SOLUTIONS Section 6.1

1. Part I True for \( n = 1 \) (trivially) because LHS = 1 and RHS = \( \frac{1}{2} \cdot 1 \cdot (1+1) = 1 \).

   Part II Assume true for \( n=k \); i.e., assume that for a particular \( k \),
   \[
   1 + 2 + 3 + \ldots + k = \frac{1}{2}k(k+1) \quad \text{(the induction hypothesis)}
   \]
   Want to prove true for \( n=k+1 \), i.e.; want to prove
   \[
   1 + 2 + 3 + \ldots + k+1 = \frac{1}{2}(k+1)(k+2).
   \]
   Here's the proof:
   \[
   1 + 2 + 3 + \ldots + k+1 = \underbrace{1 + 2 + 3 + \ldots + k} + k+1
   \]
   \[
   = \frac{1}{2}k(k+1) \quad \text{by ind hyp}
   \]
   \[
   = \frac{1}{2}k(k+1) + k+1
   \]
   \[
   = (k+1)\left(\frac{1}{2}k + 1\right) \quad \text{factor}
   \]
   \[
   = \frac{(k+1)(k+2)}{2} \quad \text{QED}
   \]

2. Part I Trivially true for \( n = 1 \) since LHS = \( 1^2 = 1 \) and RHS = \( \frac{1 \cdot 2 \cdot 3}{6} = 1 \).

   Part II Assume it's true for the particular value \( n=k \); i.e., assume that for a particular \( k \),
   \[
   1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(induction hypothesis)}
   \]
   Want to prove true for \( n=k+1 \); i.e., want to prove
   \[
   1^2 + 2^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}
   \]
   Here's the proof:
   \[
   1^2 + 2^2 + \ldots + (k+1)^2 = 1^2 + 2^2 + \ldots + k^2 + (k+1)^2
   \]
   \[
   = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by induction hypothesis}
   \]
   \[
   = (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \quad \text{(factor)}
   \]
   \[
   = (k+1)\frac{2k^2 + 7k + 6}{6} \quad \text{(algebra)}
   \]
   \[
   = \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{(factor)}
   \]
   This proves part II.

3. Part I Obviously true for \( n = 1 \) since LHS = \( 1^3 = 1 \) and RHS = \( \left(\frac{1 \cdot 2}{2}\right)^2 = 1 \).

   Part II Assume true for \( n=k \); i.e., assume that for a particular \( k \),
   \[
   1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \quad \text{(the induction hypothesis)}
   \]
   Want to prove that
   \[
   1^3 + 2^3 + \ldots + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2
   \]
proof: 
\[1^3 + 2^3 + \ldots + (k+1)^3 = 1^3 + 2^3 + \ldots + k^3 + (k+1)^3\]
\[= \left\lfloor \frac{k(k+1)}{2} \right\rfloor^2 + (k+1)^3 \quad \text{(by ind hyp)}\]
\[= (k+1)^2 \left(\frac{k^2}{4} + (k+1)\right) \quad \text{(factor)}\]
\[= (k+1)^2 \frac{k^2 + 4k + 4}{4} \quad \text{(algebra)}\]
\[= (k+1)^2 \frac{(k+1)^2}{4} \quad \text{(algebra)}\]
\[= \left\lfloor \frac{(k+1)(k+2)}{2} \right\rfloor^2 \]

4. Part I True for \( n = 1 \) since \( 11^1 - 4^1 = 7 \) which is divisible by 7.
Part II Assume \( 11^k - 4^k \) is divisible by 7. 
Try to show that \( 11^{k+1} - 4^{k+1} \) is divisible by 7.

Do some algebra to express \( 11^{k+1} - 4^{k+1} \) in terms of \( 11^k - 4^k \) so that you can use the induction hypothesis.

\[11^{k+1} - 4^{k+1} = 11 \cdot 11^k - 4 \cdot 4^k\]
\[= 11(11^k - 4^k) + 11 \cdot 4^k - 4 \cdot 4^k\]
\[= 11(11^k - 4^k) + 7 \cdot 4^k\]
\[\text{div by 7} \quad \text{div}\]
\[\text{by ind hyp} \quad \text{by 7}\]

So \( 11^{k+1} - 4^{k+1} \) is a sum of things divisible by 7 so it is divisible by 7. QED

5. Part I True for \( n = 1 \) because \( 2^2 - 1 = 2^2 - 1 = 3 \) which is div by 3.
Part II Assume \( 2^{2k} - 1 \) is divisible by 3 for some particular \( k \).
Want to show that \( 2^{2(k+1)} - 1 \) is div by 3, i.e., that \( 2^{2k+2} - 1 \) is div by 3.
Do some algebra:

\[2^{2k+2} - 1 = 2^2 \cdot 2^{2k} - 1 = 4 \cdot 2^{2k} - 1 = 4 \cdot 2^{2k} - 1 + 4 - 1 = 4 \cdot 2^{2k} - 1 + 3\]
\[\text{div by 3} \quad \text{div}\]
\[\text{by ind hyp} \quad \text{by 3}\]

So \( 2^{2k+2} - 1 \) is div by 3. This proves Part II.

6. Part I True for \( n = 35 \) since you can pay with seven 5¢ stamps.
Part II Assume true if the postage is \( n=k \) (the induction hypothesis).
Try to prove true for postage \( n=k+1 \).

case 1 Suppose postage \( k \) can be paid entirely with 5¢ stamps. It takes at least 7 of them since \( k \geq 35 \). To pay for postage \( k+1 \), replace seven of the 5¢ stamps by four 9¢ stamps. This pays for postage \( k+1 \) using only 5's and 9's.

case 2 Suppose postage \( k \) is paid using at least one 9¢ stamp. To pay for postage \( k+1 \) in this case, replace a 9¢ stamp with two 5's. This pays for postage \( k+1 \).

That proves Part II.
7. **Part I**  True for \( n = 1 \) since \( 2 > 1 \).

**Part II**  Assume \( 2^k > k \) for a particular \( k \).  Want to show that \( 2^{k+1} > k+1 \).

**proof:**

\[
2^{k+1} = 2 \cdot 2^k > 2 \cdot k \quad \text{(since } 2^k > k \text{ by ind hypothesis)}
\]

\[
= k + k \geq k + 1 \quad \text{(since } k \geq 1 \text{)}
\]

8. (a) **Part I**  If \( n = 1 \) then \( n^2 + 5n + 1 \) is 7 which is odd.

**Part II**  Assume that \( k^2 + 5k + 1 \) is odd (the induction hypothesis).

Want to show that \((k+1)^2 + 5(k+1) + 1 \) is odd.

**proof**

\[
(k+1)^2 + 5(k+1) + 1 = k^2 + 2k + 1 + 5k + 5 + 1 \quad \text{(multiply out)}
\]

\[
= k^2 + 5k + 1 + 2k + 6 \quad \text{(rearrange)}
\]

\[
= k^2 + 5k + 1 + 2(k + 3) \quad \text{odd by ind hyp even because of the 2}
\]

\[
= \text{odd} + \text{even} = \text{even} \quad \text{QED}
\]

(b) **Part II**  Assume \( k^2 + 5k + 1 \) is even.

Want to show that \((k+1)^2 + 5(k+1) + 1 \) is even.

**proof**

\[
(k+1)^2 + 5(k+1) + 1 = k^2 + 2k + 1 + 5k + 5 + 1 \quad \text{(rearrange)}
\]

\[
= k^2 + 5k + 1 + 2k + 6 \quad \text{even by ind hyp even because of the 2}
\]

\[
= \text{even} + \text{even} = \text{even} \quad \text{QED}
\]

So you can do Part II of the induction argument.

But you can't do any version of Part I (can't show \( n^2 + 5n + 1 \) is even for \( n = 1 \) or any "starting" value of \( n \) because by part (a) it is always odd).

9. **Part I**  Obviously true for \( n = 1 \).

**Part II**  Assume true for \( n = k \); i.e.; assume that

\[
(cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta
\]

Try to prove true for \( n = k+1 \); i.e., want to prove that

\[
(cos \theta + i \sin \theta)^{k+1} = \cos(k+1)\theta + i \sin(k+1)\theta
\]

**proof:**

\[
(cos \theta + i \sin \theta)^{k+1}
\]

\[
= (cos \theta + i \sin \theta)^k \cdot (cos \theta + i \sin \theta) \quad \text{(algebra)}
\]

\[
= (cos k\theta + i \sin k\theta) \cdot (cos \theta + i \sin \theta) \quad \text{(by ind hypothesis)}
\]

\[
= \cos k\theta \cos \theta - \sin k\theta \sin \theta + i (\cos \theta \sin k\theta + \sin \theta \cos k\theta) \quad \text{(algebra)}
\]

\[
= \cos (k\theta + \theta) + i \sin (\theta + k\theta) \quad \text{(trig identities)}
\]

\[
= \cos(k+1)\theta + i \sin(k+1)\theta \quad \text{(factor)}
\]
10. **Part I** True for $n = 1$ since LHS $= \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$ and RHS $= 2^1 = 2$.

**Part II** Assume true for $n=k$; i.e., assume $\binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k} = 2^k$.

Want to prove true for $n=k+1$; i.e., show that

$$\binom{k+1}{0} + \binom{k+1}{1} + \ldots + \binom{k+1}{k+1} = 2^{k+1}$$

Here's the proof (which uses Pascal's identity over and over again):

LHS of (**) 
\[
= \binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \binom{k+1}{3} + \ldots + \binom{k+1}{k} + \binom{k+1}{k+1}
\]

same as 
\[
= \binom{k}{0} \binom{k}{1} \binom{k}{2} \binom{k}{3} \ldots \binom{k}{k} \binom{k}{k+1}
\]

by Pascal

The boxed terms add up to $2^k$ by the induction hypothesis.
Similarly the unboxed terms add up to $2^k$ too.
So the LHS of (**) is $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$. QED
1. **Part I** Show true for \( n = 1 \).

\[
\left( \frac{1 + \sqrt{5}}{2} \right)^0 = 1, \quad F_1 = 1. \quad \text{So} \quad F_1 \leq \left( \frac{1 + \sqrt{5}}{2} \right)^0.
\]

And show true for \( n = 2 \).

\[
\left( \frac{1 + \sqrt{5}}{2} \right)^1 \geq 1 \quad \text{by inspection (because } \sqrt{5} \geq 1) \quad = 1
\]

and \( F_2 = 1 \) so \( F_2 \leq \left( \frac{1 + \sqrt{5}}{2} \right)^1 \).

**Part II** Assume that (*) is true for two particular consecutive values \( n = k-1 \) and \( n = k \); i.e., assume

\[
F_{k-1} \leq \left( \frac{1 + \sqrt{5}}{2} \right)^{k-2} \quad \text{and} \quad F_k \leq \left( \frac{1 + \sqrt{5}}{2} \right)^{k-1} \quad \text{(the induction hypothesis)}
\]

Try to prove (*) for \( n = k+1 \); i.e., prove that \( F_{k+1} \leq \left( \frac{1 + \sqrt{5}}{2} \right)^k \).

**Proof**

\[
F_{k+1} = F_k + F_{k-1} \quad \text{(by definition)}
\]

\[
\leq \left[ \frac{1 + \sqrt{5}}{2} \right]^{k-1} + \left[ \frac{1 + \sqrt{5}}{2} \right]^{k-2} \quad \text{(by the induction hypothesis)}
\]

\[
= \left[ \frac{1 + \sqrt{5}}{2} \right]^{k-2} \left[ \frac{1 + \sqrt{5}}{2} + 1 \right] \quad \text{(factor)}
\]

\[
= \left[ \frac{1 + \sqrt{5}}{2} \right]^{k-2} \left[ \frac{1 + \sqrt{5}}{2} \right]^2 \quad \text{(by the given identity)}
\]

\[
= \left[ \frac{1 + \sqrt{5}}{2} \right]^k \quad \text{QED (end of part II)}
\]

2. (a) **Part I** True for \( n = 2 \), because 2 is prime.

**Part II** Assume true for \( n = k \); i.e., assume \( k \) is either prime or factors into primes.

Try to prove true for \( n = k+1 \); i.e., either \( k+1 \) is prime or factors into primes.

**Proof:** If \( k+1 \) is prime you're finished.

Suppose \( k+1 \) isn't prime. You want to show that it factors into primes.

Since \( k+1 \) isn't prime it factors into \( pq \) where \( 2 \leq p \leq k \) and \( 2 \leq q \leq k \).

But you're not finished because \( p \) and \( q \) aren't necessarily primes. The induction hypothesis says that \( k \) is prime or factors into primes but you don't know about \( p \) and \( q \). So you're stuck.

(b) **Part I** Same as in (a).

**Part II** Assume true for 2,...,\( k \); i.e., assume every number from 2 through \( k \) is either prime or factors into primes. This is the strong induction hypothesis.

Try to prove true for \( n = k+1 \); i.e., try to show that \( k+1 \) is either prime or factors into primes.

As above, if \( k+1 \) is prime you're finished.

So consider the case that \( k+1 \) isn't prime. Then it factors into \( pq \) where \( 2 \leq p \leq k \) and \( 2 \leq q \leq k \). By the strong induction hypothesis \( p \) and \( q \) are themselves prime or else factor into primes. So \( k+1 \) factors into primes which proves part II.

Then, by the principle of strong induction, every integer \( \geq 2 \) is either prime or can be factored into a product of primes.
3. (a) Show that \( V = E + 1 \) for a tree with 1 edge.
A tree with 1 edge has to look like this.

\[ \begin{align*}
\text{In this case } V &= 2 \text{ so } V \text{ does equal } E + 1.
\end{align*} \]

Actually you could start by showing that it is true for a tree with zero edges.
If \( E = 0 \) then the tree consists of just a single vertex. Then \( E = 0, V = 1 \) and
\( V \) does equal \( E + 1 \).

(b) Start with a tree with \( k+1 \) edges. Delete an "outer" edge and the dangling vertex
at its end like this:

\[ \begin{align*}
\text{new graph}
\end{align*} \]

The new graph is still a tree but it has one less edge, i.e., \( E_{\text{new}} = k \).
So by the induction hypothesis, in the new tree

\[ \begin{align*}
(*) \quad V_{\text{new}} &= E_{\text{new}} + 1.
\end{align*} \]

The original tree and the new tree are related like this:

\[ \begin{align*}
V_{\text{new}} &= V_{\text{orig}} - 1 \quad \text{and} \quad E_{\text{new}} = E_{\text{orig}} - 1.
\end{align*} \]

Substitute these into (*):

\[ \begin{align*}
V_{\text{orig}} - 1 &= E_{\text{orig}} - 1 + 1
\end{align*} \]

And cancel the \(-1\)'s to get

\[ \begin{align*}
V_{\text{orig}} &= E_{\text{orig}} + 1 \quad \text{QED}
\end{align*} \]

(c) Start with a tree with \( E = k+1 \) edges and \( V \) vertices and delete an edge (no
vertices are deleted). You get two new trees; call them Tree 1 and Tree 2

Let the number of vertices and edges in Tree 1 be denoted by \( V_1 \) and \( E_1 \).
Let the number of vertices and edges in Tree 2 be denoted by \( V_2 \) and \( E_2 \).
Each of Trees 1 and 2 has \( k \) or fewer edges (since we deleted one of the original \( k+1 \)
edges and split the rest into two trees) so you can use the strong induction
hypothesis on each of Tree 1 and Tree 2 (if your induction hypothesis had just been
that a tree with \( k \) edges has \( V = E + 1 \) then you would be stuck here):

\[
\begin{align*}
V_1 &= E_1 + 1 \\
V_2 &= E_2 + 1 \\
\end{align*}
\]

Add to get

\[(**)
V_1 + V_2 = E_1 + E_2 + 2
\]

Here's how the new trees are related to the original tree with \( V \) vertices and \( E \) edges:

\[ (*) \quad V_1 + V_2 = V \quad \text{and} \quad E_1 + E_2 = E - 1 \quad \text{(remember you lost an edge)} \]

Substitute (*) into (**):

\[
\frac{V_1 + V_2}{V} = \frac{E_1 + E_2}{E-1} + 2
\]

which becomes

\[ V = E + 1 \quad \text{QED} \]
1. (a) Antireflexive because no one is her own descendent.
   Antisymmetric because if John is a descendent of Mary then Mary is certainly not a descendent of John
   Transitive (If John is a desc of Mary and Mary is a desc of Harry then John is a desc of Harry)
   (b) Reflexive, symmetric, transitive
   (c) Symmetric
      Not reflexive because 3 ≺ 3 for instance.
      Not transitive because 3 R 4 and 4 R 5 but 3 R 5 (3 · 4 is even and 4 · 5 is even but 3 · 5 is not even)
   (d) Symmetric
      Not reflexive because 7 ≺ 7 for instance.
      Not transitive because 6 R 3 and 3 R 5 but 6 R 5
      (6 + 3 < 10 and 3 + 5 < 10 but 6 + 5 is not < 10).
   (e) Antireflexive, antisymmetric, trans
   (f) Antireflexive, symmetric.
      Not trans because if L1 and L2 are two different parallel lines then L1 R L2 and L2 R L1 but L1 R L1.
   (g) Reflexive, symmetric, trans
   (h) Antireflexive, symmetric
      Not trans because if x ⊥ y and y ⊥ z then x and z are parallel or coincident, not perp.

2. Antireflexive, antisymmetric
   Not trans because rock R scissors and scissors R paper but rock not R paper

3. (a) Not reflexive Missing (2,2) (among others)
   Not antireflexive Does contain (1,1)
   Not symmetric Missing (1,2)
   Antisymmetric (2,1) is the only pair you care about, and it does not reverse.
   Trans The only hookup is 2 R 1 and 1 R 1 so you need 2 R 1 for transitivity and you have it
   (b) Not reflexive. Missing (2,2)
   Not antireflexive. Contains (1,1)
   Symmetric Antisymmetric by default There are no pairs (x,y) where x ≠ y to consider reversing
   (c) Not reflexive. Missing (2,2)
   Antireflexive
   Not symmetric Missing (3,2) for instance
   Antisymmetric
   Not transitive. Has (4,2) and (2,3) but not (4,3)

4. There are 4 pairs in A × A, namely (a,a), (b,b), (a,b), (b,a). And a set with 4 elements has 2^4 subsets. So there are 16 relations on A.

   Equivalently use (a,a), (b,b), (a,b), (b,a) as slots and fill each slot with either YES (in the relation) or NO (not in the relation. So there are 2^4 possibilities.
   Here they are.
   R_1 = φ (the empty relation where no one is related to anyone else)
   Antireflexive, symmetric, antisymmetric, trans (the last three, by default)

   R_2 = {(a,a)} symmetric, antisymmetric, trans
   R_3 = {(b,b)} same properties as R_2
   R_4 = {(a,a), (b,b)} reflexive, symmetric, antisymmetric, trans
   R_5 = {(a,b)} antireflexive, antisymmetric, trans
   R_6 = {(b,a)} same properties as as R_5
R₇ = { (a,b), (b,a) } symm, antiref
   Not trans because you have the hookup aRb, bRa but don't have aRa

R₈ = { (a,a), (a,b) } antisymm, trans
   Trans because the only hookup is aRa, aRb and you do also have aRb

R₉ = { (a,a), (b,a) } Same properties as R₈
R₁₀ = { (b,b), (a,b) } antisymm, trans
   Trans because the only hookup is aRb, bRb and you do have aRb

R₁₁ = { (b,b), (b,a) } Same as R₁₀
R₁₂ = { (a,a), (b,b), (a,b) } reflex, antisymm, trans
   Here's why R₁₂ is trans: There are two hookups to check.
   The first one is aRb, bRb and you do have aRb
   The second one is aRa, aRb and you do have aRb

R₁₃ = { (a,a), (b,b), (b,a) } reflex, antisymm, trans (just like R₁₂)

R₁₄ = { (a,a), (a,b), (b,a) } symm
   Not trans because you have the hookup bRa, aRb but not bRb

R₁₅ = { (b,b), (a,b), (b,a) } Same as R₁₄

R₁₆ = A × A = { (a,a), (b,b), (a,b), (b,a) } ref, symm, trans

5. (a) There are 5 · 5 = 25 ordered pairs of elements in U. Think of each pair as a
   slot which can be filled in two ways: Related or Unrelated (i.e., in R or not in R).
   So there are 2²⁵ relations.
   Equivalently, there are 5 · 5 = 25 pairs in A × A. There are 2²⁵ subsets of A × A so
   there are 2²²⁵ relations.
   (b) The relation must contain the five pairs (x₁, x₁),..., (x₅, x₅). The other 20
   pairs are slots that can be filled in two ways: In the relation R or not in. Answer
   is 2²⁰.
   (c) The relation must contain the six pairs (x₁, x₁),..., (x₅, x₅), (x₂, x₄). The
   other 19 pairs are slots that can be filled in two ways: Related or Unrelated.
   Answer is 2¹⁹.
   (d) There are 5 pairs of the form (xᵢ,xᵢ). Each is a slot which can be filled in two
   ways: YES or NO (in the relation vs. out of the relation) (you have this choice
   because a symmetric relation doesn't care whether or not xᵢRxᵢ).
   There are 5 · 4 pairs of the form (xᵢ,xⱼ) where i ≠ j.
   Imagine them paired up so that you have 10 pairs of matching pairs, e.g.,
   (x₁, x₃) & (x₃, x₁), (x₃, x₅) & (x₅, x₃) etc.
   For each of these 10 pairs of pairs you have two choices: BOTH IN or BOTH OUT;
   e.g., (x₁, x₃) and (x₃, x₁) are either both in the relation or both out.
   Answer is 2⁵ · 2¹⁰
   (e) This is like part (d) but now all 5 pairs of the form (xᵢ,xᵢ) must be included
   in the relation. Less choice than in part (d). The answer is 2¹⁰.
6. (a) It's really a plain graph in the sense that each edge is two-way.
(b) There's a loop at each vertex.
(c) If there is a path from \(v\) to \(w\) then there is also an edge directly from \(v\) to \(w\) (e.g., if \(v \rightarrow p, p \rightarrow q, q \rightarrow z, z \rightarrow w\) then \(v \rightarrow w\))

7. (a) Yes
Here is a relation defined on the universe of integers that is antisymmetric and also symmetric: \(x \sim y\) if \(x = y\) (where \(x\) and \(y\) are integers)
For instance \(3 \sim 3, 4 \sim 4, 5 \sim 5\) etc.
It's symmetric because if \(x \sim y\) (meaning \(x = y\)) then \(y \sim x\).
And it's antisymmetric by default since you never have \(x \sim y\) where \(x \neq y\) to begin with.
Alternatively, it is antisymmetric because if \(x \neq y\) then we have neither \(x \sim y\) nor \(y \sim x\).

Another example. Suppose the universe is \(\{a, b, c\}\).
And the relation is \(\{(a, a), (b, b)\}\), i.e., \(a \sim a\) and \(b \sim b\).
Then it is symmetric (each pairing reverses).
And also antisymmetric because for two different \(x\) and \(y\), it is true that either \(x \sim y\) or \(y \sim x\) or neither but not both. In particular, here you always have \text{NEITHER}.

(b) No. If the universe \(U\) contains \(x\) then a reflexive relation on \(U\) must have \(x \sim x\) and an antireflexive relation must \text{not} have \(x \sim x\) so a relation can't do both (unless the universe is the empty set to begin with but that would be a trick question).

(c) Yes
For example let the universe be \(\{a, b, c\}\) and let the relation be \(\{(a, b), (b, a), (a, c)\}\).
The relation is not symmetric because \(c, a\) is missing.
And it is not antisymmetric because both \((a, b)\) and \((b, a)\) are in the relation.

For example, suppose the universe is the set of all people and \(x \sim y\) means \(x\) likes \(y\).
Somewhere in the world there are pairs where the liking goes just one way, e.g.,
where John likes Mary but Mary doesn't like John. So the relation is not symmetric.
And somewhere in the world there are pairs where the liking goes both ways (e.g.,
Mary likes Sue and Sue likes Mary). So the relation is not antisymmetric.

(d) Yes. For example if the universe is \(\{1, 2, 3, 4\}\) and the relation is \(\{(2,2), (3,4)\}\) then it is not reflexive because \(R\) does not contain \((3,3)\).
And it not antireflexive because it does contain \((2,2)\).
For example if \(x \sim y\) means that \(x\) has cooked a meal for \(y\) then \(R\) is not reflexive (there are some people who have never cooked meals for themselves) and it is not antireflexive (some people \textit{have} cooked for themselves).
SOLUTIONS Section 7.2

1. (a) Yes. There is an equivalence class for each name. For instance there is a John Smith equiv class containing all the people named John Smith; there's a Mary Brown equiv class containing the people named Mary Brown etc.

(b) Yes. There is a chemistry equiv class containing all the chem majors etc. There are as many equiv classes as there are subjects chosen as majors.

(c) No. It's reflexive and symm but not transitive. If John has class with Mary and Mary has a class with Bill then John does not necessarily have a class with Bill.

(d) Yes. There are 26²⁻⁶ equivalence classes, one for people initialed A.A., another for people initialed A.B.,..., another for people initialed Z.Z. (unless there is a set of initials like X.Z. which no one has in which case there is no equivalence class named X.Z. --- it is not the case that there is a class but it's empty).

(e) Yes. There are 50 equiv classes, one containing all the Florida residents, one containing all the Illinois residents etc.

(f) No. It's reflexive and symmetric but not transitive.

(g) Yes. Each equivalence class contains exactly one person since no two people have the same SS number. There are as many classes as there are SS numbers.

(h) Yes. Each equiv class contains exactly one integer and there are as many equiv classes as there are integers.

(i) Yes. There are 26⁴ equiv classes, one containing words of the form abcz..., one containing words of the form aaaa..., etc.

(j) Yes. Two points are equiv if they are the same height above the x-axis. The equiv classes are the lines parallel to the x-axis. For example, the equiv class of the point (3,4) is the line y = 4.

(k) Yes. The traditional equivalence classes are married, divorced, widowed, never married.

(0) No. Not reflexive (you aren't married to yourself) and it isn't transitive (if A is married to B and B is married to A then it isn't true that A is married to A, and if Mary is married to John-the-Bigamist and John is married to Susan then Mary is not married to Susan).

2. (a) R is an equivalence relation because it's symmetric, reflexive and transitive:

**Reflexive** n R n for all n because sin nπ = sin nπ

**Symmetric** If n R m then m R n because if sin nπ = sin mπ then sin mπ = sin nπ

**Transitive** If n R m and m R k then n R k because if sin nπ = sin mπ and sin mπ = sin kπ then sin nπ = sin kπ.

There is only one equivalence class, namely {0, 1, 2, 3, 4, 5, ...} because sin nπ = sin mπ (= 0) for all integers n, m.

(b) R is an equivalence relation. There are two equivalence classes, {0, 2, 4, 6, ...} and {1, 3, 5, 7, ...} because cos 0 = cos 2π = cos 4π = cos 6π etc (they all equal 1) and cos π = cos 3π = cos 5π etc (they all equal -1).

3. There are 5 equiv relations

For instance R1 = { (a,a), (b,b), (c,c), (b,c), (c,b) }

i.e., a R1 a, b R1 b, c R1 c, b R1 c, c R1 b.

R4 = { (a,a), (b,b), (c,c) }, i.e., a R4 a, b R4 b, c R4 c.
4. If two equivalence classes did overlap as in the diagram below then you would have $a \sim b$ (because $a$ and $b$ are in the same cell), $b \sim c$ (because $b$ and $c$ are in the same cell) but $a \not\sim c$ (because $a$ and $c$ are not in the same cell). But this contradicts transitivity. So overlapping equivalence classes can't happen.

5. (a) Must prove that $R$ is symmetric and transitive (already have reflexive).

**Symmetry**

Suppose $x R y$. We want to show that $y R x$.

We have $x R x$ by reflexivity.

But now that we have $x R x$ and $x R y$ we also have $y R x$ by circularity, QED

**Transitivity**

Suppose $x R y$ and $y R z$. We want to show that $x R z$.

From $x R y$ and $y R z$ we have $z R x$ by circularity.
Then $x R z$ by the symmetry that we just proved. QED

(b) Suppose $R$ is an equivalence relation. We want to show that $R$ is circular.

Suppose $a R b$ and $b R c$. We want to show that $c R a$.
We have $a R c$ by transitivity and then $c R a$ by symmetry. QED