SOLUTIONS Section 5.1

1. (a) \( x_6 = 7 \)
\[ x_3 = 6 - 6x_5 - 4x_4 \]
\[ x_1 = 5 - 5x_5 - 3x_4 - 2x_2 \]
\[ x_2, x_4, x_5 \text{ free} \]
The solution can also be written as
\[ x_6 = 7 \]
\[ x_5 = t \]
\[ x_4 = s \]
\[ x_3 = 6 - 6t - 4s \]
\[ x_2 = r \]
\[ x_1 = 5 - 5t - 3s - 2r \]
(b) No sols because of row 3
(c) \( x_1 = 5, x_2 = 6 \) (no free variables)
(d) \( x_1 = 0; x_2, x_3, x_4 \text{ free} \)
The solution can also be written as \( x_1 = 0, x_2 = r, x_3 = s, x_4 = t. \)
(e) \( x_1 = 5 - 2x_2 \)
\[ x_3 = 3 \]
\[ x_2, x_4 \text{ free} \]
The solution can also be written as
\[ x_1 = 5 - 2t \]
\[ x_2 = t \]
\[ x_3 = 3 \]
\[ x_4 = s \]

2. (a) method 1 Line up \( u,v,w,y \) as cols:
\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & -1 \\
1 & 1 & 2 & 1 \\
1 & 0 & 1 & 2 \\
\end{bmatrix}
\]
Do the row ops
row3 = -row1 + row3
row4 = -row1 + row4;
row3 = -row2 + row3;
to get
\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
You can see that in the echelon form, \( C_4 = 2C_1 - C_2 \) so \( y \) is in the subspace spanned by \( u,v,w \) and in particular \( y = 2u - v \).
There are other ways to write \( y \) as a comb of \( u,v,w \). In the echelon form, \( C_4 = 3C_1 - C_3 \) so \( y = 3u - w \) etc.
The subspace is 2-dim because the first three echelon cols are dep and and the first two are ind.
method 2 Line up \( u,v,w \) (not \( y \)) as rows and row op into echelon form:
\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 \\
\end{bmatrix}
\]
row ops to
\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 2 & 1 \\
\end{bmatrix}
\]
The nonzero echelon rows are a basis for the echelon ow space and for the original
row space as well. So the space spanned by \(u,v,w\) is the set of vectors of the form 
\[
a(1,0,1,1) + b(0,1,1,0), i.e., of the form (a,b, a+b, a).
\]
By inspection, \(y\) is of this form (with \(a = 2, b = -1\)) so \(y\) is in the subspace 
spanned by \(u,v,w\).

(b) Solve \(y = au + bv + cw\)

\[
\begin{align*}
a + c &= 2 \\
b + c &= -1 \\
a + b + 2c &= 1 \\
a + c &= 2 \\
1 & 0 1 2 \\
0 & 1 1 -1 \\
1 & 1 2 1 \\
1 & 0 1 2
\end{align*}
\]
Do the row ops from part (a) to get

\[
\begin{align*}
1 & 0 1 2 \\
0 & 1 1 -1 \\
0 & 0 0 0 \\
0 & 0 0 0
\end{align*}
\]
Sol is

\[
\begin{align*}
a &= 2-c \\
b &= -1 - c \\
c &= \text{anything}
\end{align*}
\]
Since the system of equations is consistent, \(y\) is in the subspace spanned by \(u,v,w\).
Since we got more than one solution, the vectors \(u,v,w\) can't be a basis (if \(u,v,w\) were a basis then the coords of \(y\) w.r.t. \(u,v,w\) would be unique and there would have been just one solution).
To express \(y\) in terms of \(u,v,w\) choose say \(c = 0\). Then \(a = 2, b = -1\) and \(y = 2u - v\).
Or you could choose \(c = 28\). then \(a = -26, b = -29\) and \(y = -26u - 29v + 28w\) etc.

3. Start with

\[
\begin{align*}
1 & -2 3 1 \\
2 & k 6 6 \\
-1 & 3 k-3 0
\end{align*}
\]
and do row ops

\[
\begin{align*}
R2 &= -2R1 + R2 \\
R3 &= R1+ R3 \\
R3 &\leftrightarrow R2 \\
R3 &= -(4+k)R2 + R3
\end{align*}
\]
to get

\[
\begin{align*}
1 & -2 3 1 \\
0 & 1 k 1 \\
0 & 0 -k(k+4) -k \\
0 & 0 0 0
\end{align*}
\]

\[\text{case 1 } k = 0 \quad \text{Then the system is}\]

\[
\begin{align*}
1 & -2 3 1 \\
0 & 1 0 1 \\
0 & 0 0 0 \\
0 & 0 0 0
\end{align*}
\]
and it's consistent. There is a col without a pivot (the third) so there are
infinitely many solutions.

\[\text{case 2 } k = -4 \quad \text{Then the system is}\]

\[
\begin{align*}
1 & -2 3 1 \\
0 & 1 -4 1 \\
0 & 0 0 4 \\
0 & 0 0 0
\end{align*}
\]
and is inconsistent.

\[\text{case 3 } k \neq 0,4 \quad \text{Then the system is}\]
1 -2 3 | 1
0 1 k | 1
0 0 nonzero | -k
0 0 0 | 0

and there is one solution.

4. (a) False.

\[
\begin{array}{ccc|c}
1 & 0 & 2 & 4 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

There can be no sols, e.g.,
\[
\begin{array}{ccc|c}
0 & 1 & 3 & 5 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

There can be infinitely many solutions, e.g.,
\[
\begin{array}{ccc|c}
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

Can't have exactly one solution because it isn't possible for all the echelon cols to have pivots.

(b) False.

\[
\begin{array}{ccc|c}
1 & 0 & 0 & 2 \\
0 & 0 & 0 & 2 \\
\end{array}
\]

System can have no sols, e.g.,
\[
\begin{array}{ccc|c}
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
\end{array}
\]

Can have one solution, e.g.,
\[
\begin{array}{ccc|c}
0 & 1 & 3 & 6 \\
0 & 0 & 0 & 7 \\
\end{array}
\]

Can have infinitely many solutions, e.g.,
\[
\begin{array}{ccc|c}
0 & 1 & 2 & 5 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

(c) False. Can have one solution, e.g,
\[
\begin{array}{ccc|c}
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
\end{array}
\]

Can have no sols, e.g.,
\[
\begin{array}{ccc|c}
0 & 1 & 3 & 6 \\
0 & 0 & 0 & 7 \\
\end{array}
\]

Can have infinitely many solutions, e.g.,
\[
\begin{array}{ccc|c}
0 & 1 & 2 & 5 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

5. Rank \( A = 2 \), number of variables = 7 so \( n-r = 5 \). Either there are no sols or there are infinitely many with 5 free variables.

6. (a) \( x_3 = \frac{-2 - 2x_5 - 7x_4}{5} \)

\[ \begin{align*}
x_1 &= \frac{4 - x_5 - 6x_4 - 4x_3 - 3x_2}{2} \\
x_4, x_5, x_2 &= \text{free} \end{align*} \]

(b) Use row ops
\[
\begin{array}{c}
R3 = -R1 + R3 \\
R3 = -R2 + R3 \\
R4 = -R2 + R4 \\
R3 \leftrightarrow R4 \\
\end{array}
\]
to get
\[
\begin{array}{cccc|c}
2 & 8 & 5 & 2 & -6 \\
0 & 4 & 6 & 3 & 3 \\
0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Sol is
\( v = 0 \)
\[
\begin{align*}
y &= \frac{9 - 3v - 3u - 6z}{4} = \frac{9 - 3u - 6z}{4} \\
x &= \frac{8 - 2u - 5z - 3y}{2} = -5 + \frac{7}{2}z + 2u \\
u, z \text{ free}
\end{align*}
\]

(c) \( \text{row2} = -2 \text{ row1} + \text{row2} \)
\( \text{row3} = \text{row1} + \text{row3} \);
\( \text{row3} = 3 \text{ row2} + \text{row3} \);

Get
\[
\begin{array}{ccc|c}
2 & 1 & 1 & 1 \\
0 & -1 & -2 & -4 \\
0 & 0 & -4 & -4 \\
\end{array}
\]

Sol is
\[
\begin{align*}
z &= 1 \\
y &= 4 - 2z = 2 \\
x &= \frac{1 - y - z}{2} = -1 \\
\text{(no free variables)}
\end{align*}
\]

7. (a) \( z = 6 - 4w, x = 5 - 3w - 2y \)
(b) \( z = 6 - 4w \)
\[
y = \frac{1}{2} (5 - 3w - x)
\]
(c) \( w = \frac{1}{4} (6 - z) \)
\[
x = 5 - 3w - 2y = 5 - 3 \cdot \frac{1}{4} (6-z) - 2y = \frac{1}{2} + \frac{3}{4} z - 2y
\]
(d) Not possible.

8. \( A|b \) row ops to
\[
\begin{array}{ccc|c}
1 & 0 & 0 & . \\
0 & 1 & 0 & . \\
0 & 0 & 1 & . \\
0 & 0 & 0 & 0 \\
\end{array}
\]

So \( A|c \) has either no sols (if the row-opped \( c \) has a nonzero 4th entry) or one solution (if the row-opped \( c \) has a zero 4th entry).

9. (a) \( A \) has 3 echelon col with pivots; \( A|b \) row ops to something like this:
\[
\begin{array}{ccc|c}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

There is no room for an all zero echelon row so it's not possible for \( Ax = b \) to be inconsistent. There must be infinitely many solutions with 2 free variables.
(b) \( A \) has 2 echelon cols with pivots.
Could have infinitely many sols with 3 free variables. Or could have no sols if the system row ops to something like
\[
\begin{array}{ccc|c}
1 & 2 & 0 & 4 & 6 \\
0 & 0 & 1 & 5 & 7 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\] nonzero
(c) A has 3 echelon cols with pivots so $A|b$ row ops to
\[
\begin{bmatrix}
1 & 0 & 0 & | & . \\
0 & 1 & 0 & | & . \\
0 & 0 & 1 & | & . \\
0 & 0 & 0 & | & . \\
\end{bmatrix}
\]
Could be no sols, if the 4th or 5th components in the row-opped $b$ are nonzero
otherwise there is one sol.

10. Let $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. The system row ops to
\[
\begin{bmatrix}
1 & 2 & 0 & | & a-b \\
0 & 0 & 1 & | & b \\
0 & 0 & 0 & | & c-2b \\
0 & 0 & 0 & | & d-3b \\
\end{bmatrix}
\]
To get a sol you need $c-2b = 0$, $d-3b = 0$.
So any $a,b,c,d$ are OK as long as $d = 3b$, $c = 2b$.
In that case the solution is $z = b$, $x = a - b - 2y$ (y is free).

11. Solving the matrix equation $AB = I$ amounts to solving these three systems (see observation (3) about matrix multiplication in Section 1.2):
\[
(*) \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\]

The echelon form of $A$ has 3 echelon cols with pivots (since rank $A$ is 3); the systems in (*) row ops to something like
\[
\begin{bmatrix}
1 & 0 & 0 & 2 & | & . \\
0 & 1 & 0 & 3 & | & . \\
0 & 0 & 1 & 4 & | & . \\
\end{bmatrix} \quad \text{or} \quad 
\begin{bmatrix}
1 & 2 & 0 & 0 & | & . \\
0 & 0 & 1 & 0 & | & . \\
0 & 0 & 0 & 1 & | & . \\
\end{bmatrix} \quad \text{etc.}
\]
There can't be a row of 0's in the echelon form of $A$ so each of the three systems in (*) is consistent, i.e., you can solve for all the unknowns.
In fact, each system in (*) not only is consistent but has infinitely many solutions. The echelon form of $A$ has 3 cols with pivots so each system has one free variable. So there are infinitely many $\mathbf{B}$'s.
SOLUTIONS Section 5.2

1. No row ops necessary. Just back substitute.
   \[ x_5 = 0 \]
   \[ x_4 = s \]
   \[ x_3 = t \]
   \[ x_2 = -3t - 5s \]
   \[ x_1 = 4x_4 - 2x_3 - 5x_2 = 12t + 21s \]

   One way to get a basis is to let \( t = 0, s = 1 \) and then let \( t = 1, s = 0 \). This gets solutions \( u = (21, -5, 0, 1, 0) \), \( v = (13, -3, 1, 0, 0) \). A basis is \( u, v \).

   (b) System is
   \[ \begin{bmatrix}
   1 & 2 & 1 & 0 \\
   1 & 3 & 2 & 0 
   \end{bmatrix} \]
   It row ops to
   \[ \begin{bmatrix}
   1 & 2 & 1 & 0 \\
   0 & 1 & 1 & 0 
   \end{bmatrix} \]

   Sol is \( x_2 = -x_3, \ x_1 = -x_3 - 2x_2 = x_3 \).
   
   The solution has parametric equations \( x_1 = t, x_2 = -t, x_3 = t \).
   
   Subspace is 1-dim. To get a basis, just get one (nonzero) sol.
   
   If \( t = 1 \) then sol is \( u = (1, -1, 1) \); \( u \) is a basis for the subspace.

   (c) System is
   \[ \begin{bmatrix}
   2 & 1 & 1 & 0 \\
   -2 & 2 & 1 & 0 
   \end{bmatrix} \]
   It row ops to
   \[ \begin{bmatrix}
   2 & 1 & 1 & 0 \\
   0 & 0 & 0 & 0 
   \end{bmatrix} \]

   Sol is \( x_3 = 0, x_2 = 0, x_1 = 0 \).

   The subspace of solutions contains only \( \vec{0} \). It's a 0-dim subspace and it isn't considered to have a basis.

2. (a) The system
   \[ \begin{bmatrix}
   1 & 1 & 2 & 0 \\
   0 & 0 & 1 & 1 \\
   0 & 0 & 0 & 0 
   \end{bmatrix} \]
   row ops to
   \[ \begin{bmatrix}
   1 & 1 & 2 & 0 \\
   0 & 0 & 1 & 1 \\
   0 & 0 & 0 & 0 
   \end{bmatrix} \]

   Sol is \( x_3 = -x_4, \ x_1 = 2x_4 - x_2 \ (x_2, x_4 \ free) \).
   
   Null space of \( M \) is 2-dim (two free variables). To find two ind sols to be the basis vectors, set \( x_2 = 1, x_4 = 0 \) and then set \( x_2 = 0, x_4 = 1 \) to get \( u = (-1, 1, 0, 0), v = (2, 0, -1, 1) \).

   (b) Solve
   \[ \begin{bmatrix}
   0 & 1 & 0 & 0 \\
   0 & 0 & 1 & 1 \\
   0 & 0 & 0 & 0 
   \end{bmatrix} \]

   Sol is \( x_2 = x_3 = x_4 = 0, x_1 \ free \).
   
   The null space of \( M \) is 1-dim with (pick, say, \( x_1 = 1 \)) basis vector \( (1, 0, 0, 0) \).

   (c) The system of equations \( Mx = \vec{0} \) is \( x + y + 2z = 0 \). The solution is \( x = -y - 2z; y, z \ free \). Null space is 2-dim.
   
   To find basis vectors choose \( y = 0, z = 1 \) and then use \( y = 1, z = 0 \).
   A basis is \( (-2, 0, 1), (-1, 1, 0) \).

3. System is
   \[ \begin{bmatrix}
   1 & 2 & 1 & -1 \\
   2 & 5 & 3 & -1 
   \end{bmatrix} \]
   It row ops to
   \[ \begin{bmatrix}
   1 & 0 & -1 & -3 \\
   0 & 1 & 1 & 1 
   \end{bmatrix} \]

   Sol is \( x_2 = -x_3 - x_4, \ x_1 = 3x_4 + x_3 \).

   To get two ind vectors for the basis, set \( x_3 = 1, x_4 = 0 \) to get \( p = (1, -1, 1, 0) \); set \( x_3 = 0, x_4 = 1 \) to get \( q = (3, -1, 0, 1) \).

   To get an orthogonal basis use the Gram Schmidt process on \( p, q \). Take
\[ \vec{u}_1 = p \]
\[ \vec{u}_2 = q - \frac{\vec{u}_1 \cdot q}{\vec{u}_1 \cdot \vec{u}_1} \]
\[ \vec{u}_1 = \left( \frac{5}{3}, \frac{1}{3}, -\frac{4}{3}, 1 \right) \]

Orthog basis is \( \vec{u}_1, \vec{u}_2 \).

4. In each case, the set of solutions must either be a plane through the origin, a line through the origin, just the origin, or all of \( \mathbb{R}^3 \) since these are the only subspaces of \( \mathbb{R}^3 \).

(a) \( x = 0; \) \( y, z \) free. The set of solutions is the \( y, z \) plane.

(b) \( y, z \) free; \( x = -\frac{1}{2}(3y + 4z) \). The set of solutions is the plane \( 2x + 3y + 4z = 0 \).

(c) \( x, y, z \), free. The set of solutions is all of \( \mathbb{R}^3 \).

(d) \( y, z \) free; \( x = -3y \). The set of solutions is the plane \( x + 3y = 0 \).

(e) \( x = 2z, \) \( y = 3z, \) \( z \) free.

The set of solutions is the line with parametric equations \( x = 2t, y = 3t, z = t \).

5. (a) All vectors in \( \mathbb{R}^5 \) are solutions.

(b) There are no solutions.

6. For each system, \( n = 6, \) \( r = 5, \) \( n-r = 1 \).

(a) Infinitely many solutions with one free variable.

(b) Either no solutions or infinitely many solutions with one free variable.

7. (a) The cross product \( \vec{u} \times \vec{v} = (-2,-2,1) \) is perp to \( \vec{u}, \vec{v} \).

The only other perps to \( \vec{u} \) and \( \vec{v} \) are multiples of \( (-2,-2,1) \).

(b) Let \( \vec{x} = (x, y, z) \). Solve the system of equations \( \vec{u} \cdot \vec{x} = 0, \vec{v} \cdot \vec{x} = 0 \).

\[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
1 & 1 & 4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 2 & 0
\end{bmatrix}
\]

Sol is \( y = -2z, \) \( x = -2z, \) \( z \) free.

The sol is a 1-dim subspace of \( \mathbb{R}^3 \).

Set \( z = 1 \) to get \( \vec{w} = (-2,-2,1) \), a basis vector for the subspace.

The set of vectors perp to \( \vec{u} \) and \( \vec{v} \) is the set of multiples of \( (-2,-2,1) \).

8. Solve the equation \( \vec{u} \cdot (x_1, x_2, x_3, x_4) = 0 \), i.e.,

\[(*) \]
\[ x_1 + x_2 + 4x_3 - 2x_4 = 0. \]

This is a homog system of equations, one equation in 4 unknowns.

One way to write the solution is

\[ x_1 = 2x_4 - 4x_3 - x_2; \quad x_2, x_3, x_4 \text{ free.} \]

The solutions are a 3-dim subspace of \( \mathbb{R}^4 \) so you can get 3 ind vectors orthogonal to \( \vec{u} \) but no more than that. Here is one set of 3 ind solutions (actually a basis for the subspace of solutions to \( (*) \)).
Let $x_2 = 1$, $x_3 = 0$, $x_4 = 0$ to get solution $p = (-1,1,0,0)$.
Let $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ to get solution $q = (-4,0,1,0)$.
Let $x_2 = 0$, $x_3 = 0$, $x_4 = 1$ to get solution $r = (2,0,0,1)$.
Three ind solutions are $p,q,r$.

9. (a) The given equations are a homogeneous system of two equations in 6 unknowns:
\[ x_2 + x_3 - x_5 = 0 \]
\[ 2x_3 - x_4 + x_5 = 0 \]
(There are six unknowns even though they don't all actually appear in the equations because the problem is about points of the form $(x_1,\ldots, x_6)$.)
So the set of solutions is a subspace of $\mathbb{R}^6$.
The solution is
\[ x_2 = -x_3 + x_5 \]
\[ x_4 = 2x_3 + x_5 \]
\[ x_1, x_3, x_5, x_6 \text{ free} \]
To get a basis
let $x_1 = 1$, $x_3 = 0$, $x_5 = 0$, $x_6 = 0$ to get sol $(1,0,0,0,0,0)$;
let $x_1 = 0$, $x_3 = 1$, $x_5 = 0$, $x_6 = 0$ to get sol $(0,-1,1,2,0,0)$;
let $x_1 = 0$, $x_3 = 0$, $x_5 = 1$, $x_6 = 0$ to get sol $(0,1,0,1,1,0)$;
let $x_1 = 0$, $x_3 = 0$, $x_5 = 0$, $x_6 = 1$ to get sol $(0,0,0,0,0,1)$.
A basis is $(1,0,0,0,0,0), (0,-1,1,2,0,0), (0,1,0,1,1,0), (0,0,0,0,0,1)$.

(b) This is the set of points $(x_1,\ldots, x_5)$ where
\[ x_2 = x_1 \]
\[ x_3 = 2x_1 \]
\[ x_4 = 2x_1 \]
\[ x_5 \text{ anything} \]
It's a homogeneous system of 3 equations in 5 unknowns:
\[ x_1 - x_2 = 0 \]
\[ 2x_1 - x_3 = 0 \]
\[ 2x_1 - x_4 = 0 \]
The solution has free variables $x_1, x_5$ so the solutions are a 2-dim subspace of $\mathbb{R}^5$.
To get a basis, first plug in $x_1=1,x_5=0$ and then $x_1=0$, $x_5=1$. A basis is $(1,1,2,2,0), (0,0,0,0,1)$.

10. The set of vectors $\vec{x}$ such that $A \vec{x} = B \vec{x}$ is the set of solutions to the homogeneous system of equations $(A-B)\vec{x} = \vec{0}$. So the set is a subspace of $\mathbb{R}^7$.

11. Solution is $y = 0$, $x$ free. The space of solutions is 1-dim. A basis vector is $i = (1,0)$. 
SOLUTIONS Section 5.3

1. Solve \[
\begin{bmatrix}
1 & 1 & 0 & 0 & | & 0 \\
0 & 1 & 0 & 1 & | & 0
\end{bmatrix}
\]
Solution is

\[x_2 = -x_4\]
\[x_1 = -x_2 = x_4\]
\[x_3, x_4\] free

If \(x_3 = 1, x_4 = 0\) then the sol is \(u = (0,0,1,0)\).
If \(x_3 = 0, x_4 = 1\) then the sol is \(v = (1,-1,0,1)\).
Orthog complement is a 2-dim subspace with basis \(u,v\).

2. Find the orthog comp of the subspace spanned by \(p,q,r\). Solve
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & | & 1 \\
1 & 1 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 1 & 1 & 0 & | & 0
\end{bmatrix}
\]
Row op to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & | & 1 \\
0 & 1 & 0 & 0 & 0 & | & -1 \\
0 & 0 & 1 & 1 & 0 & | & 0
\end{bmatrix}
\]
Sol is \(x_3 = -x_4\)
\[x_2 = x_6\]
\[x_1 = -x_6\]
\[x_4, x_5, x_6\] free

So you can say that the answer is the set of vectors of the form \((x_1,\ldots,x_6)\) where \(x_3 = -x_4, x_2 = x_6\) and \(x_1 = -x_6\). Or you can pick 3 ind sols like \(u = (0,0,-1,1,0,0),\)
\(v = (0,0,0,1,0,0), w = (-1,1,0,0,1,1)\) and say that the answer is the 3-dim subspace with basis \(u,v,w\).

3. The echelon form of \(A\) is
\[
\begin{bmatrix}
1 & 2 & 0 & 3 & 0 \\
0 & 0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
(a) The row space is a 3-dim subspace of \(R^5\) with basis
\((1,2,0,3,0), (0,0,1,4,0), (0,0,0,0,1)\).
(b) Col space is a 3-dim subspace of \(R^4\) with basis \((1,2,0,1), (0,1,1,1), (0,0,1,0)\),
the original cols corresponding to the echelon cols with pivots.
(c) \(A\) has 5 cols and rank 3 so \(n = 5, r = 3\) and dim of null space is \(n-r = 2\).

To find the basis, solve the system
\[
\begin{bmatrix}
1 & 2 & 0 & 3 & 0 & | & 0 \\
0 & 0 & 1 & 4 & 0 & | & 0 \\
0 & 0 & 0 & 0 & 1 & | & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]
Solution is
\[x_1 = -3x_4 - 2x_2\]
\[x_3 = -4x_4\]
\[x_5 = 0\]
\[x_2, x_4\] free
If $x_2 = 1$, $x_4 = 0$ then the solution is $u = (-2, 1, 0, 0, 0)$.

If $x_2 = 0$, $x_4 = 1$ then the solution is $v = (-3, 0, -4, 1, 0)$.

A basis for the null space is $u, v$.

(d) The row space of $A$ and the null space of $A$ are orthogonal complements. So the orthogonal complement of the null space is the row space. From part (a), the orthogonal complement is a 3-dimensional subspace of $\mathbb{R}^5$ with basis $(1, 2, 0, 3, 0), (0, 0, 1, 4, 0), (0, 0, 0, 0, 1)$.

4. (a) First of all, the second column of $A$ is a multiple of the first, so the column space is spanned by columns 1 and 3 alone (or by columns 2 and 3 alone).

To get the orthogonal complement of the space spanned by columns 1 and 3 of $A$, line up the spanning vectors as rows of a matrix and find the null space. So let

$$B = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and solve $Bx = 0$. By inspection,

$$x_1 = 0, \quad x_2 = -\frac{4}{3} x_3 - \frac{5}{3} x_4.$$ 

Any two independent solutions can serve as a basis.

One possibility is $(0, -4, 3, 0), (0, -5, 0, 3)$.

(b) Use the Gram Schmidt process on $x_1 = (0, -4, 3, 0), x_2 = (0, -5, 0, 3)$.

New orthogonal basis is

$$u_1 = x_1 = (0, -4, 3, 0)$$

$$u_2 = \frac{x_2 - (x_2 \cdot u_1)}{u_1} = (0, -5, 0, 3) - \frac{20}{25} (0, -4, 3, 0) = (0, -\frac{9}{5}, -\frac{12}{5}, 3)$$

5. First of all, the hyperplane is a subspace of $\mathbb{R}^5$ because it's the set of solutions to a homogeneous system (one equation in five unknowns). In particular, the coefficient matrix $A$ is the $1 \times 5$ matrix $\begin{bmatrix} 2 & 3 & 5 & -7 & 1 \end{bmatrix}$. The hyperplane is the null space of $A$.

The set of vectors orthogonal to the hyperplane is the orthogonal complement of the hyperplane, namely, the row space of $A$. This row space consists of all multiples of the one row $\mathbf{n} = (2, 3, 5, -7, 1)$. So $\mathbf{n}$ and multiples of $\mathbf{n}$ are the only vectors orthogonal to the hyperplane. QED

By the way, the hyperplane has dimension 4 because rank $A = 1$ and $n = 5$ (or equivalently because when you solve the system of one equation in five unknowns, there are 4 free variables). In general, a hyperplane in $\mathbb{R}^n$ has an equation of the form $a_1 x_1 + \ldots + a_n x_n = 0$ and is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The orthogonal complement is a 1-dimensional subspace (a "line through the origin") with basis vector $(a_1, \ldots, a_n)$.

6. No. Line $L_2$ does not contain all vectors orthogonal to all the vectors in $L_1$. It only contains some of them. It's the plane through the origin perp to $L_1$ that is the subspace containing all of them. So the orthogonal complement of $L_1$ is a plane (see the diagram). Similarly, the orthogonal complement of $L_2$ is the plane through the origin perp to $L_2$. 
7. (a) $u$ is orthog to $w$ because $u$ is orthogonal to every vector in $W$.
(b) Can't tell. Since $u$ is not in the orthogonal complement of $W$ you know that $u$ cannot be orthog to every vector in $W$. In other words, there is at least one vector in $W$ that $u$ is not orthog to. But that "at least one" vector might not be $w$.

The diagram shows a subspace $W$ in $\mathbb{R}^3$, a plane through the origin, and its orthogonal complement, a line through the origin perp to the plane. The vector $w$ is in $W$. There are two $u$ vectors, neither in the orthog comp of $W$. I tried to make the first look like it is not orthog to $v$ and the second look like it is orthog to $v$.

8. No. the row space of a matrix and its null space are orthogonal complements. so a vector in the null space has to be orthog to everything in the row space. But $(2,1,1)$ is not orthog to $(3,1,2)$. 
1. M violates (7) on the invertible list so nothing on the list holds. For instance, M is not invertible.

2. (a) \[
\begin{vmatrix}
2 & -5 & 4 \\
1 & -3 & 2 \\
0 & 1 & 1
\end{vmatrix} = -1 \text{ so there is just one sol.}
\]
\[
x = \begin{vmatrix}
2 & -5 & 4 \\
1 & -3 & 2 \\
0 & 1 & 1
\end{vmatrix}^{-1} = 2, \quad y = \begin{vmatrix}
2 & 5 & 4 \\
1 & 1 & 2 \\
0 & 7 & 1
\end{vmatrix}^{-1} = 3, \quad z = \begin{vmatrix}
2 & -5 & 5 \\
1 & -3 & 1 \\
0 & 1 & 1
\end{vmatrix}^{-1} = 4
\]

(b) \[
\begin{vmatrix}
0 & 2 & 4 \\
1 & 1 & 4 \\
1 & 0 & 2
\end{vmatrix} = 0. \text{ System has either no sols or infinitely many. Use row ops to find out which it is. The system row ops to}
\]
\[
\begin{vmatrix}
1 & 0 & 1 & 4 \\
0 & 1 & 2 & 10 \\
0 & 0 & 0 & 2
\end{vmatrix}
\]
No solutions because of the last line.

(c) Same coeff matrix as part (b). System row ops to
\[
\begin{vmatrix}
1 & 0 & 1 & 4 \\
0 & 1 & 2 & 6 \\
0 & 0 & 0 & 0
\end{vmatrix}
\]
Infinitely many sols.
\[
x = 4 - 2z, \quad y = 6 - 2z, \quad z \text{ free}
\]

3. (a) Matrix of coeffs is \[
M = \begin{bmatrix}
3 & 1 \\
2 & -1
\end{bmatrix}.
\]
There is exactly one sol because |M| \neq 0.
\[
\begin{vmatrix}
3 & 1 \\
5 & -1
\end{vmatrix} = \frac{8}{5}, \quad \begin{vmatrix}
3 & 3 \\
2 & 5
\end{vmatrix} = -\frac{9}{5}
\]
Sol is \[
x = \begin{vmatrix}
3 & 1 \\
5 & -1
\end{vmatrix}^{-1} = \frac{8}{5}, \quad y = \begin{vmatrix}
3 & 3 \\
2 & 5
\end{vmatrix}^{-1} = -\frac{9}{5}
\]

(b) System is \[
M^* = b \text{ where } M = \begin{bmatrix}
3 & 1 \\
2 & -1
\end{bmatrix} \text{ and } b = \begin{bmatrix}
3 \\
5
\end{bmatrix}.
\]
Solution is \[
x = M^{-1} b = \begin{bmatrix}
3 & 1 \\
2 & -1
\end{bmatrix}^{-1} \begin{bmatrix}
3 \\
5
\end{bmatrix} = \begin{bmatrix}
\frac{8}{5} \\
-\frac{9}{5}
\end{bmatrix}
\]

(c) \[
M | b \text{ row ops to } \begin{vmatrix}
1 & 0 & 0 & 8/5 \\
0 & 1 & 1 & -9/5
\end{vmatrix}
\]
Sol is \[
x = \frac{8}{5}, \quad y = -\frac{9}{5}.
\]

4. \[
x = \begin{vmatrix}
c & b \\
\hline
f & e
\end{vmatrix} = \frac{ec - bf}{ae - bd}, \quad y = \begin{vmatrix}
a & c \\
\hline
d & f
\end{vmatrix} = \frac{af - cd}{ae - bd}
\]
assuming \(ae - bd \neq 0\), i.e., assuming \[
\begin{vmatrix}
a & b \\
\hline
d & e
\end{vmatrix} \neq 0.
\]
5. A can't be invertible (if it were, then Ax=b would have had one solution).

(a) Can't be just one solution because A is not invertible. Can't be no solutions since a homog system always has at least the trivial solution. So must be infinitely many solutions. There would be either 1 or 2 or 3 free variables (there would be 3 free variables iff A were the zero matrix).
(b) Can't be just one sol because A is not invertible. There are either no solutions or infinitely many.

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix}
\]

If A|c row opped to something like 0 0 1 6 then there would be no solutions.

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
5 \\
7
\end{bmatrix}
\]

If A|c row opped to something like 0 0 1 6 then there would be infinitely many solutions.

(c) |A| = 0 since A is not invertible.
(d) Rank can't be 3. Rank must be 2 or 1 or 0 (in the special case that A is the zero matrix).

6. The rows of A are orthonormal (by inspection) so the matrix A is orthogonal. So A is invertible and in particular $A^{-1} = A^T$.

Solution is

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = A^T \begin{bmatrix}
2 \\
3 \\
4
\end{bmatrix} = \begin{bmatrix}
1/\sqrt{6} & -2/\sqrt{5} & 1/\sqrt{30} \\
2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\
1/\sqrt{6} & 0 & 5/\sqrt{30}
\end{bmatrix} \begin{bmatrix}
2 \\
3 \\
4
\end{bmatrix}
\]

\[
x = \frac{1}{\sqrt{6}} - \frac{6}{\sqrt{5}} + \frac{4}{\sqrt{30}}
\]

\[
y = \frac{4}{\sqrt{6}} + \frac{3}{\sqrt{5}} + \frac{8}{\sqrt{30}}
\]

\[
z = \frac{2}{\sqrt{6}} + \frac{20}{\sqrt{30}}
\]
SOLUTIONS Section 6.1

1. Want a matrix $M$ so that

$$
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
    x_4 \\
    2x_1 - 3x_2 \\
    5x_1 + 6x_3 + 7x_4 \\
\end{bmatrix}
$$

So $M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & -3 & 0 & 0 \\ 5 & 0 & 6 & 7 \end{bmatrix}$

2. $M \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ so the image is $(10,10)$.

To get the pre-image of $(0,1)$ solve $M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 1/5 \end{bmatrix}$

So the pre-image is $(-2/5, 1/5)$.

3. (a) $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ so $A$ sends point $(x,y)$ to point $(2x,2y)$.

Every point moves away from the origin so that it is twice as far from the origin as it used to be.

$B$ maps $(x,y)$ to $(0,y)$ so $B$ projects points on the $y$-axis. The image of the given figure is a segment on the $y$-axis.

$C$ sends $(x,y)$ to $(x,-y)$ so $C$ reflects things in the $x$-axis.

(b) The opposite of expanding radially is contracting radially so

$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

The operator $B$ isn't one-to-one since many points project to the same image on the $y$-axis. No inverse.

The opposite of reflecting in the $x$-axis is to reflect back again so $C^{-1} = C$.

(c) $B$ projects onto the $y$-axis.

$B^2$ projects twice. But after a point is projected once, it's already on the $y$-axis so the second projection doesn't move it any more. So $B^2$ is the same mapping as $B$. Similarly $B^{100} = B$.

(d) $C$ reflects in the $x$-axis. $C^2$ reflects twice which means that $(x,y)$ is eventually sent back to $(x,y)$ again. So $C^2 = I$.

(e) The range is the set of outputs.

With $A$, every point is an output; in particular $(a,b)$ is the image of $(1/2a , 1/2b)$.

The range of $A$ is $\mathbb{R}^2$. 
With B, the outputs are all the points on the y-axis.
The range of B is the y-axis.
With C, every point is an output, corresponding to its reflection as input.
The range of C is \( \mathbb{R}^2 \).

4. The vector \( \mathbf{u} = (1, 7) \) points along the line.
The line is a 1-dim subspace of \( \mathbb{R}^2 \) with basis \( \mathbf{u} \).

I want the projection of point \( \mathbf{x} = (x, y) \) onto the line:
\[
\mathbf{p} = \frac{u \cdot x}{u \cdot u} u = \frac{x+7y}{50} \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \quad \mathbf{q} = \frac{7x+49y}{50} \begin{pmatrix} 1 \\ 7 \end{pmatrix}
\]
So \( M \) sends \( (x, y) \) to \( \begin{pmatrix} x+7y \\ 7x+49y \end{pmatrix} \). So \( M = \begin{pmatrix} 1/50 & 7/50 \\ 7/50 & 49/50 \end{pmatrix} \).

5. (a) The lefthand diagram shows \( p+q \) and \( T(p+q) \).
The middle diagram shows \( T(p) \) and \( T(q) \).
The righthand diagram shows \( T(p) + T(q) \).

(b) The two points \( T(p+q) \) in the lefthand diagram and \( T(p)+T(q) \) in the righthand diagram are not the same.
So \( T \) does not have the property \( T(u+v) = T(u)+T(v) \) for all \( u, v \) (my points \( p \) and \( q \) are a counterexample).
So \( T \) is not linear.

6. Let
\[
A = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Then \( A \) rotates and \( B \) reflects so \( BA \) first rotates and then reflects. Answer is
\[
BA = \begin{pmatrix} -1/2 & 1/2 \sqrt{3} \\ 1/2 \sqrt{3} & 1/2 \end{pmatrix}
\]

7. No. The vectors that map to \( v \) are the solutions to the system of equations
\( Mx = v \). The system can have no solutions, one solution or infinitely many. So either nothing maps to \( v \) or one vector maps to \( v \) or infinitely many map to \( v \). But \( v \) can't have exactly 13 pre-images.

8. (a) \( A = \begin{pmatrix} 1 & 5 & -2 \\ 4 & 3 & 7 \\ 2 & \pi & 8 \end{pmatrix} \)
(b) Let
\[ G = \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} p & q & r \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ 3 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}. \]

Then
- \( G \) maps \( i,j,k \) to \( u,v,w \) respectively.
- \( G^{-1} \) maps \( u,v,w \) to \( i,j,k \) respectively (inverse exists because \( u,v,w \) are ind).
- \( H \) maps \( i,j,k \) to \( p,q,r \) respectively.
- So \( HG^{-1} \) maps \( u,v,w \) to \( p,q,r \). QED

9. (a) \( A \) maps \( (x,y,z) \) to \( (x,0,0) \) so \( A \) projects a point onto the \( x \)-axis.
- Range of \( A \) is the \( x \)-axis, null space is the \( y,z \) plane (because those are the points that map to the origin), \( \text{rank } A = \dim \text{range } A = 1 \).
- \( B \) maps \( (x,y,z) \) to \( (x,y,0) \) so \( B \) projects a point onto the \( x,y \) plane.
- Range of \( B \) is the \( x,y \) plane, null space is the \( z \)-axis, \( \text{rank } B = \dim \text{range } B = 2 \).
- \( C \) reflects a point in the \( x,y \) plane.
  - Range of \( C \) is \( \mathbb{R}^3 \), null space contains only \( \vec{0} \) and \( \text{rank } C = 3 \).

(b) Solution to
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
is \( x=0; \ y,z \) free.

The set of points where \( x=0 \) and \( y \) and \( z \) can be anything is the \( y,z \) plane. So the null space is the \( y,z \) plane.

10. The cols of \( A \) are ind (no col is a combination of the preceding cols). So the range is a 3-dim subspace of \( \mathbb{R}^3 \). So the range is \( \mathbb{R}^3 \). The three cols of \( A \) are a basis but so are any 3 ind vectors in \( \mathbb{R}^3 \), including \( i,j,k \).

The cols of \( B \) are dependent but the last two cols are ind. The range is a 2-dim subspace of \( \mathbb{R}^3 \) with basis vectors \( u = (1,1,0) \), \( v = (1,1,1) \). The range is the plane in 3-space determined by points \( u,v \) and the origin (i.e., determined by arrows \( u \) and \( v \) attached to the origin).

11. (a) No because \( Mu \neq \vec{0} \).

(b) \textit{Answer I} Try to solve \( Mx = \vec{u} \) to see if there is a pre-image \( \vec{x} \). The system
\[
\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}
\]
has no solution because the last two equations are \( z = 2 \) and \( 2z = 3 \) (no \( z \) can do that). So \( \vec{u} \) is not in the range.

If you don't spot immediately that the system is inconsistent, you row op to echelon form
The last row shows that the system is inconsistent.

**answer 2** See if \( u \) is in the col space of \( M \).
I can tell by inspection that it isn't; there is no way you can get
\[
a(1,0,0) + b(2,0,0) + c(1,1,2) = (1,2,3).
\]

You would need \( c = 2 \) and also \( c = \frac{3}{2} \) to do it.

Or you can use row operations on
\[
\begin{bmatrix}
  1 & 2 & 1 & 1 \\
  0 & 0 & 1 & 2 \\
  0 & 0 & 2 & 3 \\
\end{bmatrix}
\]
to get
\[
\begin{bmatrix}
  1 & 2 & 1 & 1 \\
  0 & 0 & 1 & 3 \\
  0 & 0 & 0 & -1 \\
\end{bmatrix}
\]
In the echelon form, col 4 is not a combination of the other cols.
So back in the original, \( u \) is not a combination of the cols of \( M \).

12. (a) Solve \( Mx = \vec{0} \). The system row ops to

\[
\begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 1 & 3 & 0 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Solution is \( x = -2z, y = -3z, z \) free. The set of solutions is a 1-dim subspace of \( \mathbb{R}^3 \). Any nonzero solution, say \((-2,-3,1)\), can serve as the one basis vector.
(b) **method 1** The range is the col space. To get a basis for the col space of $M$, do the row ops $R_1 \leftrightarrow R_2$, $R_3 = -2R_1 + R_3$, $R_4 = -R_1 + R_4$, $R_4 = -R_2 + R_4$ to get

$$
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

Echelon cols 1 and 2 have pivots so cols 1 and 2 of the original matrix are a basis for the col space. So a basis for the range is $u = (0,1,2,1)$, $v = (1,0,0,1)$.

**method 2** The range is the set of $\vec{b}$'s that make $M\vec{x} = \vec{b}$ consistent. So solve

$$
\begin{bmatrix}
0 & 1 & 3 & | & b_1 \\
1 & 0 & 2 & | & b_2 \\
2 & 0 & 4 & | & b_3 \\
1 & 1 & 5 & | & b_4
\end{bmatrix}
$$

Use the same row ops as method 1 to get

$$
\begin{bmatrix}
1 & 0 & 2 & | & b_1 \\
0 & 1 & 3 & | & b_2 \\
0 & 0 & 0 & | & -2b_2 + b_3 \\
0 & 0 & 0 & | & -b_1 - b_2 + b_4
\end{bmatrix}
$$

For the system to be consistent you need $-2b_2 + b_3 = 0$ and $-b_1 - b_2 + b_4 = 0$. The range is the set of points $(b_1, b_2, b_3, b_4)$ such that $b_3 = 2b_2$ and $b_4 = b_1 + b_2$. If you can't see a basis by inspection, write these equations as

$$
\begin{align*}
b_1 &= t \\
b_2 &= s \\
b_3 &= 2s \\
b_4 &= s + t
\end{align*}
$$

The range is the set of points of the form

$$
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}
$$

and a basis is $(1,0,0,1)$, $(0,1,2,1)$.

**Question** I got different answers from methods 1 and 2. So isn't one of them wrong?

**Answer** No. They are both right. A subspace has lots of bases.

**method 3** Take the cols of $M$, line them up as rows and row op into echelon form:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

The nonzero echelon rows are a basis for the original row space so they are a basis for the col space of $M$.

13. (a) $M$ is invertible. Range is all of $\mathbb{R}^{10}$, null space contains only $\vec{0}$, $|M| \neq 0$.

(b) Range is a 7-dim subspace of $\mathbb{R}^{10}$, null space is a 3-dim subspace of $\mathbb{R}^{10}$, $M$ isn't invertible, $|M| = 0$.

14. (a) In the echelon form of $M$, 5 of the 9 cols have pivots. The range is a 5-dim
subspace of $\mathbb{R}^5$ so the range is $\mathbb{R}^5$.

Null space is a 4-dim subspace of $\mathbb{R}^9$ because there are 4 free variables when you
solve $M\vec{x} = \vec{0}$. Also because of the formula: dim of the null space is $n-r$ where $n = 9$,
$r = 5$.

(b) In the echelon form of $M$, 3 of the 9 cols have pivots. Range is a 3-dim
subspace of $\mathbb{R}^5$. Null space is a 6-dim subspace of $\mathbb{R}^9$.

15. (a) False. The range is a 4-dim subspace of $\mathbb{R}^6$ which is not the same as $\mathbb{R}^4$.

(b) True. The range is a 6-dim subspace of $\mathbb{R}^6$, and the only 6-dim subspace of $\mathbb{R}^6$
is $\mathbb{R}^6$ itself.

16. (a) Not possible. First of all, $A\vec{0} = \vec{0}$ so $\vec{0}$ is always an output.

Secondly, the set of outputs (the range) is a subspace. And the set containing
just $b$ is not a subspace (not closed).

(b) This is like asking how many sols there are to the system $A\vec{x} = \vec{b}$.
Since we know that $b$ is an output of $A$ we know that there is at least one sol.
The possibilities are either exactly one sol or infinitely many. So either there is
just one input that produces $b$ or there are infinitely many inputs that produce $b$.

(c) Now we know that the range of $A$ is all of $\mathbb{R}^4$. So $A$ is invertible. So the system
$A\vec{x} = \vec{0}$ has exactly one sol. So there is just one input that produces $b$.

17. (a) No. If $M$ turns into $M_1$ after row ops, the systems of equations $M\vec{x} = \vec{0}$ and
$M_1\vec{x} = \vec{0}$ have the same sols.

(b) Yes. The range is the col space, and row ops can change the col space (see row
op rule (2) in Section 3.1).

18. (a) The set of vectors of the form $(a,a,0,b)$ is a subspace with basis
$u = (1,1,0,0), v = (0,0,0,1)$.

The set of outputs of a matrix is the col space. So you want a matrix whose column
space has basis $u,v$. One way to do it is to have cols $u$ and $v$ and then for the other
two cols use any combinations of $u$ and $v$ so that $u$ and $v$ are a maximal ind set of
cols. Or just make the last two cols $\vec{0}$.

There are lots of answers. One possibility is
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

(b) Not possible. The set of vectors of the form $(a,a,2,b)$ is not a subspace (not
closed under scalar multiplication) (not closed under addition) (doesn't contain $\vec{0}$).
So it can't be the range of any matrix operator.

19. (a) $\text{Rank}(AB) \leq 4$ and $\text{rank}(AB) \leq 7$. So the best conclusion is $\text{rank}(AB) \leq 4$.

(b) $\text{Rank}(AB) = 4$.

(c) Since you end up with a contradiction, what this proves is that $A$ can't be
invertible in the first place.

Here's another way to see it. If $A$ were invertible with rank 4 then it would have
to be $4 \times 4$. Since $AB$ exists, $B$ has to have 4 cols. But then it can't have rank 7.
So $A$ can't be invertible.

20. The matrix $I$ in question must be $3 \times 3$. And $B$ must be $4 \times 3$.

$\text{r}(AB) = \text{r}(I) = 3$

$\text{r}(AB) \leq \text{r}(B)$ (rule about rank of a product)

So $\text{r}(B) \geq 3$.

But rank $B$ can't be $> 3$ since $B$ only has 3 cols. So rank $B = 3$. 
21. (a) The range of $A$ is a 4-dim subspace of $\mathbb{R}^7$.

(b) The vectors $A\hat{x}_1, \ldots, A\hat{x}_6$ are in the range of $A$. So they are six vectors in a 4-dim subspace of $\mathbb{R}^7$. But 6 vectors in a 4-dim subspace must be dep. QED.

22. Let $A = \begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

Then
- $A$ sends $i,j$ to $u,p$ respectively
- $B$ sends $i,j$ to $v,q$ respectively
- $A^{-1}$ sends $u,p$ to $i,j$ respectively

So $BA^{-1}$ sends $u,p$ to $v,q$. So

\[ M = BA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \times \frac{1}{4} \begin{bmatrix} 1 & -6 \\ 0 & 4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1/4 & 2/4 \\ 3/4 & 2/4 \end{bmatrix} \]

check $Mu = \begin{bmatrix} 1/4 & 2/4 \\ 3/4 & 2/4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = v$

$Mp = \begin{bmatrix} 1/4 & 2/4 \\ 3/4 & 2/4 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = q$ QED
1. Let \( P = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \). Then \( P^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \).

New matrix for the trans is \( P^{-1}MP = \begin{bmatrix} 5/2 & -1/2 \\ 9/2 & -1/2 \end{bmatrix} \).

Trans sends \((x,y)\) to \((x, 3x+y)\) and sends \((X,Y)\) to \(\left(\frac{5}{2}X - \frac{1}{2}Y, \frac{9}{2}X - \frac{1}{2}Y\right)\).

2. (a) Let \( P = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \). Then \( P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \) and

\( M = P^{-1}AP \)

\( A = PMP^{-1} = \begin{bmatrix} 7 & -14 \\ 2 & -5 \end{bmatrix} \)

(b) \( \begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \) so \( T(u+3v) = 6u + v \).

(c) Convert \( u+3v \) to the \( i,j \) basis.

\( u+3v = (2,0) + 3(2,1) = (8,3) \).

\( \begin{bmatrix} 7 & -14 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 1 \end{bmatrix} \) so \( T(u+3v) = T(8i+3j) = 14i + j \)

Check that the answers to (b) and (c) agree:

\( 6u+v = 6(2,0) + (2,1) = 14i + j \). QED

3. (a) The coords of \( \vec{u} \) w.r.t. basis \( \vec{u},\vec{v},\vec{w} \) are 1,0,0 (because \( u = 1u + 0v + 0w \)).

\( M = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \) so \( T(u) = 2u + 5v + 8w = -22i + 29j + 18k \)

**Question** What's wrong with finding \( M \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \)? It comes out to be \( \begin{bmatrix} 38 \\ 71 \\ 50 \end{bmatrix} \) so why isn't the answer \( T(u) = (38,71,50) \).

**Answer** Two things wrong here.

Remember that \( M \) is the matrix of the transformation w.r.t. basis \( u,v,w \). So when you find \( M \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \) it means that you are finding \( T(u+4v+6w) \). But the problem was to find \( T(u) \) not \( T(u+4v+6w) \). And furthermore the 38,71,50 you get out means that the image of \( u+4v+6w \) is 38u+71v+50w, not (38,71,50).

(b) Let \( P = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -8 & 2 \\ 4 & 1 & 2 \\ 6 & -2 & 2 \end{bmatrix} \)

I'll think of \( M \) as the new matrix of \( T \). I want the old matrix (i.e., w.r.t. \( i,j,k \)).

new = \( P^{-1} \) old \( P \) so the old matrix is \( PMP^{-1} \).

As a check, I used Mathematica to compute \( PMP^{-1} \) and then I found \( PMP^{-1} \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \) and I got \( \begin{bmatrix} -22 \\ 29 \\ 18 \end{bmatrix} \) as I should.

4. **step 1** One possibility is \( u = (1,3), v = (-3,1) \)

**step 2** The reflection of \((X,Y)\) is \((X,-Y)\). You want a matrix \( B \) such that
B \[
\begin{bmatrix}
X \\
Y
\end{bmatrix}
\] = \[
\begin{bmatrix}
X \\
-Y
\end{bmatrix}
. 
So \( B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \).

**step 3** Let \( P = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \). Then \( P^{-1} = \begin{bmatrix} 1/10 & 3/10 \\ -3/10 & 1/10 \end{bmatrix} \), \( P^{-1}AP = B \) so

\[ A = PBP^{-1} = \begin{bmatrix}
-8/10 & 6/10 \\
6/10 & 8/10
\end{bmatrix}\]

check: \( A \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \) so the reflection of (-5,5) in the line \( y=3x \) is (7,1). If you plot the points, they do look like reflections.

5. (a) \( A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)

(b) Let \( P = \begin{bmatrix} 1 & -5 \\ 4 & 2 \end{bmatrix} \). Then \( B = P^{-1}AP = \frac{1}{22} \begin{bmatrix} -18 & -20 \\ -8 & 18 \end{bmatrix} \)

(c) \( B \begin{bmatrix}
X \\
Y
\end{bmatrix} = \frac{1}{22} \begin{bmatrix} -18 & -20 \\ -8 & 18 \end{bmatrix} \begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
\frac{-18}{22}X - \frac{20}{22}Y \\
\frac{-8}{22}X + \frac{18}{22}Y
\end{bmatrix} \)

so \( T \) sends \((X,Y)\) to \((-\frac{18}{22}X - \frac{20}{22}Y, -\frac{8}{22}X + \frac{18}{22}Y)\)

6. (a) Points like (3,2) and (2,3) are reflections of one another in the line \( y=x \)
In general, \( T \) reflects points in the line \( y=x \) (see the diagram).

(b) By inspection, the matrix of \( T \) w.r.t. basis \( i,j \) is \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

Let \( P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \).

The matrix for \( T \) w.r.t. basis \( u,v \) is

\[ P^{-1}AP = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \]

(c) \( \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \) so \( T(2u-v) = 3u-4v \).

The diagram shows the \( X,Y \) coord system with basis \( u,v \). The \( Y \)-axis is the same as the \( X \)-axis.
I plotted the input point \((2,-1)\) and the output point \((3,-4)\). They do look like reflections in the line \( y=x \).
As a further check, \( 2-v = (3,2) \) and \( 3u-4v \) does out to be \((2,3)\), as it should.
7. (a) Let \( P_1 = [p \ q \ r] = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \)

Similarly, let \( P_2 = [u \ v \ w] \).

Let \( M_{ijk} \) be the matrix of the trans w.r.t. basis \( i,j,k \).

Then
\[
M_{pqr} = P_1^{-1} M_{ijk} P_1 \\
M_{uvw} = P_2^{-1} M_{ijk} P_2
\]

So
\[
M_{ijk} = P_1^{-1} M_{pqr} P_1^{-1} \\
M_{uvw} = P_2^{-1} P_1 M_{pqr} P_1^{-1} P_2
\]

(b) I want to write \( M_{uvw} \) as \( Q^{-1} M_{pqr} Q \). From part (a), I have
\[
M_{uvw} = P_2^{-1} P_1 M_{pqr} P_1^{-1} P_2
\]

But \( P_2^{-1} P_1 \) can be written as \((P_1^{-1} P_2)^{-1}\) by inverse rules so
\[
M_{uvw} = (P_1^{-1} P_2)^{-1} M_{pqr} P_1^{-1} P_2 \text{ which is of the form } Q^{-1} M_{pqr} Q \text{ where } Q = P_1^{-1} P_2.
\]

8. Suppose \( A \) and \( B \) are similar. Then \( B = P^{-1}AP \) for some \( P \).

But multiplying by invertible matrices (namely \( P^{-1} \) and \( P \)) doesn't change rank. So \( B \) has same rank as \( A \).

9. There is a matrix \( Q \) such that \( A = Q^{-1}BQ \) because \( A \) and \( B \) are similar. So
\[
A^T = (Q^{-1}BQ)^T \text{ take } T \text{ on both sides} \\
= Q^TB^T(Q^{-1})^T \text{ T rule} \\
= Q^TB^T(Q^T)^{-1} \text{ inverse rule}
\]

There are two ways to continue now.
version 1 Solve for \( B \) to get
\[
B^T = (Q^T)^{-1} A^T Q^T
\]
This makes \( A^T \) and \( B^T \) similar because \( B^T = R^{-1} A^T R \) where \( R = Q^T \).

version 2 Rewrite as
\[
A^T = (Q^{-1})^{-1} R^{-1} A^T R \text{ take } T \text{ on both sides} \\
= (Q^{-1})^{-1} B^T (Q^T)^{-1}
\]
This makes \( A^T \) and \( B^T \) similar because \( A^T = P^{-1} B^T P \) where \( P = (Q^T)^{-1} \).
SOLUTIONS review problems for Chapters 5 and 6

1. (a) Start with:

\[
\begin{pmatrix}
1 & -1 & -1 & 1 \\
1 & 1 & 1 & 0 \\
1 & -1 & 1 & 0 \\
1 & 1 & -1 & 0 \\
\end{pmatrix}
\]

Do row ops:
row2 = -row1 + row2
row3 = -row1 + row3
row4 = -row1 + row4
row4 = -row2 + row4
row4 = row3 + row4

to get:

\[
\begin{pmatrix}
1 & -1 & -1 & 1 \\
0 & 2 & 2 & -1 \\
0 & 0 & 2 & -1 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

The last line makes the system inconsistent.

(b) The vector \( \mathbf{i} \) = (1,0,0,0) is not in the subspace spanned by:

\( \mathbf{u}_1 = (1,1,1,1), \quad \mathbf{u}_2 = (-1,1,-1,1), \quad \mathbf{u}_3 = (-1,1,1,-1). \)

(c) method 1 (better)

The least squares solution is the projection of \( \mathbf{i} \) onto the col space.

The cols \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \) are orthogonal so

\[
\mathbf{i}_{\text{proj}} = \frac{\mathbf{u}_1 \cdot \mathbf{i}}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{i}}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \frac{\mathbf{u}_3 \cdot \mathbf{i}}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3 + \frac{\mathbf{u}_4 \cdot \mathbf{i}}{\mathbf{u}_4 \cdot \mathbf{u}_4} \mathbf{u}_4
\]

So the least squares sol to the inconsistent system is \( a = 1/4, \quad b = -1/4, \quad c = -1/4. \)

method 2

Let \( A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \). The normal equations are:

\[
\begin{bmatrix}
4 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}
\]

Least squares solution is \( a = 1/4, \quad b = -1/4, \quad c = -1/4. \)

(d) The original system asked you to find \( a, b, c \) so that \( a \mathbf{u}_1 + b \mathbf{u}_2 + c \mathbf{u}_3 \) is \( \mathbf{i} \). It turned out to be impossible but using \( a = 1/4, \quad b = -1/4, \quad c = -1/4 \) will at least make \( ||a \mathbf{u}_1 + b \mathbf{u}_2 + c \mathbf{u}_3 - \mathbf{i}|| \) minimum (i.e., they make the vector \( a \mathbf{u}_1 + b \mathbf{u}_2 + c \mathbf{u}_3 \) as close to \( \mathbf{i} \) as possible).

2. \( A = Q^{-1}BQ \) for some \( Q \).

\( Q, B \) and \( Q^{-1} \) are invertible and the product of invertibles is invertible. So \( A \) is invertible.

3. \( |M| = 0, \quad \text{rank } M = 0, \quad M \) maps everything to \( \mathbf{0} \) so range contains only \( \mathbf{0} \) and the null space is all of \( \mathbb{R}^4 \).
4. There are many answers. Here are some possibilities.

(a) Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \).

There is one solution if \( \vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) and no solutions if \( \vec{b} = \begin{bmatrix} 6 \\ 7 \end{bmatrix} \).

(b) Let \( A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix} \)

If \( \vec{b} = \begin{bmatrix} a \\ b \end{bmatrix} \) then the solution is \( x_1 = a - x_3, x_2 = b - 4x_3; x_3 \) free.

(c) Let \( A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \)

If \( \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \) then there are no solutions. If \( \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \) then there are infinitely many solutions, namely \( x_1 = 2 - 2x_2, x_3 = 3, x_2 \) free.

(d) Let \( A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) (any invertible matrix will work).

(e) Not possible. You can't avoid having a solution when \( \vec{b} = \vec{0} \) because \( Ax = \vec{0} \) always has at least the trivial solution.

5. (a) \( \text{rank } AB = 6 \)  (b) \( \text{rank } AB \leq 6 \)

6. (a) \( \text{rank } = \text{number of echelon cols with pivots } = 3 \)

(b) The range is the col space. Basis vectors are the original cols 1,4,6 in \( M \).

(c) Solve \( Mx = \vec{0} \). Sol is

\[
\begin{align*}
x_6 &= 0 \\
x_4 &= -6x_5 \\
x_1 &= -5x_5 - 3x_3 - 2x_2 \\
x_2, x_3, x_5 &\text{ free}
\end{align*}
\]

Set \( x_2 = 1, x_3 = 0, x_5 = 0 \) to get \( u = (-2,1,0,0,0,0) \).

Set \( x_2 = 0, x_3 = 1, x_5 = 0 \) to get \( v = (-3,0,1,0,0,0) \).

Set \( x_2 = 0, x_3 = 0, x_5 = 1 \) to get \( w = (-5,0,0,-6,1,0) \).

Basis for the null space is \( u,v,w \). The null space is 3-dim.

(d) No. For one reason, \( \vec{u} \) doesn't satisfy the system \( M\vec{x} = \vec{0} \) because it doesn't satisfy the equivalent system \( \text{ech } M \vec{x} = \vec{0} \). For another reason, part (c) gave the solution to \( Mx = \vec{0} \) and \( u \) doesn't fit (among other things, its 6th component isn't 0).

(e) 0. Since \( \text{rank } M = 3 \), the largest order of a nonzero subdet is 3. Subdets larger than 3 \( \times \) 3 must be 0.

(f) Can't do. At least one 3 \( \times \) 3 subdet is nonzero but you can't tell which one(s).

(g) Can't do it without knowing what row ops it took to get \( M \) into echelon form.

To solve \( M\vec{x} = \vec{b} \), I want to do row ops not just to \( M \) but to \( M|\vec{b} \). Unless I know what the row ops do to \( b \), I can't solve the system of equations.
7. Let \( P \) be the basis changing matrix whose columns are the \( i,j,k \) coordinates of the new basis vectors. Then

\[
B = P^{-1} A P \\
|B| = |P^{-1} A P| \\
= |P^{-1}| |A| |P| \quad \text{det rule} \\
= |A| \quad \text{because } |P^{-1}| = \frac{1}{|P|}
\]

Note: \( P^{-1} A P \) does not equal \( P^{-1} P A \); you can't change the order of the factors in a matrix product. So in the matrix product \( P^{-1} A P \), the \( P^{-1} \) and \( P \) do not cancel out.

But \( |P^{-1}| |A||P| \) does equal \( |P^{-1}||P||A| \) because determinants are numbers and you can change the order of the factors in a product of numbers. So in the product \( |P^{-1}||A||P| \), \( |P^{-1}| \) and \( |P| \) do cancel out.

8. The rank of \( M \) is either 0 or 1 but not 2. So the range of \( M \) is either a 0-dim or a 1-dim subspace of \( \mathbb{R}^2 \). So at best, the outputs of \( M \) can spread out along a line through the origin. But they can't spread out as much as the triangular region in the righthand diagram. The correct image of the lefthand region would have to lie along a line through the origin.

9. (a) The lefthand diagram shows the input \( \text{in1} = 3u + 2v \) and the corresponding output \( \text{out1} = u - v \).

(b) The righthand diagram shows the input \( \text{in2} = 3i + 2j \) and the corresponding output \( \text{out2} = 3j \).

I would have put all of this in one picture but it got too crowded.

Note: There is no connection between \( \text{in1} \) and \( \text{in2} \) or between \( \text{out1} \) and \( \text{out2} \). There was never intended to be any connection.

(c) Let the columns of \( P \) be the coordinates of \( u \) and \( v \) (w.r.t. \( i,j \)). Then

\[ A = P^{-1} B P \]

\[ \begin{array}{c}
\text{in1} \\
\text{out1}
\end{array} \quad \text{Problem 9(a)} \]

\[ \begin{array}{c}
\text{in2} \\
\text{out2}
\end{array} \quad \text{9(b)} \]

10. Find a basis for the orthog complement of the space spanned by \( u,v,w \).

Let \( A \) have rows \( u,v,w \). Solve the system of equations \( \overrightarrow{Ax} = \overrightarrow{0} \)

(Equivalently, solve the system \( \overrightarrow{u} \cdot \overrightarrow{x} = 0, \overrightarrow{v} \cdot \overrightarrow{x} = 0, \overrightarrow{w} \cdot \overrightarrow{x} = 0 \).

\[
\begin{array}{ccc|c}
1 & 3 & 2 & 0 \\
2 & 6 & 9 & 5 \\
-1 & -3 & 3 & 0 \\
\hline
0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Row ops to

\[
\begin{array}{ccc|c}
1 & 3 & 2 & 0 \\
1 & 3 & 0 & 0 \\
-1 & -3 & 3 & 0 \\
\hline
0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Sol is

\[
x_3 = -\frac{1}{3}x_4 \\
x_1 = -x_4 - 3x_2
\]

Set \( x_2 = 1, x_4 = 3 \) (avoids fractions) to get solution \( p = (-3,0,-1,3) \)
Set \( x_2 = 1, \ x_4 = 0 \) to get solution \( q = (-3, 1, 0, 0) \)

A basis for the space of solutions is \( p, q \).
So you can take \( p \) and \( q \) as the answer.