SOLUTIONS Section 7.1

1. \( |z| = \sqrt{25 + 9} = \sqrt{34} \)
   
   \( \bar{z} = 5 + 3i \)

   \( zw = 2 - 42i \)

   \( \frac{1}{z} = \frac{1}{5-3i} \cdot \frac{5+3i}{5+3i} = \frac{5+3i}{34} = \frac{5}{34} + \frac{3}{34}i \)

   \( z^2 = 16 - 30i \)

2. \( z \) must be real.

3. \( z \) must be pure imag (e.g., \( 3i, 6i, -2i \)).

4. \( A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^5 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = A \)

   The products repeat with period 4.

   By \( A^{243} \) you've gone through 60 cycles plus 3 rounds (because \( 243 = 60 \cdot 4 + 3 \)) so you land back at \( A^3 \). So \( A^{243} = A^3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, A^{244} = A^4, A^{245} = A \).

5. (a) \( \sqrt{53} \) (b) \( 1 \) (c) \( 3 \) (d) \( 2 \) (e) \( 7 \) (f) \( 1 \) (g) \( \sqrt{41} \)

   (h) \( \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \)

6. (a) \( zw = (2-i)(-3+4i) = -2 + 11i \), \( \bar{zw} = -2 - 11i \)

   \( \bar{z} \bar{w} = (2+i)(-3-4i) = -2 - 11i \)

   (b) \( zw = -2 + 11i \), \( |zw| = \sqrt{125} = 5\sqrt{5} \)

   \( |z| \cdot |w| = \sqrt{5} \cdot \sqrt{25} = 5\sqrt{5} \)

7. You don't have to actually multiply \( \pi + \sqrt{2}i \) and \( \sqrt{17} + i \) together. By (5), the magnitude of the product is the product of the separate magnitudes. So

   \( \text{mag of the product} = \text{mag of } z \cdot \text{mag of } \sqrt{17} + i \)

   \( = \sqrt{\pi^2 + 2} \cdot \sqrt{18} \)
SOLUTIONS Section 7.2

1. \( A^* = \begin{bmatrix} 2 & 3+2i \\ i & 4-6i \end{bmatrix} \)

2. (a) 6-2i  (b) 7  (c) 6i

3. \((A^*BC)^* = C^*B^*A^{**} = C^*B^*A\)

4. \( A^* = \begin{bmatrix} a & d & \bar{g} \\ \bar{b} & \bar{e} & \bar{h} \\ -c & f & \bar{k} \end{bmatrix} \). To find \(|A^*|\) I'll expand down col 1.

\[ |A^*| = \begin{vmatrix} a & d & \bar{g} \\ \bar{b} & \bar{e} & \bar{h} \\ -c & f & \bar{k} \end{vmatrix} = a(\bar{e}k - \bar{f}h) - b(dk - fg) + c(dh - eg) \]

by conjugate rules

The expression under the conjugate sign on the righthand side happens to be \(|A|\) expanded across row 1. So

\[ |A^*| = \text{det of } A \quad \text{QED} \]

5. (a) True. The diagonal entries are still real and matching entries \(a+bi\) and \(-a-bi\) are now \(-a-bi\) and \(-a+bi\) which are still conjugates.

(b) True. \(H^*\) is \(H\) so of course it's Herm.

6. method 1 (good for 3 \( \times \) 3's only)

Let

\( A = \begin{bmatrix} a1 + a2i & a2 + b2i & a3 + b3i \\ a4 + b4i & a5 + b5i & a6 + b6i \\ a7 + b7i & a8 + b8i & a9 + b9i \end{bmatrix} \)

where \(a1, ..., a9, b1, ..., b9\) are all real.

Then

\( A + A^* = \begin{bmatrix} 2a1 & a2+a4 + (b2-b4)i & \text{etc} \\ a2+a4 - (b2-b4)i & 2a5 & \text{etc} \\ \text{etc} & \text{etc} & 2a9 \end{bmatrix} \)

You can see that \(A+A^*\) is Herm.

method 2 (good for \(n \times n\)'s in general)

I'll show that \((A + A^*)^* = A + A^*\).

\[(A + A^*)^* = A^* + A^{**} \quad * \text{ rule}\]
\[= A^* + A \quad * \text{ rule}\]
\[= A + A^* \]

7. Want to show that \((A^*HA)^* = A^*HA\).

\((A^*HA)^* = A^{**}H^*A^{**} = A^*H^*A = A^*HA \quad \text{by * rules and the hypothesis that H is Herm})

8. Want to show that \((H^{-1})^* = H^{-1}\).

\[(H^{-1})^* = (H^*)^{-1} \quad * \text{ rule}\]
\[= H^{-1} \quad H \text{ is Herm so } H^* = H \]


method 2 (good for \(3 \times 3\)'s) Take the typical complex matrix \(A\) from method 1 of #6 and compute \(A - A^*\) to see that it is skew Herm.

10. \(K^2)^* = (KK)^* = K^*K^* = (-K)(-K) = K^2\) so \(K^2\) is Herm.

\((K^3)^* = (KKK)^* = K^*K^*K^* = (-K)(-K)(-K) = -K^3\) so \(K^3\) is skew Herm.
1. No since \( u \cdot v = (-i)(-1) + (1)(i) = 2i, \) not 0 (remember to conjugate the components of the first vector before you multiply and add).

2. \( u = (1+2i, 3i, 4) \)
   
   \( u \cdot v = (2+i)(3-i) - 3i+4i(1-2i) = 15 + 2i \)  
   
   **Warning**  
   Don't write \( u \cdot v = \bar{u} \cdot \bar{v} \)

   \( v \cdot u = \bar{u} \cdot \bar{v} = 15 - 2i \)

   \[ \|u\| = \sqrt{4 + 1 + 9 + 16} = \sqrt{30} \]

   \[ \|v\| = \sqrt{9 + 1 + 1 + 1 + 4} = 4 \]

   \[ |u \cdot v| \) (meaning the mag of the complex number \( u \cdot v \)) = \sqrt{225 + 4} = \sqrt{229} \]

   \( v_{\text{unit}} = \left( \frac{3-i}{4}, -\frac{i}{4}, \frac{1-2i}{4} \right) \)

3. (a) \( \sqrt{1 + 1} = \sqrt{2} \)  
   (b) \( \sqrt{4 + 9 + 1} = \sqrt{14} \)  
   (c) \( \sqrt{9 + 1 + 4 + 16} = \sqrt{30} \)

4. (a) \( (u+v) \cdot (u-v) = u \cdot u - v \cdot v + v \cdot u - u \cdot v \)
   
   \[ = \|u\|^2 - \|v\|^2 + v \cdot u - u \cdot v \]

   \[ = 9 - 49 + 6 + 2i - (6-2i) \]

   \[ = -40 + 4i \]

   (b) \( \|u + iv\|^2 = (u + iv) \cdot (u + iv) \)

   \[ = u \cdot u + iv \cdot u + u \cdot iv + iv \cdot iv \]

   \[ = u \cdot u + i(v \cdot u) + i(v \cdot u) + i \cdot i(v \cdot v) \]

   \[ = \|u\|^2 - i(v \cdot u) + i(u \cdot v) + \|v\|^2 \]

   \[ = 9 - i(6 + 2i) + i(6 - 2i) + 49 \]

   \[ = 62 \]

   \[ \|u + iv\| = \sqrt{62} \]

   (c) \( (2-3i)u \cdot iu + v \cdot iu = 2-3i \)  

   \[ = (2+3i)i\|u\|^2 + i(6 + 2i) = -29 + 24i \]

   (d) \( \|6iu\| = |6i|\|u\| \) (where \( |6i| \) means the mag of the number \( 6i \))

   \[ = 6 \times 3 = 18 \]

   (e) \( \|(2-3i)u\| = |2-3i|\|u\| = 3 \sqrt{13} \)

5. (a) True  
   (b) False. It multiplies the norm by the mag of \( 3i \) which is 3.

6. (a) True because the diagonals of a rhombus are perp.  
   (b) If \( \|u\| = \|v\| \) then \( (u+v) \cdot (u-v) = u \cdot u - v \cdot v \)

   \[ = \|u\|^2 - \|v\|^2 \]

   \[ = 0 \]

   so \( u+v \) and \( u-v \) are orthog.

   (c) The step \( (u+v) \cdot (u-v) = u \cdot u - v \cdot v \) breaks down in \( \mathbb{C}^n \). What you have instead is \( (u+v) \cdot (u-v) = u \cdot u + v \cdot u - u \cdot v + v \cdot v \).

   In \( \mathbb{R}^n \), \( u \cdot v \) and \( v \cdot u \) are equal so those terms cancel out. But they aren't necessarily equal in \( \mathbb{C}^n \).

   For a counterexample, choose \( u \) and \( v \) so that \( u \cdot v \) is non-real so that it doesn't equal \( v \cdot u \). For instance let \( u = (i,0) \) and \( v = (1,0) \). Then \( \|u\| = \|v\| \) (both are 1) but \( (u+v) \cdot (u-v) = (1+i, 0) \cdot (-1+i, 0) = (1-i)(-1+i) = 2i, \) not 0.
7. \[ \|x + iy\|^2 = (x + iy) \cdot (x + iy) \]
   \[ = x \cdot x + (iy) \cdot (iy) + (iy) \cdot x + x \cdot (iy) \]
   \[ = x \cdot x + (-i)(i)(y \cdot y) + (-i)(y \cdot x) + i(x \cdot y) \]
   \[ = \|x\|^2 + \|y\|^2 + i(x \cdot y - y \cdot x) \]

8. Let \( x \cdot y = a + bi \). Then \( y \cdot x = a - bi \) and

   \[ \frac{1}{2} i(y \cdot x) - \frac{1}{2} i(x \cdot y) = \frac{1}{2} i(a-bi) - \frac{1}{2} i(a+bi) = b \text{ which is the imag part of } x \cdot y \]

9. (a) Let \( x = au + bv \). Then

   \( (2+3i, 6-7i) = a(i,0) + b(0,i) \)

   \[ 2+3i = ia, a = \frac{2+3i}{i} = -2i+3 \text{ (note that } 1/i = -i) \]

   \[ 6-7i = ib, b = \frac{6-7i}{i} = -6i -7 \]

   (b) \( P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}, \quad P^{-1} \begin{bmatrix} 2+3i \\ 6-7i \end{bmatrix} = \begin{bmatrix} 3-2i \\ -7-6i \end{bmatrix} \)

   so the new coords of \( x \) are \( 3-2i \) and \( -7-6i \)

(c) \( u \) and \( v \) are orthog and unit vectors so

   \( x = (u \cdot x)u + (v \cdot x)v = (3-2i)u + (-7-6i)v \)

   \textbf{warning} It’s OK to use the formula \( x = \frac{u \cdot x}{u \cdot u} u + \frac{v \cdot x}{v \cdot v} v \) but make sure you get \( u \cdot u = 1 \). It is \textit{wrong} to say \( u \cdot u = i^2 = -1 \).

10. It computes \( v \cdot u \) or equivalently, \( -u \cdot v \).
SOLUTIONS Section 7.4

1. Let $U_1$ and $U_2$ be unitary. Want to show that $(U_1 U_2) (U_1 U_2)^* = I$.

   $(U_1 U_2) (U_1 U_2)^* = U_1 U_2 U_2^* U_1^*$ \hspace{1cm} * rule
   
   $= U_1^* U_1 U_2 U_2^*$ \hspace{1cm} since $U_2$ is unitary
   
   $= I$ \hspace{1cm} since $U_1$ is unitary

2. The cols of $U$ are orthonormal. So the rows of $U^T$ are orthonormal so $U^T$ is also unitary.

3. (a) Not unitary since $||M|| = 2$ not 1.
   
   (b) $||M|| = 1$ which is inconclusive.
   
   (c) Not unitary since $||M|| = \sqrt{5}$ not 1.
   
   (d) $||M|| = 1$, inconclusive.
   
   (e) $||M|| = 1$, inconclusive.

4. Not unitary since the cols are no longer unit vectors (they have norm 3 now).

5. $(U^{-1})^* = (U^{-1})^{-1}$ \hspace{1cm} * rule

   $= U^{-1} (U^{-1})^{-1}$ \hspace{1cm} since $U$ is unitary
   
   $= U^{-1} U$
   
   $= I$

6. If $U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $U_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ then $U_1 + U_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

   $U_1$ and $U_2$ are unitary but $U_1 + U_2$ is not.
SOLUTIONS review problems for Chapter 7

1. \( u \cdot v = (-2i)(3+4i) + (-6)(5+6i) = -22 - 42i \)
   \( v \cdot u = u \cdot v = -22 + 42i \)
   \( \|u\| = \sqrt{4 + 36} = \sqrt{40} \)
   \( \|v\| = \sqrt{9 + 16 + 25 + 36} = \sqrt{86} \)

2. Let the scalar \( w \cdot v \) be called \( k \) temporarily. Then
   \[
   (v - kw) \cdot w = v \cdot w - (kw) \cdot w = v \cdot w - \frac{k}{k} (w \cdot w)
   \]
   \[
   = v \cdot w - 9k
   \]
   \[
   = v \cdot w - 9 (w \cdot v)
   \]
   plug the \( k \) back in
   \[
   = v \cdot w - 9 (v \cdot w)
   \]
   \[=-8(v \cdot w)\]

3. \( H = H^* \) so \( |H| = |H^*| = |H| \).
   So \( |H| \) equals its conjugate. So \( |H| \) is real.

4. Want to show that \( (U^{-1}HU)^* = U^{-1}HU \).
   \( (U^{-1}HU)^* = U^* H^* (U^{-1})^* \) \( * \) rule
   \( = U^{-1} H (U^*)^* \) since \( U \) is unitary and \( H \) is Herm
   \( = U^{-1}HU \)

5. Let
   \[
   A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}
   \]
   Then \( A \) and \( B \) are Herm but \( AB = \begin{bmatrix} 14 & 34 \\ 14 & 34 \end{bmatrix} \) is not Herm.

6. (a) \textit{method 1} Each col has norm 1. And the dot product of any col with any other col is 0. So the cols are orthonormal which makes \( A \) unitary.

   \[ \text{method 2} \]
   Do the same thing as method 1 but with rows.

   \[ \text{method 3} \]
   \[
   A^*A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} = I
   \]
   So \( A \) is unitary.

   (b) \( A^{-1} \) is \( A^* \), which I found above.

7. (a) Not possible. First col isn't a unit vector.
    (b) Let \( x = -3i \), \( y \) any real number
    (c) Let \( x = 3i \), \( y \) any pure imaginary.
8. \( \|u\| = \sqrt{1 + 1 + 4 + 9 + 4} \) so \( u_{\text{unit}} = \left( \frac{i}{\sqrt{19}}, \frac{i}{\sqrt{19}}, \frac{2}{\sqrt{19}}, \frac{2+3i}{\sqrt{19}} \right) \).

9. The diagonal entries are real by property (5) of dot products.
   The matching entries like \( u \cdot w \) and \( w \cdot u \) are conjugates by property (1) of dot products.
   So \( A \) is Hermitian.