CHAPTER 8 EIGENVALUES AND EIGENVECTORS

SECTION 8.1 INTRODUCTION

Definition of eigenvalue and eigenvector

Let $M$ be a square matrix. Suppose $\vec{x}$ is a nonzero column vector, $\lambda$ is a scalar and

$$M\vec{x} = \lambda \vec{x}$$

Then $\lambda$ is called an eigenvalue of $M$ and $\vec{x}$ is the corresponding eigenvector. In other words, an eigenvector of $M$ is a nonzero vector that maps to a multiple of itself.

Note that $M\vec{0} = \vec{0}$ so $\vec{0}$ always maps to a multiple of itself, but $\vec{0}$ doesn't count as an eigenvector because eigenvectors must be nonzero.

Eigenspaces

Suppose $u$ and $v$ are eigenvectors of an $n \times n$ matrix corresponding to the eigenvalue $\lambda$. Let $k$ be a scalar. The numbers $u$ and $k$ are also eigenvectors corresponding to $\lambda$, provided that they are not $\vec{0}$ (i.e., provided $v \neq -u$ and $k \neq 0$).

The set of all eigenvectors corresponding to eigenvalue $\lambda$, together with the zero vector, is a subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$ called the eigenspace corresponding to $\lambda$.

Proof

Let $u$ and $v$ be eigenvectors corresponding to eigenvalue $\lambda$.

I want to show that if $u + v$ and $ku$ are nonzero then they are also eigenvectors corresponding to $\lambda$.

$$M(u+v) = Mu + Mv \quad \text{matrix algebra}$$
$$= \lambda u + \lambda v \quad u \text{ and } v \text{ are eigenvector corr to } \lambda$$
$$= \lambda(u + v) \quad \text{vector algebra}$$

So provided that $u+v$ is not $\vec{0}$, it is an eigenvector corr to $\lambda$. Also

$$M(ku) = k(Mu) \quad \text{matrix algebra}$$
$$= k(\lambda u) \quad u \text{ is an eigenvector corr to eigenvalue } \lambda$$
$$= \lambda(ku) \quad \text{vector algebra}$$

So provided that $ku$ is not $\vec{0}$, it's an eigenvector corr to $\lambda$.

The awkward status of the zero vector

It turns out that nonzero vectors with the property that $M\vec{x} = \lambda \vec{x}$ are useful. But the fact that $\vec{0}$ always has this property is never interesting or useful which is why mathematicians do not consider the vector $\vec{0}$ to be an eigenvector of a matrix $M$.

This unfortunately means that the set of all eigenvectors of $M$ corresponding to an eigenvalue $\lambda$ is not a subspace because a subspace must contain $\vec{0}$.

So the eigenspace of $M$ corresponding to eigenvalue $\lambda$ is defined as the set of all eigenvectors corresponding to $\lambda$ plus the non-eigenvector $\vec{0}$.

Example

Suppose $M$ reflects points in the x-axis in $\mathbb{R}^2$. (Fig 1).

If $u$ is on the x-axis then $Mu = u$ so $u$ is an eigenvector corresponding to eigenvalue 1.

If $v$ is on the y-axis then $Mv = -v$ so $v$ is an eigenvector corresponding to eigenvalue $-1$.

If $w$ is on neither axis then $Mw$ is not a multiple of $w$, so $w$ is not an eigenvector.

The eigenvalues are 1 and $-1$. The corresponding eigenspaces are the x-axis and y-axis respectively.
mathematical catechism

question 1 What does it mean to say that $A$ has eigenvalue $\lambda$.
answer It means that there is a nonzero vector $u$ such that $Au = \lambda u$.

question 2 What does it mean to say that $A$ has eigenvector $u$.
answer It means that $u \neq 0$ and there is a scalar $\lambda$ such that $Au = \lambda u$.

question 3 What does it mean to say that $A$ has eigenvector $u$ with corresponding eigenvalue $\lambda$.
answer It means that $u \neq 0$ and $Au = \lambda u$.

PROBLEMS FOR SECTION 8.1

1. Let

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Are $u$ and $v$ eigenvectors of $A$? If so, find the corresponding eigenvalues.

2. Let $u$ be a nonzero vector. Spot eigenvalues and eigenvectors if

(1) $Mu = 6u$
(2) $Mu = 0$
(3) $ABu = 3u$
(4) $ABu = 3Bu$
(5) $ABu = 3Au$
(6) $2Mu = u$
(7) $M(u+v) = 3u + 3v$
(8) $(A + B)u = Au + Bu$
(9) $(A+B)u = 2u$

3. Suppose $u$ is an eigenvector of $M$ corresponding to eigenvalue $\lambda$. Show that $u$ is also an eigenvector of $3M$ and find the corresponding eigenvalue.

4. Find the eigenvalues and eigenvectors of the $n \times n$ matrix $I$.

5. Remember that the null space of a matrix operator $M$ is the set of vectors that map to $0$.

(a) If the null space contains more than just $0$ show that it is an eigenspace of $M$ and find the corresponding eigenvalue.

(b) By the way, what kind of matrix has a null space that contains more than $0$?

6. Suppose $u$ is an eigenvector of matrix $M$ corresponding to eigenvalue $\lambda = 5$. What can you conclude about $u_{\text{unit}}$?
7. Suppose $A, B, C$ are $2 \times 2$ matrices.
   A projects points onto the $x$-axis as illustrated in the first diagram.
   B reflects points in a line $L$ through the origin as illustrated in the middle diagram.
   C expands radially by a factor of 2 as illustrated in the third diagram.
   Think geometrically to find eigenvalues and eigenspaces of $A, B, C$.

8. (a) Can an eigenvalue of a matrix $M$ have more than one corresponding eigenvector.
    (b) Can an eigenvector of a matrix $M$ have more than one corresponding eigenvalue.

9. Let $A$ be square with eigenvalue $\lambda$ and corresponding eigenvector $u$.
    Let $B = P^{-1}AP$.
    Show that $B$ also has eigenvalue $\lambda$ and find the corresponding eigenvector.

10. Can two eigenspaces (corresponding to different eigenvalues) overlap (i.e., have something in common)?

11. Suppose 0 is an eigenvalue of $M$. What does this have to do with invertibility.
    Suggestion: Remember what it means for 0 to be an eigenvalue and then use the invertible matrix rule.
SECTION 8.2 FINDING EIGENVALUES AND EIGENVECTORS

finding eigenvalues

To find eigenvalues of an $n \times n$ matrix $M$, solve this equation for $\lambda$:

$$|M - \lambda I| = 0$$

i.e., find $\lambda$'s that make the determinant of the matrix $M - \lambda I$ zero.

Here's why.

To find eigenvalues of $M$ you want to find $\lambda$'s so that the equation $M\vec{x} = \lambda \vec{x}$ has a nonzero solution for $\vec{x}$. The equation can be written as

$$(M - \lambda I)\vec{x} = \vec{0}$$

It's a square system and by Cramer's rule it has just one solution iff $|M - \lambda I| \neq 0$.

It also is a homogenous system and always has (at least) the trivial solution $\vec{x} = \vec{0}$. To get a nonzero solution in addition to the trivial solution you must avoid having just one solution, i.e., you must avoid $|M - \lambda I| \neq 0$. So you want $|M - \lambda I| = 0$. QED

characteristic polynomial and characteristic equation

If $M$ is $n \times n$ then $|M - \lambda I|$ is an $n$-th degree polynomial in the variable $\lambda$ (you'll see this when you start doing examples), called the characteristic polynomial of $M$, and $|M - \lambda I| = 0$ is called the characteristic equation.

the multiplicity of an eigenvalue

Suppose the characteristic equation factors into

$$(\lambda - 8)^4 (\lambda - 6)^3 (\lambda - 7) = 0$$

Then $\lambda = 8$ is called a 4-fold root of the equation, or a 4-fold eigenvalue or an eigenvalue with multiplicity 4. Similarly 6 is a 3-fold eigenvalue and 7 is a 1-fold eigenvalue.

how many eigenvalues does a matrix have

An $n \times n$ matrix has at least one and at most $n$ eigenvalues, and their multiplicities add up to $n$. We say that the matrix has $n$ eigenvalues counting multiplicity meaning that a 3-fold eigenvalue counts 3 times.

This holds because the characteristic equation is an $n$-th degree polynomial equation in variable $\lambda$ and that kind of an equation always has $n$ (possibly non-real) roots counting multiplicity.

For example, if $M$ is $6 \times 6$ then $M$ can have one eigenvalue with multiplicity 6.

or $M$ can have one 5-fold eigenvalue and one 1-fold eigenvalue.

or $M$ can have six 1-fold eigenvalues, etc.

finding eigenvectors after using (1) to find eigenvalues

If $\lambda$ is an eigenvalue of $M$ then the corresponding eigenvectors are found by solving $M\vec{x} = \lambda \vec{x}$ for $\vec{x}$. In other words:

The eigenspace corresponding to eigenvalue $\lambda$ is the set of solutions to the homogenous system

$$(M - \lambda I)\vec{x} = \vec{0}.$$
Furthermore:

(2)

If $\lambda$ is a $k$-fold eigenvalue of $M$ then there are $k$ or fewer corresponding independent eigenvectors (proof omitted).

In particular, a 1-fold eigenvalue has one corresponding ind eigenvector; a 3-fold eigenvalue may have 1, 2 or 3 corresponding ind eigenvectors, etc.

example 1

Let

$$M = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

To find the eigenvalues and a maximal number of independent eigenvectors for each eigenvalue, begin with

$$M - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 4 \\ -2 & 8-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{bmatrix}$$

The characteristic poly is

$$|M-\lambda I| = (5-\lambda) \left[ (8-\lambda)(5-\lambda) - 4 \right] + 2 \left[ -2(5-\lambda) - 8 \right] + 4 \left[ -4 - 4(8-\lambda) \right]$$

(I expanded the det across row 1)

$$= (5-\lambda) \left[ \lambda^2 - 13\lambda + 36 \right] + 4[\lambda - 9] + 16[\lambda - 9]$$

$$= (5-\lambda) (\lambda - 4)(\lambda - 9) + 20(\lambda - 9)$$

$$= (\lambda - 9) \left[ (5-\lambda)(\lambda - 4) + 20 \right]$$

$$= (\lambda - 9)(-\lambda^2 + 9\lambda)$$

$$= -\lambda(\lambda - 9)^2$$

The characteristic equ is $-\lambda(\lambda - 9)^2 = 0$ and the roots are $\lambda = 0, 9, 9$.

0 is 1-fold so there is one corresponding eigenvector.

9 is 2-fold so there are either one or two corresponding independent eigenvectors.

case where $\lambda = 0$

To find the eigenvectors, solve $(M - \lambda I)x = 0$ for $x$ which in this case means

solving $Mx = \vec{0}$. Start with

$$\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and row op to get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution is

$$x_2 = -\frac{1}{2}x_3$$

$$x_1 = -x_3$$

Choose any one solution. If $x_3 = -2$ then $x_2 = 1$, $x_1 = 2$ so an eigenvector is

$$u = (2,1,-2).$$

(All other eigenvectors corresponding to $\lambda=0$ are multiples of $u$.)
case where $\lambda = 9$

Solve $(M - 9I)x = \vec{0}$, i.e., solve

\[
\begin{array}{ccc|c}
  -4 & -2 & 4 & 0 \\
  -2 & -1 & 2 & 0 \\
  4 & 2 & -4 & 0 \\
\end{array}
\]

Each equation says $-2x_1 - x_2 + 2x_3 = 0$.
Use say $x_1$ and $x_3$ as free variables; then $x_2 = 2x_3 - 2x_1$.
Since there are two free variables there are two ind eigenvectors.
Set $x_1 = 1$, $x_3 = 0$ to get eigenvector $v = (1, -2, 0)$.
Set $x_1 = 0$, $x_3 = 1$ to get eigenvector $w = (0, 2, 1)$.

example 2
Let

\[
M = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

Find eigenvalues, and for each $\lambda$, find as many ind eigenvectors as possible.

solution $|M - \lambda I| = \begin{vmatrix}
-\lambda & -1 \\
1 & -\lambda
\end{vmatrix} = \lambda^2 + 1$

The sols to $\lambda^2 + 1 = 0$ are $\lambda = \pm i$; each $\lambda$ is 1-fold.
To find the eigenvectors corresponding to $\lambda = i$, solve $(M - iI)x = \vec{0}$,

\[
\begin{array}{cc|c}
  -i & -1 & 0 \\
  1 & -i & 0 \\
\end{array}
\]
Both equations say $x = iy$.
Set $y = 1$ to get eigenvector $u = (i, 1)$

To find the eigenvectors corresponding to $\lambda = -i$, solve $(M + iI)x = \vec{0}$,

\[
\begin{array}{cc|c}
  i & -1 & 0 \\
  1 & i & 0 \\
\end{array}
\]

The first equation says $x = y/i$. The second says $x = -iy$. Since $1/i = -i$, these are the same equation. Set $y = 1$ to get eigenvector $v = (-i, 1)$.

Note that a real matrix may have non-real eigenvalues in which case the eigenvectors will be non-real also.

matrices with a complete set of eigenvectors
If an $n \times n$ matrix $M$ has $n$ ind eigenvectors then $M$ is said to have a complete set of eigenvectors, namely, enough eigenvectors to make a basis for $C^n$ (or, if real, for $R^n$).

This happens iff each k-fold eigenvalue of $M$ has k corresponding ind eigenvectors.
As a special case, if all the eigenvalues of $M$ are 1-fold then $M$ must have a complete set of eigenvectors.

Here's how to get the $n$ independent eigenvectors when they do exist.
Suppose $M$ is $6 \times 6$ with

- 3-fold eigenvalue $\lambda_1$
- 2-fold eigenvalue $\lambda_2$
- 1-fold eigenvalue $\lambda_3$. 
And suppose that each k-fold $\lambda$ produces k ind eigenvectors. Then there are three ind eigenvectors $u,v,w$ corresponding to $\lambda_1$; there are two ind eigenvectors $p,q$ corresponding to $\lambda_2$; and there is one ind eigenvector $r$ corresponding to $\lambda_3$. It can be shown (proof omitted here) that the six eigenvectors $u,v,w,p,q,r$ are ind.

The matrix $M$ in example 1 has a 1-fold eigenvalue 0 with eigenvector $u = (2,1,-2)$ and a 2-fold eigenvalue 9 with ind eigenvectors $v = (1,-2,0)$, $w = (0,2,1)$. So $M$ has a complete set of eigenvectors, namely $u,v,w$, available to be a basis for $\mathbb{R}^3$.

eigenvalues and eigenvectors of a diagonal matrix

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. By inspection,

$A_i = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2i$, $A_j = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2j$, $A_k = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = 3k$.

So 2 is an eigenvalue with corresponding eigenvectors $i,j$ (a 2-dim eigenspace) and 3 is an eigenvalue with corresponding eigenvector $k$ (a 1-dim eigenspace).

invertible rule

Let $M$ be $n \times n$. The following are equivalent; i.e., either all are true or all are false.

(1) $M$ is invertible (nonsingular).
(2) $|M| \neq 0$.
(3) Echelon form of $M$ is $I$.
(4) Rows of $M$ are independent.
(5) Cols of $M$ are independent.
(6) Rank of $M$ is $n$.
(7) $Mx = \vec{0}$ has only the trivial solution $x = \vec{0}$.

In other words, the null space of $M$ contains only $\vec{0}$.

In other words, if $x \neq \vec{0}$ then $Mx \neq \vec{0}$.

In other words, the operator $M$ maps $\vec{0}$, and nothing else, to $\vec{0}$.

(8) The operator $M$ is one-to-one, i.e., $M$ doesn't send two inputs to the same output.
(9) The range of $M$ is $\mathbb{R}^n$.
(10) The eigenvalues of $M$ are all nonzero.

Having (10) on the list means every non-invertible matrix has 0 as an eigenvalue and no invertible matrix has 0 as an eigenvalue.

proof that (10) belongs on the list

I'll show that not having (10) is equivalent to not having (2).

0 is an eigenvalue of $M$ iff the equation $|M - \lambda I| = 0$ has $\lambda = 0$ as one of its sols

iff $|M - 0I| = 0$

iff $|M| = 0$

leading term of the characteristic poly

If $M$ is $3 \times 3$ then the characteristic poly of $M$ begins with $-\lambda^3$ (as in example 1). If $M$ is $4 \times 4$ then the characteristic poly of $M$ begins with $\lambda^4$. 
In general suppose $M$ is $n \times n$.
If $n$ is even then the characteristic poly of $M$ begins with $\lambda^n$.
If $n$ is odd then the characteristic poly of $M$ begins with $-\lambda^n$.

**proof.**

Let $M = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$

Then

$$|M - \lambda I| = \begin{vmatrix} a_{11} - \lambda & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} - \lambda \end{vmatrix}$$

Compute the determinant by finding the sum of products consisting of one entry from each row and col where each product in the sum is prefixed a sign according to the how-many-inversions rule.

One of the products in the sum is

$$\text{(a}_{11} - \lambda)(a_{22} - \lambda) \ldots (a_{nn} - \lambda),$$

the product consisting of all the diagonal entries of $A - \lambda I$. This product always gets a plus sign since there are no inversions. The $\lambda^n$ term in the sum comes entirely from multiplying out this product. And if there are an even number of factors, i.e., if $n$ is even, then you get $\lambda^n$ when you multiply out; if $n$ is odd, you get $-\lambda^n$.

**sum and product of the eigenvalues**

1. The product of the eigenvalues of $M$ is $|M|$ provided that a $k$-fold $\lambda$ is counted $k$ times in the product. For example, if the eigenvalues are $-2, 3, 3$ then $|M| = -18$.
2. The trace of a square matrix $M$ is defined as the sum of its diagonal entries. The sum of the eigenvalues of $M$ is trace $M$ provided that a $k$-fold $\lambda$ is counted $k$ times in the sum.

For example if $M = \begin{bmatrix} 2 & \cdot & \cdot \\ \cdot & 4 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$ then trace $M = 6$ and the sum of the eigenvalues must be 6.

**proof in the $3 \times 3$ case**

Let

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

Suppose $M$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

I'm going to compute the characteristic poly $|M - \lambda I|$ in two different ways.

For the first way, expand the determinant using the original definition of a determinant in Section 1.3.

\[ (*) \quad |M - \lambda I| = \begin{vmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & k - \lambda \end{vmatrix} = (a-\lambda)(e-\lambda)(k-\lambda) - (a-\lambda)hf + \text{four more terms} \]

For the second way, remember that $\lambda_1, \lambda_2, \lambda_3$ are roots of the characteristic poly so the polynomial has factors $\lambda - \lambda_1, \lambda - \lambda_2, \lambda - \lambda_3$. And

\[ (**) \quad |M - \lambda I| = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \]
The minus sign is there because if \( M \) is \( n \times n \) where \( n \) is odd then the leading coeff of the characteristic poly is \(-\lambda^R\).

The two versions in (*) and (**) must agree.

Equate coeffs of \( \lambda^2 \) from the two versions in (*) and (**) : The \( \lambda^2 \) coeff in (*) is \( a+e+k \). The \( \lambda^2 \) coeff in (**) is \( \lambda_1 \lambda_2 + \lambda_3 \). So \( \lambda_1 \lambda_2 + \lambda_3 = a + e + k \), proving (2).

Equate the constant terms in (*) and (**) : The constant term in (**) is \( \lambda_1 \lambda_2 \lambda_3 \).

The constant term in (*) includes \( aek - ahf \) plus four more terms. Instead of doing algebra to get it, notice that for any polynomial you can get the constant term by setting the variable equal to 0. So to get the constant term in (*), set \( \lambda = 0 \) in \( |M-\lambda I| \). The constant term is \( |M| \). So \( |M| = \lambda_1 \lambda_2 \lambda_3 \), proving (1).

**Mathematical Catechism**

**Question 1** What does it mean to say that a square matrix has a complete set of eigenvectors.

**First Answer** It means that if \( M \) is \( n \times n \) then \( M \) has \( n \) independent eigenvectors.

**Second Answer** It means that each \( k \)-fold eigenvalue has \( k \) corresponding independent eigenvectors.

**Question 2** What is the characteristic polynomial of a square matrix \( M \).

**Answer** \( |M-\lambda I| \)

**Question 3** What is the characteristic equation of a square matrix \( M \).

**Answer** \( |M-\lambda I|=0 \)

**Question 4** What does it mean to say that \( \lambda = 7 \) is a 3-fold eigenvalue of \( A \).

**Answer** It means that \((\lambda-7)^3\) is a factor of the characteristic poly \(|A-\lambda I|\).

**Problems for Section 8.2**

1. Find eigenvalues and corresponding independent eigenvectors. Does the matrix have a complete set of eigenvectors.

   (a) \[
   \begin{bmatrix}
   0 & 3 \\
   2 & -1 
   \end{bmatrix}
   \]

   (b) \[
   \begin{bmatrix}
   i & 1 \\
   1 & -i 
   \end{bmatrix}
   \]

   (c) \[
   \begin{bmatrix}
   2 & 2 & -6 \\
   2 & -1 & -3 \\
   -2 & -1 & 1 
   \end{bmatrix}
   \]

   (the eigenvalues turn out to be \( \lambda = -2, -2, 6 \))

   (d) \[
   \begin{bmatrix}
   0 & 0 & 1 \\
   0 & 1 & 0 \\
   1 & 0 & 0 
   \end{bmatrix}
   \]

   (e) \[
   \begin{bmatrix}
   4 & 0 & 0 \\
   0 & -5 & 0 \\
   0 & 0 & \pi 
   \end{bmatrix}
   \]

2. The matrix that rotates points in \( R^2 \) by \( \theta \) degrees counterclockwise is

   \[
   M = \begin{bmatrix}
   \cos \theta & -\sin \theta \\
   \sin \theta & \cos \theta 
   \end{bmatrix}
   \]

   (a) Find the eigenvalues (the algebra comes out nicer than you would expect).

   (b) Pick one of your eigenvalues and find the corresponding eigenvector(s).

3. Look at example 1 where I found eigenvectors \( u,v,w \). Find a dozen other eigenvectors.

4. Suppose \( M \) is \( 6 \times 6 \) and \( \lambda \) is one of its eigenvalues. If \( u \) and \( v \) are eigenvectors corr to \( \lambda \) and \( u \) and \( v \) happen to be ind, what can you conclude about the multiplicity of \( \lambda \).
5. Suppose $M$ is $5 \times 5$. Is it possible for $M$ to not have eigenvalues or eigenvectors?

6. Suppose
   
   \[ A \text{ has characteristic poly } (\lambda-3)^5(\lambda-2)^2(\lambda-4) \]
   \[ B \text{ has characteristic poly } -(\lambda+1)(\lambda-2)(\lambda+3). \]
   
   Find the size of each matrix.
   Find their eigenvalues.
   What can you conclude about the number of ind eigenvalues for each eigenvalue.
   Do the matrices have a complete set of eigenvalues.

7. (a) Suppose the book's answer says that the eigenvector of $M$ corresponding to the 1-fold eigenvalue $\lambda$ is $(i,1)$ but your answer is $(-1,i)$. Are you a dope?
    (b) Suppose the book's answer says that the eigenvectors of $M$ corresponding to the 2-fold eigenvalue $\lambda$ are $u = (1,1,1,1)$, $v = (2,0,3,1)$. Your eigenvectors are $u = (1,1,1,1)$ (so far so good) and $p = (1,2,1,1)$. Are you right.

8. What can you conclude about eigenstuff if
   (a) $|A - 2I| = 0$  (b) $|A - 2I| = 6$

9. Find eigenvalues and eigenvectors for
   (a) the $3 \times 3$ identity matrix $I$  (b) the $3 \times 3$ zero matrix
   Find them by inspection and then, for practice, with the method used in example 1 (although that is overkill).

10. (a) Factor $P^{-1}AP - \lambda I$.
    (b) Use part (a) to show that similar matrices have the same characteristic polynomials.
    (c) Show that similar matrices have the same eigenvalues.

11. Suppose $A$ is $5 \times 5$, $9$ is an eigenvalue, and $A - 9I$ has rank 2.
    (a) How many independent eigenvectors are there corresponding to eigenvalue 9.
    (b) How many total eigenvectors are there corresponding to eigenvalue 9.

12. (a) When is 0 an eigenvalue of a matrix $M$.
    (b) If 0 is an eigenvalue of an $n \times n$ matrix $M$ with rank $r$, what is the dimension of the corresponding eigenspace?

13. Suppose $M$ is invertible and has eigenvector $u$ with corresponding eigenvalue $\lambda$.
    Find some eigenstuff for $M^{-1}$.

14. Suppose the matrix $AB$ has eigenvalue 5. What determinant must therefore be 0.

15. Let $A = \begin{bmatrix} 0 & \pi & 0 \\ 3 & \sqrt{2} & 7 \\ 1 & 2 & 1 \end{bmatrix}$
    (a) Find the product of the eigenvalues of $A$.
    (b) Find the sum of the eigenvalues of $A$.

16. (a) True or False. If $u,v,w$ are ind vectors in $\mathbb{R}^5$ and $p,q$ are also ind vectors in $\mathbb{R}^5$ then $u,v,w,p,q$ are ind.
    (b) True or False. If $u,v,w$ are ind eigenvectors of $M$ corresponding to $\lambda=3$, and $p,q$ are ind eigenvectors of $M$ corresponding to $\lambda=\pi$ then $u,v,w,p,q$ are ind.

17. On an exam, a student was given a $2 \times 2$ matrix and told to find eigenvalues and eigenvectors.
She found the eigenvalues correctly.

Then she tried to solve the system of equations \((M - \lambda I)\vec{x} = \vec{0}\) to find the eigenvectors corresponding to the first eigenvalue \(\lambda\). After doing some algebra she ended up writing "the only solution I can find is \(x_1=0, x_2=0\) and since \((0,0)\) does not count as an eigenvector, this \(\lambda\) doesn't have any corresponding eigenvectors".

Could she be right. Could some mean teacher have made up an exam question like that.
EIGENVALUES AND EIGENVECTORS OF HERMITIAN MATRICES

**lemma** (sliding property of Herms)

If $H$ is an $n \times n$ Hermitian matrix and $u$ and $v$ are in $\mathbb{C}^n$ and written as column vectors then

$$Hu \cdot v = u \cdot Hv$$

**proof**

$$Hu \cdot v = (Hu)^* v \quad \text{the dot product } p \cdot q \text{ is the same as the matrix product } p^* q \quad (\S 7.3)$$

$$= u^* H^* v \quad \text{by } * \text{ rule}$$

$$= u^* Hv \quad \text{since } H \text{ is Herm}$$

$$= u \cdot Hv \quad \text{Section 7.3 again}$$

**properties of eigenvalues and eigenvectors of Hermitian (and symmetric) matrices**

Let $H$ be an $n \times n$ Hermitian matrix.

1. The eigenvalues are real.
2. Eigenvectors corresponding to different eigenvalues are orthogonal.
3. Each $k$-fold eigenvalue has $k$ corresponding ind eigenvectors.
4. $H$ has a complete set of orthonormal eigenvectors (i.e., has $n$ of them).

And here's how to find the complete set.

Suppose $H$ is $4 \times 4$ with 1-fold eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Pick corresponding eigenvectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$. They will automatically be orthog by property (2).

Normalize them and they will still be eigenvectors corresponding to $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (multiples of eigenvectors are still eigenvectors) and still be orthog. So $\vec{u}_1^{\text{unit}}, \vec{u}_2^{\text{unit}}, \vec{u}_3^{\text{unit}}, \vec{u}_4^{\text{unit}}$ are a complete set of orthonormal eigenvectors for $H$.

Or suppose $H$ is $4 \times 4$ with 3-fold eigenvalue $\lambda_1$ and 1-fold eigenvalue $\lambda_2$. Pick 3 independent eigenvectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ corresponding to $\lambda_1$. Use the Gram Schmidt process to exchange them for 3 orthog vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ in the eigenspace. Pick any eigenvector $\vec{w}$ corresponding to $\lambda_2$. By (2), $\vec{w}$ is orthog to $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Normalize $\vec{w}, \vec{u}_1, \vec{u}_2, \vec{u}_3$ and they will be a complete set of orthonormal eigenvectors of $H$.

**proof of (1)**

Let $\lambda$ be an eigenvalue of $H$. I want to show that $\lambda$ is real.

There is a nonzero vector $x$ such that $Hx = \lambda x$. Then

$$x \cdot Hx = Hx \cdot x \quad \text{(by the sliding property of Herms)}$$

$$x \cdot \lambda x = \lambda x \cdot x \quad (Hx = \lambda x)$$

$$\lambda (x \cdot x) = \overline{\lambda} (x \cdot x) \quad \text{(dot rules)}$$

$$\lambda = \overline{\lambda} \quad \text{(it's OK to cancel the } x \cdot x \text{'s since } x \neq 0 \text{ so } x \cdot x \neq 0)$$

So $\lambda$ is real since it equals its conjugate.

**proof of (2)**

Let $\vec{u}_1$ and $\vec{u}_2$ be eigenvectors corresponding to different eigenvalues $\lambda_1, \lambda_2$.

Write them as column vectors. Then
\[ u_1 \cdot Hu_2 = Hu_1 \cdot u_2 \quad \text{(sliding property)} \]
\[ u_1 \cdot (\lambda_2 u_2) = \lambda_1 (u_1 \cdot u_2) \quad \text{(Hu}_1 = \lambda_1 u_1, \ Hu_2 = \lambda_2 u_2 \) \]
\[ \lambda_2 (u_1 \cdot u_2) = \lambda_1 (u_1 \cdot u_2) \quad \text{(dot rules plus the fact that, by (1), } \lambda_1 \text{ is real)} \]
\[ (\lambda_2 - \lambda_1) (u_1 \cdot u_2) = 0 \]
So \( \lambda_2 = \lambda_1 \) or \( u_1 \cdot u_2 = 0 \).
But \( \lambda_2 \) and \( \lambda_1 \) are different eigenvalues so it must be that \( u_1 \cdot u_2 = 0 \).
So \( u_1 \) and \( u_2 \) are orthogonal, QED

**proof of (3)** too hard

**warning**

Herms are not the only matrices with properties (1), (2), (3) or (4). But it can be shown that they are the only ones with all four properties simultaneously.

**PROBLEMS FOR SECTION 8.3**

1. Suppose \( A \) is Herm and has eigenvalue 2 with corresponding eigenvector \( u = (1,3) \) and also has eigenvalue 3 with corresponding eigenvector \( v = (2,y) \). Find \( y \).

2. Suppose that among the eigenvectors of a matrix \( M \) are \( u = (2,3) \) and \( v = (4,5) \). Can you tell if \( M \) is Hermitian.

3. Can you tell if \( M \) is Hermitian if its eigenvalues are
   (a) 2,3,4  
   (b) 2±i, 6
SECTION 8.4 DIAGONALIZING A SQUARE MATRIX

the diagonalizing process

Diagonalizing a square matrix A means finding a matrix P (if possible) so that $P^{-1}AP$ is diagonal.

Here's how to do it.

Suppose a $3 \times 3$ matrix A has the three independent eigenvectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$ with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (not necessarily distinct, e.g., could have $\lambda_2 = \lambda_3$). Let

$$P = \begin{bmatrix}
u_1 & v_1 & w_1 \\
u_2 & v_2 & w_2 \\
u_3 & v_3 & w_3
\end{bmatrix} \text{ (the basis changing matrix)}$$

Then

$$P^{-1}AP = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix} \text{ (the basis changing matrix)}$$

call this $\Lambda$

We say that $P$ diagonalizes $A$.

This means that if a transformation is represented by matrix $A$ w.r.t. the usual basis $i,j,k$ then with respect to a a new basis consisting of eigenvectors of $A$, the transformation is represented by the diagonal matrix $\Lambda$.

The process of diagonalizing $A$ is also referred to as finding a diagonal matrix similar to $A$ (similar matrices were defined in Section 6.2).

Furthermore it can be shown that (1) is the only way to diagonalize $A$, meaning that if say $Z^{-1}AZ = \begin{bmatrix}
2 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 7
\end{bmatrix}$ then 2,5,7 are eigenvalues of $A$ and the columns of $Z$ are the corresponding eigenvectors.

when is a matrix diagonalizable

The matrix $A$ can be diagonalized iff it has a complete set of eigenvectors. Remember that this happens iff each k-fold $\lambda$ has k independent eigenvectors. And in particular, it happens if all the eigenvalues are 1-fold.

A matrix that doesn't have "enough" eigenvectors can't be diagonalized.

proof of (1)

Look at the underlying transformation $T$, the transformation represented by matrix $A$ w.r.t. basis $i,j,k$ and represented by $P^{-1}AP$ w.r.t. the basis $u,v,w$. I want to show that $P^{-1}AP$ is $\Lambda$.

You know that $T(u) = \lambda_1 u$
With respect to basis \( u, v, w \), the coords of \( u \) are 1,0,0 and the coords of \( \lambda_1 u \) are \( \lambda_1, 0, 0 \). So

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
\lambda_1 \\
0 \\
0
\end{bmatrix}
\]

Similarly,

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
\lambda_2 \\
0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\lambda_3
\end{bmatrix}
\]

The three equations in (2) and (3) force \( P^{-1} AP \) to be \( \Lambda \).

**diagonalizing Hermitian (and symmetric) matrices**

Suppose \( H \) is Hermitian.

We know from the last section that \( H \) has a complete set of eigenvectors so \( H \) is diagonalizable. Furthermore we know that \( H \) has a complete set of orthonormal eigenvectors so it is always possible to choose a unitary \( P \) for the diagonalizing process. And we know that \( H \) has real eigenvalues so \( \Lambda \) is real. All in all:

If \( H \) is Herm then \( H \) is diagonalizable; i.e., you can find a \( P \) so that \( P^{-1} \Omega HP = \hat{O} \). Furthermore \( \hat{O} \) is real and \( P \) can be chosen to be unitary (if you play your cards right).

Here’s a reminder of how to get a unitary \( P \). For each 1-fold eigenvalue of \( H \), pick an eigenvector. For each \( k \)-fold eigenvalue of \( H \), \( k > 1 \), pick \( k \) ind eigenvectors and Gram Schmidt them. This produces a complete set of orthogonal eigenvectors. Normalize them (they’re still eigenvectors) and use them as cols of \( P \).

**warning**

1. Herms are not the only diagonalizable matrices. They are not even the only matrices which can be diagonalized with a unitary \( P \). You may be lucky enough to get a complete set of eigenvectors (maybe even orthogonal ones) for a non-Herm.

   But Herms are the only matrices that can be diagonalized so that \( P \) is unitary and \( \Lambda \) is real (proof in problem 7).

2. If you want to make \( P \) a unitary matrix (in the real case, an orthogonal matrix) don’t forget to normalize the orthogonal eigenvectors.

   But there’s no point normalizing non-orthog eigenvectors so don’t go on a normalizing binge.

**example 1**

Let

\[
A = \begin{bmatrix}
5 & -2 & 4 \\
-2 & 8 & 2 \\
4 & 2 & 5
\end{bmatrix}
\]

Diagonalize \( A \) if possible.

**solution**

\( A \) is symmetric so it is diagonalizable. To get a diagonalization, first find the eigenvalues and a complete set of eigenvectors.

From example 1 in the preceding section, \( A \) has these eigenvalues and eigenvectors:

\( \lambda_1 = 0 \) (1-fold) with corresponding eigenvector \( u = (2,1,-2) \)

\( \lambda_2 = 9 \) (2-fold) with corresponding eigenvectors \( v = (0,2,1), w = (1,-2,0) \).
There are a variety of diagonalizations possible, depending on which eigenvectors are used and on the order in which they are lined up as cols of \( P \).

If the cols of \( P \) are \( u,v,w \) in that order then the diagonal entries in \( \Lambda \) are 0,9,9 in that order. So one answer to the question is

\[
P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{ where } P = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ -2 & 1 & 0 \end{bmatrix}
\]

As a check, you can actually compute \( P^{-1}AP \) to see that it is diagonal but it's not necessary to do that. If your algebra is correct then \( P^{-1}AP \) is guaranteed to be the diagonal matrix \( \Lambda \).

Similarly, if the cols of \( P \) are \( v,u,w \) in that order then the diagonal entries in \( \Lambda \) are 9,0,9 in that order. So another answer to the question is

\[
P^{-1}AP = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{ where } P = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -2 \\ 1 & -2 & 0 \end{bmatrix}
\]

Multiples of \( u \) are also eigenvectors corresponding to \( \lambda_1 \) and combinations of \( v \) and \( w \) are also eigenvectors corresponding to \( \lambda_2 \). So I can also use \( 2u, \ 3v, \ v - 4w \) as cols of \( P \) and get

\[
P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{ where } P = \begin{bmatrix} 4 & 0 & -4 \\ 2 & 6 & 10 \\ -4 & 3 & 1 \end{bmatrix}
\]

etc.

element 1 continued

If possible, diagonalize \( A \) with a unitary \( P \).

**Solution** Possible since \( A \) is symmetric. Use the Gram Schmidt process on \( v,w \) to get

\[
\vec{u}_1 = v = (0,2,1)
\]

\[
\vec{u}_2 = w - \frac{\vec{u}_1 \cdot \vec{w}}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = w - \frac{-4}{5} u_1 = (1, -\frac{2}{5}, \frac{4}{5})
\]

Normalize \( \vec{u} \), \( \vec{u}_1 \), \( \vec{u}_2 \) and use them as the cols for \( P \). Then \( P \) is unitary (actually, orthogonal) and

\[
P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{ where } P = \begin{bmatrix} 2/3 & 0 & 5/\sqrt{45} \\ 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4\sqrt{45} \end{bmatrix}
\]

**Warning**

1. If the question says "diagonalize the matrix \( M \)" then you must find \( \Lambda \) and find \( P \) and write \( P^{-1}MP = \Lambda \).

2. It isn't necessary to get orthonormal eigenvectors unless you are specifically requested to diagonalize \( A \) using a unitary (or orthogonal) \( P \). In that case don't forget to get eigenvectors that are both orthogonal and unit length.

3. In example 1, don't write \( A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \) when you mean \( P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \).

Don't forget the \( P \) and \( P^{-1} \).
**example 2**

Suppose $M$ is $3 \times 3$ with eigenvalues $7,6,5$ and corresponding eigenvectors $u = (2,1,3), \ v = (1,4,2), \ w = (4,1,6)$.

Then $M$ is diagonalizable and

$$P^{-1}MP = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ where } P = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 4 & 1 \\ 3 & 2 & 6 \end{bmatrix}$$

There is no way to diagonalize $M$ with a unitary (or orthogonal) $P$ since the eigenvectors corresponding to different $\lambda$'s did not turn out to be orthogonal (and cannot be re-chosen to be orthog).

**powers and roots of a diagonalizable matrix**

Suppose

$$A = P\Lambda P^{-1} \text{ where } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Then

$$A^n = P\Lambda^n P^{-1} = P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1}$$

And

$$\text{a square root of } A = P \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} P^{-1}$$

Similarly for an $n \times n$ matrix.

**proof of (4)**

$$A^n = (P\Lambda P^{-1})^n = P\Lambda P^{-1} P\Lambda P^{-1} P\Lambda P^{-1} \ldots P\Lambda P^{-1} P\Lambda P^{-1} = P\Lambda^n P^{-1}$$

And by the nature of matrix multiplication, the $n$-th power of a diagonal matrix is found by raising the diagonal entries to the $n$-th power. So

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

**example 3**

Let

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

One way to find $A^{100}$ is to multiply $A$ by itself 100 times. Another way is to first diagonalize $A$.

The eigenvalues are $-3,2$ with eigenvectors $(1,-1), (3,2)$. So

$$A = P\Lambda P^{-1} \text{ where } P = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

So
\[ A^{100} = P^{100} P^{-1} \]
\[ = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-3)^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{bmatrix} \]
\[ = \begin{bmatrix} \frac{2}{5} \cdot 3^{100} + \frac{3}{5} \cdot 2^{100} & -\frac{3}{5} \cdot 3^{100} + \frac{3}{5} \cdot 2^{100} \\ -\frac{2}{5} \cdot 3^{100} + \frac{2}{5} \cdot 2^{100} & \frac{3}{5} \cdot 3^{100} + \frac{2}{5} \cdot 2^{100} \end{bmatrix} \]

And one square root of \( A \) is
\[ P \begin{bmatrix} i\sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} P^{-1} \]

Another square root of \( A \) is
\[ P \begin{bmatrix} i\sqrt{3} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} P^{-1} \]

etc.

**summary of some important ideas (you should know the answers to these questions)**

**question 1** What does it mean to say that a matrix \( A \) is diagonalizable.

**answer 1** It means that there exists a matrix \( P \) such that \( P^{-1}AP \) is diagonal.

**question 2** What does it mean to diagonalize \( A \) with a unitary matrix.

**answer 2** It means finding a unitary matrix \( P \) such that \( P^{-1}AP \) is diagonal.

**PROBLEMS FOR SECTION 8.4**

1. Diagonalize \( M \) if possible and do it with a unitary \( P \) if possible.
   
   (a) \[ \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \]
   
   (b) \[ \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \]
   
   (c) \[ \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \]
   
   (eigenvalues are 2,1,1)

2. Find a diagonal matrix \( D \) and an invertible matrix \( Q \) (if possible) so that \( Q^{-1}AQ = D \). And make \( Q \) unitary if possible.
   
   (a) \( A \) has 2-fold eigenvalue \( \lambda_1 = 2 \) with eigenvectors \( u = (1,0,0,0), v = (2,3,1,0) \) and 2-fold eigenvalue \( \lambda_2 = 3 \) with eigenvectors \( w = (0,0,0,2), x = (0,1,-3,1) \).
   
   (b) \( A \) has 2-fold eigenvalue \( \lambda_1 = 0 \) with eigenvectors \( u = (1,0,0,0), v = (2,3,1,0) \) and 2-fold eigenvalue \( \lambda_2 = 3 \) with eigenvectors \( w = (1,1,0,0), x = (0,0,0,1) \).

3. Let \( A \) be \( 3 \times 3 \). True or False.
   
   (a) If \( A \) has 3 distinct eigenvalues then \( A \) can be diagonalized.
   
   (b) If \( A \) has only 2 distinct eigenvalues then \( A \) can't be diagonalized.

4. Suppose \( A \) and \( B \) can be simultaneously diagonalized meaning that there is a matrix \( P \) so that \( P^{-1}AP = \Lambda_1 \) and \( P^{-1}BP = \Lambda_2 \) where \( \Lambda_1 \) and \( \Lambda_2 \) are diagonal.

Show that \( A \) and \( B \) commute (i.e., \( AB = BA \)).
5. Suppose $A$ is $3 \times 3$ with eigenvalues $2, 3, 4$ and corresponding eigenvectors $u = (1, 2, 3)$, $v = (5, 6, 7)$, $w = (9, 10, 11)$.
   (a) Diagonalize $A$ in several ways.
   (b) Find $|A|$.

6. Let $K$ be skew Hermitian.
   (a) Show that $iK$ is Hermitian
   (b) Show that $K$ can be diagonalized so that $P$ is unitary and $\Lambda$ is diagonal with pure imaginary entries.

7. It's possible for a lucky non-Herm to be diagonalizable with a unitary $P$.
   But show that only Herms can be diagonalized with a unitary $P$ and a real $\Lambda$.
   In other words, show that if $P^{-1}AP = \Lambda$ where $P$ is unitary and $\Lambda$ is diagonal with real entries, then $A$ is Herm.

8. Find $A^\infty$ if $A$ is $2 \times 2$ with eigenvalues $\lambda_1 = .2$ and $\lambda_2 = 1$ and corresponding eigenvectors $u = (1, 3)$, $v = (2, 4)$. 
REVIEW PROBLEMS FOR CHAPTER 8

1. Let \( M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \). Diagonalize \( M \) if possible.

2. Suppose \( A \) has eigenvalue \( \lambda \) with corresponding eigenvector \( u \).
   (a) Is \( 2\lambda \) also an eigenvalue of \( A \).
   (b) Is \( 2u \) also an eigenvector of \( A \).

3. Suppose \( AB \) has eigenvalue \( \lambda \neq 0 \) with corresponding eigenvector \( u \). Show that \( \lambda \) is also an eigenvalue of \( BA \) and find the corresponding eigenvector.

4. Suppose \( u \) is an eigenvector of \( A \) with corresponding eigenvalue 3, and \( v \) is an eigenvector of \( A \) with corresponding eigenvalue 0.
   (a) Show that \( u \) is in the range of \( A \).
   (b) \( v \) is in what well-known subspace associated with \( A \).

5. You want to make up an exam question so that the eigenvectors and eigenvalues come out to be "even". In particular you want \( M \) to be \( 2 \times 2 \) with eigenvalues 3,5 and corresponding eigenvectors (2,3) and (1,0). Make up such a matrix \( M \).

6. Suppose the eigenvalues of a \( 3 \times 3 \) matrix \( M \) are \( 2 \pm 3i, 4 \).
   Decide if \( M \) might, must, can't be (a) Hermitian (b) invertible (c) unitary

7. Let \( A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \)
   (a) Diagonalize \( A \).
   (b) Use part (a) to find \( A^8 \).

8. You already know that a noninvertible matrix always has eigenvalue 0 (from the invertible rule in §8.2).
   Confirm this again by thinking about the product of the eigenvalues.

9. Let \( A \) be \( n \times n \).
   Use the characteristic polynomial to show if \( A \) has eigenvalue 3 then \( 8A \) has eigenvalue \( 8 \cdot 3 = 24 \) (in general, multiplying a matrix by \( k \) will multiply its eigenvalues by \( k \)).

10. Suppose \( M \) is a \( 20 \times 20 \) matrix
    1 is a \( 12 \)-fold eigenvalue with 12 ind corresponding eigenvectors
    -1 is an \( 8 \)-fold eigenvalue with 8 ind corresponding eigenvectors
    Find \( M^{-1} \) (and make the answer as simple as possible).
    Suggestion: Think diagonalization.