2, -1/2), \ v = (1/2, 1/2). \)

(c) The (old) matrix for \( q \) is \( A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & -5 \end{bmatrix} \)

Let \( P \) be the basis changing matrix from part (b) that converts \( x,y \) coords to \( X,Y \) coords.
The matrix for \( q \) in the new coord system is \( P^TAP = \begin{bmatrix} -7/4 & 3/2 \\ 3/2 & -1/4 \end{bmatrix} \).

So \( q = -\frac{7}{4}x^2 + 3xy - \frac{1}{4}y^2 \) again.

(d) The old matrix is \( A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & -5 \end{bmatrix} \); \( q = \begin{bmatrix} x \\ y \end{bmatrix}A \begin{bmatrix} x \\ y \end{bmatrix} \)

(e) The new matrix is \( B = \begin{bmatrix} -7/4 & 3/2 \\ 3/2 & -1/4 \end{bmatrix} \); \( q = \begin{bmatrix} x \\ y \end{bmatrix}B \begin{bmatrix} x \\ y \end{bmatrix} \)

(f) \( P^TAP = B \)

7. (a) \( q = (2X-Y)^2 + 4(2X-Y)(X+3Y) - (X+3Y)^2 = 11x^2 + 10xy - 20y^2 \)

(b) Let \( P \) be the usual basis changing matrix (which converts from new to old coords). We know that

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}
\]

\text{this is the basis-changing matrix } P

The new basis vectors are \( u = (2,1) \), \( v = (-1,3) \).

(c) The (old) matrix for \( q \) is \( A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \).

Let \( P \) be the basis changing matrix from part (b) that converts \( X,Y \) coords to \( x,y \) coords.

The matrix for \( q \) in the new coord system is

\( P^TAP = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 5 & -20 \end{bmatrix} \)

So \( q = 11x^2 + 10xy - 20y^2 \) again.

8. The new basis vectors \( u \) and \( v \) are unit vectors (because the scale in the new coord system is the same as the scale in the old system) pointing along the X-axis and Y-axis respectively:

\( u = (1,0) \)

\( v = (1,1)_\text{unit} = \left( \frac{1}{\sqrt{2}} , \frac{1}{\sqrt{2}} \right) = \left( \frac{1}{2} \sqrt{2} , \frac{1}{2} \sqrt{2} \right) \)

Let

\( P = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \)

Problem 8

(a) Old matrix = \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), new matrix = \( p^T \) old \( P = \begin{bmatrix} 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix} \)

\( q = x^2 + y^2 + \sqrt{2}xy \)

(b) New matrix for \( q = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} = p^T \text{ old } P \)

old = \( (p^T)^{-1} \) new \( p^{-1} = (p^{-1})^T \) new \( p^{-1} = \begin{bmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2} \end{bmatrix} \)

\( q = -\sqrt{2}y^2 + \sqrt{2}xy \)

(c) From part (a), \( x^2 + y^2 = x^2 + y^2 + \sqrt{2}xy \). So the circle has equation
\[ x^2 + y^2 + \sqrt{2}xy = 1 \] in the new coord system.

9. (a) method 1

When you change scales like this, the new coords \(X, Y\) and old coords \(x, y\) are related by \(X = \frac{1}{12}x, Y = 2y\) (Section 24). So \(x = 12X, y = \frac{1}{2}Y\) and

\[
q = 2(12X)^2 + 3(12X)(\frac{1}{2}Y) + 4(\frac{1}{2}Y)^2 = 288X^2 + 18XY + Y^2
\]

method 2

The new basis vectors are \(u = 12i, v = \frac{1}{2}j\). So \(P = \begin{bmatrix} 12 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\)

New matrix = \(P^T \begin{bmatrix} 2 & 3/2 \\ 3/2 & 4 \end{bmatrix} P = \begin{bmatrix} 288 & 9 \\ 9 & 1 \end{bmatrix}\) so \(q = 288X^2 + 18XY + Y^2\)
SOLUTIONS Section 9.2

1. (a) (i) \( q \) has matrix \( A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \).

The eigenvalues of \( A \) are \( 2, -3 \) so \( q = 2x^2 - 3y^2 \).
The corresponding eigenvectors are \((2,1), (-1,2)\).
Orthonormal eigenvectors are \( u = \left( \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right), v = \left( -\frac{1}{\sqrt{5}}, 2\sqrt{5} \right) \).
The coord system in which \( q \) is \( 2x^2 - 3y^2 \) has basis vectors \( u, v \).

(ii) \( \begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} X \\ Y \end{bmatrix} \) where \( P = \begin{bmatrix} u & v \end{bmatrix} \) so the change of variable is
\[
x = \frac{2}{\sqrt{5}} X - \frac{1}{\sqrt{5}} Y, \quad y = \frac{1}{\sqrt{5}} X + \frac{2}{\sqrt{5}} Y
\]

(iii) The old matrix for \( q \) was found in part (i); \( q = \begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} \)

(iv) The new matrix for \( q \) is \( B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \); \( q = \begin{bmatrix} X & Y \end{bmatrix} B \begin{bmatrix} X \\ Y \end{bmatrix} \)

(v) \( B = P^T A P \) where \( P = \begin{bmatrix} u & v \end{bmatrix} \).

(b) Use the orthonormal coord system from part (a) with basis \( u, v \). In the new system, the equation is \( 2x^2 - 3y^2 = 2 \).
The graph is a hyperbola whose major axis is the \( X \)-axis, the line \( y = \frac{1}{2} x \). The vertices are \((\pm 1,0)) \) which in the original coord system are \( \left( \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \) and \( \left( -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right) \).

2. (a) Let \( q = 2x^2 + 2y^2 + 3z^2 + 4xy + 2xz + 2yz \). Matrix for \( q \) is \( \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} \).

Eigenvalues are 2, 5, 0 with corresponding eigenvectors \( u = (-1,-1,2), \ v = (1,1,1), \ w = (1,-1,0) \)

They are already orthog. Normalize them to get orthonormal basis vectors. In this new orthonormal coord system, \( q = 2x^2 + 5y^2 \) so the equation is \( 2x^2 + 5y^2 = 9 \).
It's an elliptic cylinder with its axis along the \( Z \)-axis (see the diagram).

(b) If you use another method of diagonalizing, the new coord system won't necessarily be orthonormal. And if it isn't then you can't tell a circular cylinder from an elliptical cylinder (great tragedy).
(c) Let \( P = \begin{bmatrix} \text{unit} & \text{unit} & \text{unit} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{3} & 0 \end{bmatrix} \)

Then

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
\]

so

\[
x = -\frac{1}{\sqrt{6}} X + \frac{1}{\sqrt{3}} Y + \frac{1}{\sqrt{2}} Z,\quad y = -\frac{1}{\sqrt{6}} X + \frac{1}{\sqrt{3}} Y - \frac{1}{\sqrt{2}} Z,\quad z = \frac{2}{\sqrt{6}} X + \frac{1}{\sqrt{3}} Y.
\]

To solve for \( X,Y,Z \) in terms of \( x,y,z \) take advantage of the fact that \( P^{-1} = P^T \) since \( P \) is an ortho matrix. So

\[
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

\[
X = -\frac{1}{\sqrt{6}} x - \frac{1}{\sqrt{3}} y + \frac{2}{\sqrt{6}} z,\quad Y = \frac{1}{\sqrt{3}} x + \frac{1}{\sqrt{3}} y + \frac{1}{\sqrt{3}} z,\quad Z = \frac{1}{\sqrt{6}} x - \frac{1}{\sqrt{2}} y
\]

(d) Same q as in part (b) so the equation is \( 2x^2 + 5y^2 = -9 \). Graph is empty (no points satisfy the equation).

---

3. (a) Let \( A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \).

Do ops \( R_2 = -2R_1 + R_2 \), \( C_2 = -2C_1 + C_2 \) on \( A \) to get \( \begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix} \).

Do the col op on \( I \) to get \( P = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \).

Then \( q = x^2 - 6y^2 \) in the coord system with basis vectors \( u = (1,0), v = (-2,1) \).

(b) \( \begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \) so \( x = X - 2Y, y = Y \).

The check is that \( (X-2Y)^2 + 4(X-2Y)Y - 2Y^2 \) does equal \( x^2 - 6y^2 \).

(c) \( P^TAP = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix} \) which is what I got with the row/col op method.

4. Do these row/col ops on \( A \).

\( R_2 = -R_1 + R_2, \ C_2 = -C_1 + C_2 \)

\( R_3 = -2R_1 + R_3, \ C_3 = -2C_1 + C_3 \)

\( R_3 = -\frac{3}{2} R_2 + R_3, \ C_3 = -\frac{3}{2} C_2 + C_3 \)

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}
\]

and get
This is enough to diagonalize $q$ but if you want $\pm 1$'s keep going.

Divide Row 2 by $\sqrt{2}$, divide Col 2 by $\sqrt{2}$. Multiply Row 3 by $\sqrt{2}$, multiply Col 3 by $\sqrt{2}$ and get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Do all the col ops to $I$ to get

$$P = \begin{bmatrix} 1 & -1\sqrt{2} & -1/\sqrt{2} \\ 0 & 1\sqrt{2} & -3/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

In the coord system with the cols of $P$ as the basis vectors, $q = x^2 + y^2 + z^2$.

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x \\ Y \\ Z \end{bmatrix}$ so $x = \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} Y - \frac{1}{\sqrt{2}} Z, y = \frac{1}{\sqrt{2}} Y - \frac{3}{\sqrt{2}} Z, z = \sqrt{2} Z$.

5. Use row/col ops to diagonalize.

$$R_3 = -R_1 + R_3; C_3 = -C_1 + C_3$$
$$R_3 = -3R_2 + R_3; C_3 = -3C_2 + C_3$$

Matrix becomes $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -7 \end{bmatrix}$.

The diagonal has 2 positive and 1 negative entry. So there are 2 positive eigenvalues and one negative eigenvalue.

6. First, use the given information to get a diagonal version of $q$.

$$q = \frac{1}{3} x^2 - 4 y^2$$ in a coord system with basis $u = (\begin{smallmatrix} -4/5 \\ 3/5 \end{smallmatrix}), v = (\begin{smallmatrix} 3/5 \\ 4/5 \end{smallmatrix})$.

(Remember to normalize the orthogonal eigenvectors.)

Now that $q$ is diagonal, here are two ways to get the diagonal coeffs to be $\pm 1$'s.

*method 1* (row/col ops) The matrix for $q$ w.r.t. basis $u,v$ is $A = \begin{bmatrix} 1/3 & 0 \\ 0 & -4 \end{bmatrix}$.

Do the row/col ops

- multiply row 1 by $\sqrt{3}$; mult col 1 by $\sqrt{3}$
- divide row 2 by 2; divide col 2 by 2.

on $A$ to get $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Let $P = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$. Do the col ops on $P$ to get

$$Q = \begin{bmatrix} -4\sqrt{3}/5 & 3/10 \\ 3\sqrt{3}/5 & 4/10 \end{bmatrix}.$$ 

Then $q = x^2 - y^2$ in a coord system with basis vectors

$u_1 = (\begin{smallmatrix} -4\sqrt{3}/5 \\ 3\sqrt{3}/5 \end{smallmatrix}), v_1 = (3/10, 4/10)$ (the cols of $Q$).

*method 2* for continuing (mostly by inspection and a little algebra)

$$q = \frac{1}{3} x^2 - 4 y^2$$ in a coord system with basis $u = (\begin{smallmatrix} -4/5 \\ 3/5 \end{smallmatrix}), v = (\begin{smallmatrix} 3/5 \\ 4/5 \end{smallmatrix})$.

$q$ can be rewritten as $\left(\frac{1}{\sqrt{3}} x\right)^2 - (2\sqrt{3}/5 y)^2$.

Let $X_1 = \frac{1}{\sqrt{3}} x, Y_1 = 2\sqrt{3}/5 y$. Then $q = X_1^2 - Y_1^2$ in a new $X_1, Y_1$ coord system.
Now I just have to get the new basis vectors $u_1$ and $v_1$.

The change of variable from $X,Y$ to $X_1,Y_1$ just changed the scales on the $X$-axis and $Y$-axis (see (5), (6), (7) in Section 2.4):

$X_1$-scale $= \sqrt{3}$ $X$-scale

$Y_1$-scale $= \frac{1}{2}$ $Y$-scale

$u_1 = \sqrt{3} \ u = (-4\sqrt{3}/5, 3\sqrt{3}/5)$

$v_1 = \frac{1}{2} \ v = (3/10, 4/10)$

7. (a) $q = (x^2 + 4xy + 4y^2) - 2y^2 - 4y^2$

$= (x+2y)^2 - 6y^2$

$= X^2 - 6Y^2$

where

$X = x + 2y$, $Y = y$

$$
\begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
$$

$p^{-1}$

The new basis vectors are $u = (1,0)$, $v = (-2,1)$

(b) $q = 2(x^2 + \frac{3}{2}xy) + y^2$

$= 2(x^2 + \frac{3}{2}xy + \frac{9}{16}y^2) + y^2 - \frac{9}{8}y^2$

$= 2(x + \frac{3}{4}y)^2 - \frac{1}{8}y^2$

$= 2X^2 - \frac{1}{8}Y^2$

where

$X = x + \frac{3}{4}y$, $Y = y$

$$
\begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
1 & \frac{3}{4} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
$$

$p^{-1} = \begin{bmatrix}
1 & \frac{3}{4} \\
0 & 1
\end{bmatrix}$, $p = \begin{bmatrix}
1 & -\frac{3}{4} \\
0 & 1
\end{bmatrix}$

The new basis vectors are $u = (1,0)$, $v = (-3/4, 1)$

(c) $q = 3 \begin{bmatrix} x^2 + (2y+6z)x \end{bmatrix} - 6y^2 + z^2$

The coeff of $x$ in the brackets is $2y+6z$. Take half, square it and add it on to complete the square.

$= 3 \begin{bmatrix} x + y + 3z \end{bmatrix}^2 - 6y^2 + z^2 - 3(y+3z)^2$

$= 3 \begin{bmatrix} x + y + 3z \end{bmatrix}^2 - 9y^2 - 18yz - 26z^2$

The coeff of $y$ in the second bracket is $2z$. Take half, square it and add it on to complete the square.
\[ q = 3 \left[ x + y + 3z \right]^2 - 9 \left[ y^2 + 2zy + z^2 \right] - 26z^2 + 9z^2 \\
= 3 \left[ x + y + 3z \right]^2 - 9 \left[ y + z \right]^2 - 26z^2 + 9z^2 - 17z^2 \\
= 3x^2 - 9y^2 - 17z^2 \]

where

\[ X = x + y + 3z, \quad Y = y + z, \quad Z = z. \]

Then

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\
y \\
z \end{bmatrix},
\]

\[
p^{-1} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(I used Mathematica to invert)}
\]

so the new basis vectors are \( u = (1,0,0), \quad v = (-1,1,0), \quad w = (-2,-1,1). \)

(d) \[ q = 3 \left[ x^2 + \frac{2y+5z}{3} x \right] + y^2 + 4z^2 + 6yz \]

In the bracket, the coeff of \( x \) is \( \frac{2y+5z}{3} \). Take half, square it and add it on to complete the square.

\[
q = 3 \left[ x + \frac{2y+5z}{6} \right]^2 + y^2 + 4z^2 + 6yz - \frac{3}{6} \left( \frac{2y+5z}{6} \right)^2 \\
= 3 \left[ x + \frac{2y+5z}{6} \right]^2 + \frac{2}{3} y^2 + \frac{13}{3} yz + \frac{23}{12} z^2 \\
= 3 \left[ x + \frac{2y+5z}{6} \right]^2 + \frac{2}{3} \left[ y^2 + \frac{13}{2} z y \right] + \frac{23}{12} z^2 
\]

In the second bracket, the coeff of \( y \) is \( \frac{13}{2} z \). Take half, square it and add it on to complete the square.

\[
q = 3 \left[ x + \frac{2y+5z}{6} \right]^2 + \frac{2}{3} \left[ y + \frac{13}{4} z \right]^2 + \frac{23}{12} z^2 - \frac{2}{3} \left( \frac{13}{4} z \right)^2 \\
= 3 \left[ x + \frac{2y+5z}{6} \right]^2 + \frac{2}{3} \left[ y + \frac{13}{4} z \right]^2 - \frac{41}{8} z^2 
\]

Let \( X = x + \frac{1}{3} y + \frac{5}{6} z, \quad Y = y + \frac{13}{4} z, \quad Z = z. \) Then \( q = 3x^2 + \frac{2}{3} y^2 - \frac{41}{8} z^2. \)

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 5/6 \\ 0 & 1 & 13/4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\
y \\
z \end{bmatrix}.
\]

Invert \( p^{-1} \) to get \( P \). The new basis vectors are the cols of \( P \).
1. *method 1* Do these row/col ops

\[
\begin{align*}
R_2 &= -\frac{1}{2} R_1 + R_2, & C_2 &= -\frac{1}{2} C_1 + C_2 \\
R_3 &= -\frac{1}{2} R_1 + R_3, & C_3 &= -\frac{1}{2} C_1 + C_3 \\
R_3 &= R_2 + R_3, & C_3 &= C_2 + C_3
\end{align*}
\]

A turns into

\[
\begin{bmatrix}
2 & 0 & 1 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The diagonal entries are ≥ 0 so A is positive semi-definite (and one diagonal entry actually is 0 so A is not positive definite).

*method 2* The αθm's are \(\det[2] = 2\), \[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]

= 1, \(|A| = 0\).

The αθm's are ≥ 0 so A is positive semi-def (and not positive definite).

*method 3*

\[
\begin{vmatrix}
2-\lambda & 1 & 1 \\
1 & 1-\lambda & 0 \\
1 & 0 & 1-\lambda
\end{vmatrix}
= (2-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda) = (1-\lambda)(\lambda^2 - 3\lambda)
\]

Eigenvalues are 1, 0, 3, all ≥ 0 and one actually is 0 so A is positive semi-def (and not positive definite).

2. Many methods available. For 2 × 2 matrices using αθm's is fastest.

(a) \(\det[2] = 2\), \[
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

= 3. The αθm's are positive so matrix is pos def.

(b) \(\det[1] = 1\), \[
\begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
\]

= -3. Matrix is indefinite.

3. False (it's true for a diagonalized q, with only square terms, but not necessarily true otherwise). As a counterexample, let q = 2x^2 + 4xy + y^2. Then Q is negative when x = -1, y = 1 so q is not positive definite.

4. (a) Let q be the quadratic form with matrix A. Then -A has quadratic form -q.

For example, if \(A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}\) then the quadratic form with matrix A is \(ax^2 + 2bxy + cy^2\) and the quadratic form with matrix -A is \(-ax^2 - 2bxy - cy^2\).

If A is neg definite then \(q < 0\) (except at the origin) so -q > 0 (except at the origin) so -A is positive definite.

(b) The αθm's of A have signs + - + - + etc.

What happens to the αθm's when you multiply A by -1.

Remember that when you multiply an n × n matrix by -1, the determinant is multiplied by \((-1)^n\). That means it stays the same if n is even and changes sign if n is odd.

So the first αθm, a 1 × 1 det which was negative, changes sign.

The second αθm, a 2 × 2 det which was positive, doesn't change sign.

The third αθm, a 3 × 3 det which was negative, changes sign. And so on.

So the αθm's of -A are all positive. So -A is positive definite.

5. *method 1* Let q be the quadratic form corresponding to A. Then 3A corresponds to quadratic form 3q.

If q > 0 (except at the origin) then 3q > 0 (except at the origin)

So if A is positive definite then 3A is also positive definite.

*method 2* The αθm's of A are positive

When you multiply A by 3, you multiply the first αθm by 3, the second αθm by 3^2, the third αθm by 3^3 etc. The new αθm's are still positive so 3A is positive definite too.
6. (a) A is negative definite (the diagonal form of \( q \) has all negative coeffs). So the eigenvalues are negative (not necessarily the numbers \(-2, -3, -5\), but all negative).

\[ |A| \text{ is negative.} \]

\text{\textit{reason 1}} it's the product of the 3 negative eigenvalues

\text{\textit{reason 2}} \(|A|\) is the 3rd \(\text{OPM}\) and the signs of the \(\text{OPM}\)'s are \(-+−\)

(b) A is indefinite; A has two positive eigenvalues and one negative eigenvalue; \(|A|\) is the product of the eigenvalues so it's negative.

7. (a) \((B^T B)^T = B^T B^T = B^T B\). So \(B^T B\) is symmetric.

(b) Let \(\vec{x}\) be the col vector of variables. Then \(q = x^T (B^T B) x\)

(c) \(q = x^T (B^T B) x = (Bx)^T Bx \) \(\text{(T rule)}\)

\[ = \|Bx\|^2 \] \((\text{connection between dotting vectors and multiplying matrices})\)

\[ \geq 0 \] \((\text{since norms are never negative})\)

So \(q\) is positive semi-definite.

8. (a) True. \text{\textit{reason 1}} \(|A|\) is one of the \(\text{OPM}\)'s and all the \(\text{OPM}\)'s are positive.

\text{\textit{reason 2}} All the eigenvalues of \(A\) are positive and \(|A|\) is the product of the eigenvalues.

(b) False. A counterexample is \(A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\).

\(|A|\) is positive but \(A\) is negative definite (its \(\text{OPM}\)'s are \(-2, +6\)).

9. The matrix for \(q\) is \(\begin{bmatrix} 2 & h/2 \\ h/2 & 3 \end{bmatrix}\). The \(\text{OPM}\)'s are 2 and \(6 - \frac{1}{4} h^2\).

(a) \(q\) is positive def iff \(6 - \frac{1}{4} h^2 > 0, \; -\sqrt{24} < h < \sqrt{24}\)

(b) \(q\) is positive semi-def iff \(6 - \frac{1}{4} h^2 > 0, \; -\sqrt{24} \leq h \leq \sqrt{24}\)

(c) \(q\) is positive semi-def but not pos def iff \(h = \pm \sqrt{24}\)

(d) \(q\) is indefinite iff \(6 - \frac{1}{4} h^2 < 0, \; h > \sqrt{24}\) or \(h < -\sqrt{24}\)

(e) Can never have \(q\) negative definite.

10. (a) \text{\textit{proof 1}}

\text{Step 1} The eigenvalues of \(A\) are nonzero. That's because a positive definite matrix always has positive eigenvalues which means that as a by-product, the eigenvalues are nonzero.

\text{Step 2} A matrix with nonzero eigenvalues is invertible (see the latest invertible list in Section 8.2).

\text{\textit{proof 2}} The \(\text{OPM}\)'s of \(A\) are positive. But the last \(\text{OPM}\) is \(|A|\) itself. So \(|A| \neq 0\) so \(A\) is invertible.

(b) The converse (that if \(A\) is invertible then it is positive definite) is false. As a counterexample, the matrix \(-I\) is invertible but isn't positive def (it's neg def).

(c) First I'll show that the eigenvalues of \(A^{-1}\) and \(A\) are reciprocals of one another.

Suppose \(\lambda\) is an eigenvalue of \(A\) with corresponding eigenvector \(u\).

Then \(A u = \lambda u\) where \(u \neq \vec{0}\) so \(u = A^{-1} \lambda u = \lambda A^{-1} u\).

\(\lambda \neq 0\) since \(A\) is invertible so it's safe to divide by \(\lambda\) to get \(A^{-1} u = \frac{1}{\lambda} u\).
This makes \( \mathbf{u} \) an eigenvector of \( \mathbf{A}^{-1} \) with corresponding eigenvalue \( 1/\lambda \).
So \( \mathbf{A} \) and \( \mathbf{A}^{-1} \) have reciprocal eigenvalues.

So if the eigenvalues of \( \mathbf{A} \) are positive then the eigenvalues of \( \mathbf{A}^{-1} \) must also be positive. So if \( \mathbf{A} \) is positive definite then so is \( \mathbf{A}^{-1} \) QED.

11. (a) True. Here's a proof by contradiction.
   The quadratic form with matrix \( \mathbf{A} \) is
   \[
   q = ax^2 + dy^2 + fz^2 + 2bx y + 2cxz + 2eyz.
   \]
   Suppose \( d \) was negative. Then \( q \) would be negative when \( x=0, y=1, z=0 \) which contradicts the fact that \( \mathbf{A} \) is positive definite.
   Suppose \( d \) was zero. Then \( q = 0 \) when \( x=0, y=0, z=1 \) which again contradicts the fact that \( \mathbf{A} \) is positive definite.
   So \( d \) must be positive. Similarly, \( a \) and \( f \) must be positive.

   (b) False. Here's a counterexample. Let
   \[
   \mathbf{A} = \begin{bmatrix}
   1 & 2 & . \\
   2 & 1 & . \\
   . & . & 1
   \end{bmatrix}
   \]
   be positive but \( \mathbf{A} \) is not positive definite. One reason is that the second \( \mathbf{0} \) term is not positive. Another reason is that the quadratic form with matrix \( \mathbf{A} \) is
   \[
   q = x^2 + y^2 + z^2 + 4xy + \text{who cares. It is negative when } x=1, y=-1, z=0.
   \]
SOLUTIONS review problems for Chapter 9

1. q has matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

   (a) (i) The new basis just amounts to a change of scale on the old axes (see "special case where only the scale changes" in §2.4).
   So $X = \frac{1}{2} x$, $Y = 3y$; $x = 2X$, $y = \frac{1}{3} y$. So
   
   $q = 2x^2 - 4xy + 5y^2 = 2(2X)^2 - 4(2X)(\frac{1}{3}Y) + 5(\frac{1}{3}Y)^2 = 8X^2 - \frac{8}{3} XY + \frac{5}{9} Y^2$

   (ii) Let $P = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$. Then
   
   new matrix for $q$ is $P^TAP = \begin{bmatrix} 8 & -4/3 \\ -4/3 & 5/9 \end{bmatrix}$. So $q = 8x^2 - \frac{8}{3} XY + \frac{5}{9} Y^2$.

(b) method 1

   Eigenvalues are $\lambda$ are 6,1 with corresponding eigenvectors $(1,-2)$, $(2,1)$. They are already orthogonal. Normalize them to get

   $u = (\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$, $v = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$

   With basis $u,v$, $q = 6X^2 + Y^2$.

   If you use the eigenvectors $(1,-2)$ and $(2,1)$ as the basis, without normalizing them, then $q$ is still diagonal but it's $30X^2 + 5Y^2$, not $6X^2 + Y^2$.

   method 2

   Do row/col op $R_2 = R_1 + R_2$, $C_2 = C_1 + C_2$. A turns into $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

   Do the col op to I and get $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. With new basis $u = (1,0)$, $v = (1,1)$ (don't normalize) the quadratic form is $q = 2x^2 + 3y^2$.

   If you want to normalize $u$ and $v$ (if you have some desperate compulsion to normalize) then you can use new basis vectors $u_{\text{unit}}$, $v_{\text{unit}}$ but then $q$ is $2x^2 + \frac{3}{2}y^2$, not $2x^2 + 3y^2$.

   method 3

   $q = 2(x^2 - 2xy) + 5y^2$

   $= 2(x^2 - 2xy + y^2) + 5y^2 - 2y^2$

   $= 2(x-y)^2 + 3y^2$

   $= 2x^2 + 3y^2$

   where $X = x-y$, $Y = y$.

   Then

   $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

   so

   $p^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   The new basis vectors are $u = (1,0)$, $v = (1,1)$.

(c) The best system in which to graph is the orthonormal system from method 1. The equation is $6X^2 + Y^2 = 7$ and because it's an orthonormal system you know that the graph is an ellipse (not a circle).

   The major axis is the $Y$-axis, the line through the origin pointing like vector $(2,1)$. So the major axis is line $y = \frac{1}{2} x$ (see the diagram).

   The minor axis is the $X$-axis, the line through the origin pointing like vector $(1,-2)$. So the minor axis is the line $y = -2x$.

   The vertices of the ellipse are points C and D. In the new coord system, D has coords $X=0$, $Y=\sqrt{7}$.

   $p\begin{bmatrix} 0 \\ \sqrt{7} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{7} \end{bmatrix} = \begin{bmatrix} 2\sqrt{7}/\sqrt{5} \\ \sqrt{7}/\sqrt{5} \end{bmatrix}$
So the old coords of D are \( x = \frac{2\sqrt{7}}{\sqrt{5}} , y = \frac{\sqrt{7}}{\sqrt{5}} \).

Similarly, C has new coords \( X = 0 , Y = -\sqrt{7} \) and old coords \( x = \frac{-2\sqrt{7}}{\sqrt{5}} , y = \frac{-\sqrt{7}}{\sqrt{5}} \).

Problem 1(c)

2. (a) method 1 The pm's are 2,3,4 so A is positive definite.

method 2 Use row/col ops

\[
\begin{align*}
\text{add } &\frac{1}{2} \text{ row 1 to row 2, add } \frac{1}{2} \text{ col 1 to col 2} \\
\text{add } &-\frac{1}{2} \text{ row 1 to row 3, add } -\frac{1}{2} \text{ col 1 to col 3} \\
\text{add } &\frac{1}{3} \text{ row 2 to row 3, add } \frac{1}{3} \text{ col 2 to col 3}
\end{align*}
\]

New matrix is \[
\begin{bmatrix}
2 & 0 & 0 \\
0 & \frac{3}{2} & 0 \\
0 & 0 & \frac{4}{3}
\end{bmatrix}
\]. Diagonal entries are positive so A is pos def.

(b) Continue from the row/col ops in part (a).

\[
\begin{align*}
\text{divide row 1 by } &\sqrt{2} , \text{ divide col 1 by } \sqrt{2} \\
\text{multiply row 2 by } &\sqrt{2}/3 , \text{ multiply col 2 by } \sqrt{2}/3 \\
\text{multiply row 3 by } &\sqrt{3}/4 , \text{ multiply col 3 by } \sqrt{3}/4
\end{align*}
\]

All the row/col ops put together turn A into I.

Do all the col ops to I to get \[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{(2\sqrt{3})} & -\frac{\sqrt{3}}{6} \\
0 & \frac{2}{3} & \frac{\sqrt{3}}{6} \\
0 & 0 & \frac{\sqrt{3}}{2}
\end{bmatrix}
\]

Then \( P^T A P = I \)

3. \( q = \vec{x}^T A \vec{x} \)

4. There is some (invertible) P so that A = \( P^T B P \). Then

\[
|AB| = |P^T B P| = |P^T| \cdot |B| \cdot |P| \cdot |B| \quad \text{(det rule)}
\]

\[
\geq 0 \quad \text{(the product of real squares is \( \geq 0 \))}
\]

(Didn't need the fact that P is invertible.)

5. \( M^T A M = B \) (The quadratic form that has matrix A w.r.t. the standard basis has matrix B w.r.t. the new basis composed of the cols of M.)

6. old = \[
\begin{bmatrix}
1 & 3/2 \\
3/2 & -1
\end{bmatrix}
\], new = \[
\begin{bmatrix}
-2 & 0 \\
0 & 3
\end{bmatrix}
\].

Let P have cols u and v. Then \( P^T \) old P = new.