1. (a) Look at strings of 1's and 2's. Let \( y_n \) be the number of strings whose sum is \( n \) (call them good \( n \)-strings).

Each good \( n \)-string begins with 1 or 2.

So each good \( n \)-string (where \( n \geq 3 \)) is one of the following two types:

(i) 1 followed by a string with sum \( n-1 \), i.e., a good \( (n-1) \)-string
(ii) 2 followed by a string with sum \( n-2 \), i.e., a good \( (n-2) \)-string

There are \( y_{n-1} \) strings of type (i).

There are \( y_{n-2} \) strings of type (ii).

So
\[
y_n = y_{n-1} + y_{n-2} \quad \text{for } n \geq 3
\]

This is a second order recursion so I need two IC:
\[
y_1 = 1 \quad (\text{only the string 1 is OK})
\]
\[
y_2 = 2 \quad (\text{the strings 2, 11 are OK})
\]

(b) \( y_3 = y_2 + y_1 = 2 + 1 = 3 \)
\( y_4 = y_3 + y_2 = 3 + 2 = 5 \)
\( y_5 = y_4 + y_3 = 5 + 3 = 8 \)
\( y_6 = y_5 + y_4 = 8 + 5 = 13 \)

There are 13 ways to climb a staircase with 6 steps.
(Check: The thirteen strings are 121, 2112, 2211, 1122, 1212, 2121, 11112, 11121, 11211, 12111, 21111, 111111, 222)

2. Let \( u_n \) be the number of \( n \)-words with an even number of A's (good \( n \)-words).

\textbf{Warning} Don't leave this line out. It doesn't make sense to have a recursion relation involving \( u_n \) unless you state at the beginning what \( u_n \) represents.

The good \( n \)-words (with \( n \geq 2 \)) split into these types:

(a) A followed by a good \( (n-1) \)-word
(b) A followed by a bad \( (n-1) \)-word, i.e., an \( (n-1) \)-word with an odd number of A's

There are \( 25u_{n-1} \) words of type (a) (25 possibilities for the non-A and \( u_{n-1} \) possibilities for the good \( (n-1) \)-word).

The number of words of type (b)
\[
\text{= total number of (n-1)-words - number of good (n-1)-words}
\]
\[
\text{= } 26^{n-1} - u_{n-1}
\]

So \( u_n = 25u_{n-1} + 26^{n-1} - u_{n-1} \), i.e.,
\( u_n = 26^{n-1} + 24u_{n-1} \) for \( n \geq 2 \).

This is first order recursion. Need one IC.
The IC is \( u_1 = 25 \) (each of the 1-words B,C,...,Z has an even number of A's).

3. (a) There are \( 10 \cdot u_{n-1} \) strings of type (a) (D can be picked in any of 10 ways and the legal \( (n-2) \)-string in any of \( u_{n-1} \) ways).

There are \( 10 \cdot 3 \cdot u_{n-2} \) strings of type (b).

So all in all,
\[
u_n = 10u_{n-1} + 30u_{n-2} \quad \text{for } n \geq 3.
\]

This is a second order recursion so I need two IC.
4. Let $u_n$ be the number of messages of length $n$.
A message of length $n$ begins with either $a$ or $ab$ or $bc$.
So a message of length $n$ (where $n \geq 3$) must be one of the following types:
(i) $a$ followed by a message of length $n-1$
(ii) $ab$ followed by a message of length $n-2$
(iii) $bc$ followed by a message of length $n-2$
So
$$u_n = 10u_{n-1} + 30u_{n-2}$$
for $n \geq 3$

This is a second order recurrence. The two IC are
$$u_1 = 10, \quad u_2 = 100$$

(b) $u_3 = 10u_2 + 30u_1 = 1300$
$$u_4 = 10u_3 + 30u_2 = 16,000$$

5. The tiling of an $n$-board must begin, say at the left end of the board, with

or

or

So for $n \geq 3$, each tiling must be one of the following three types:
(i) plus a tiling for an $(n-2)$-board
(ii) plus a tiling for an $(n-1)$-board
(iii) plus a tiling for an $(n-2)$-board

The number of tilings of type (i) is $y_{n-2}$
The number of tilings of type (ii) is $y_{n-1}$
The number of tilings of type (iii) is $y_{n-2}$
So
$$y_n = 2y_{n-2} + y_{n-1}$$
for $n \geq 3$

This is second order recurrence. The two IC are
$$y_1 = 1, \quad y_2 = 3$$
(see the diagram below)

one way to tile a 1-board
3 ways to tile a 2-board
6. Let \( a_n \) be the number of \( n \)-symbol sentences.
For \( n \geq 3 \), a sentence must begin with two letters or with a letter followed by a blank.
So every sentence of length \( n \), \( n \geq 3 \), must be one of these two types:
(a) letter followed by \# followed by an \((n-2)\)-symbol sentence
(b) letter followed by an \((n-1)\)-symbol sentence

For example, the 5-symbol sentence Q#CAT is Q# followed by the 3-symbol sentence CAT so it's type (a). The 7-symbol sentence THE#CAT is T followed by the 6-symbol sentence HE#CAT so it's type (b).

There are \( 26a_{n-2} \) sentences of type (a)
There are \( 26a_{n-1} \) sentences of type (b)
So
\[
a_n = 26a_{n-1} + 26a_{n-2} \quad \text{for } n \geq 3
\]
This is second order rr. The two IC are
\[
a_1 = 26 \quad \text{(any letter is a 1-symbol sentence)}
\]
\[
a_2 = 26 \cdot 26 \quad \text{(any string of 2 letters is OK)}
\]

7. Each good \( n \)-string must be one of these two types:
(a) nonzero followed by a good \((n-1)\)-string
(b) 0 followed by any string of 0's, 1's, 2's of length \( n-1 \)

For example, the good 6-string 020012 is 0 followed by the string 20012 so it's type (b). The good 7-string 3012202 is 3 followed by the good 6-string 012202 so it's type (a).
There are \( 3a_{n-1} \) strings of type (a) (the nonzero can be picked in 3 ways and the good \((n-1)\)-string in \( a_{n-1} \) ways).
There are \( 3^{n-1} \) strings of type (b) (each spot in the \((n-1)\)-string can be picked in 3 ways).
So
\[
a_n = 3a_{n-1} + 3^{n-1} \quad \text{for } n \geq 2 \quad \text{(a first order rr)}
\]
The IC is
\[
a_1 = 4 \quad \text{(there are four good strings of length 1, namely 0,1,2,3)}
\]
Then
\[
a_2 = 3a_1 + 3^1 = 12 + 3 = 15
\]
(check: The 15 strings are 00, 01, 02, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33)
\[
a_3 = 3a_2 + 3^2 = 45 + 9 = 54
\]
\[
a_4 = 3a_3 + 3^3 = 162 + 27 = 189
\]

8. (a) The rr is \( b_n = 2^{n-2} + 2b_{n-1} \) for \( n \geq 3 \). But it's wrong because the two categories overlap. The string 00100 is in both categories.
(b) The rr is \( b_n = 2^{n-2} + b_{n-1} \) for \( n \geq 3 \). But it's wrong because the two categories don't include all good \((n-1)\)-strings. The good string 0100 is in neither category.
(c) Every good string must begin with either 1 or 01 or 00.
So every good \( n \)-string must be one of these three types
(i) 1 followed by a good \((n-1)\)-string
(ii) 01 followed by a good \((n-2)\)-string
(iii) 00 followed by any \((n-2)\)-string
There are \( b_{n-1} \) strings of type (i).
There are \( b_{n-2} \) strings of type (ii).
There are \( 2^{n-2} \) strings of type (iii) (each of the \( n-2 \) spots can be filled in 2 ways).
So
\[
b_n = b_{n-1} + b_{n-2} + 2^{n-2} \text{ for } n \geq 3
\]
This is a second order rr. The two IC are
\[
b_1 = 0 \text{ (can't have consecutive 0's if you have only one digit)}
b_2 = 1 \text{ (the only good string is 00)}
\]
So
\[
b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3
\]
\[
b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8
\]
\[
b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19
\]
9. (a) There are \( n \) spots to fill.
First can be filled in any of 4 ways.
Second can be filled in any of 3 ways (any letter except the one in the 1st spot)
Third can be filled in any of 3 ways (any letter except the one in the 2nd spot)
etc.
Answer is \( y_n = 4 \cdot 3^{n-1} \)

(b) \textit{method 1}
Each good \( n \)-string begins with A or B or C or D. So it must be one of the following four types.
(i) A followed by a good \( (n-1) \)-string beginning with \( \overline{A} \)
(ii) B followed by a good \( (n-1) \)-string beginning with \( \overline{B} \)
(iii) C followed by a good \( (n-1) \)-string beginning with \( \overline{C} \)
(iv) D followed by a good \( (n-1) \)-string beginning with \( \overline{D} \)
By symmetry, of all the good \( (n-1) \)-strings, \( \frac{1}{4} \) begin with A, \( \frac{1}{4} \) begin with B, \( \frac{1}{4} \) begin with C and \( \frac{1}{4} \) begin with D.
So the number of strings of type (i) is \( \frac{3}{4} y_{n-1} \) and similarly for (ii)-(iv).
So
\[
y_n = 4 \cdot \frac{3}{4} y_{n-1}
y_n = 3 y_{n-1} \text{ for } n \geq 2
\]
This is a first order rr. The IC is
\[
y_1 = 4 \text{ (the good strings A,B,C,D)}
\]

\textit{method 2}
Every good \( n \)-string begins with a good \( (n-1) \) string followed by one of the three letters that the good \( (n-1) \) string didn't end with.
That makes \( y_n = 3y_{n-1} \)

(c) Part (a) says
\[
y_n = 4 \cdot 3^{n-1}
\]
\[
3y_{n-1} = 3 \cdot 4 \cdot 3^{n-2} = 4 \cdot 3^{n-1}
\]
So \( y_n \) does equal \( 3y_{n-1} \). And \( y_1 = 4 \cdot 3^0 = 4 \). So the formula in (a) satisfies the rr and IC from (b).
10. A good subset of \( S_n \) either contains the last number \( n \) or it doesn't contain \( n \).
(The same argument works if you focus on any one particular number. It doesn't have
 to be the last one, \( n \)).

So a good subset must be one of the following types:

(i) It contains \( n \) in which case it can't contain \( n-1 \) so the rest of it is a
  good subset of \( \{1, \ldots, n-2\} \).
(ii) It doesn't contain \( n \) in which case it must be a good subset of \( \{1, \ldots, n-1\} \).

For example, look at \( S_7 = \{1, 2, \ldots, 7\} \).

Some good subsets of type (i) are \( \{1, 7\}, \{7\}, \{2, 4, 7\} \).
Some good subsets of type (ii) are \( \{2, 5\} \) and \( \{1, 4, 6\} \).

There are \( y_{n-2} \) subsets of type (i).
There are \( y_{n-1} \) subsets of type (ii).

So \( y_n = y_{n-1} + y_{n-2} \) for \( n \geq 3 \).

This is a second order \( \text{rr} \) so I need two IC>

\( S_1 = \{1\} \). It has good subsets \( \emptyset \) and \( \{1\} \).

\( S_2 = \{1, 2\} \). It has good subsets \( \emptyset \), \( \{1\} \), \( \{2\} \).

So the IC are \( y_1 = 2 \), \( y_2 = 3 \)

Then \( y_3 = 5 \), \( y_4 = 8 \), \( y_5 = 13 \), \( y_6 = 21 \), \( y_7 = 34 \), \( y_8 = 55 \).

11. money after \( n \) months = old principal after \( n-1 \) months
    + new $200 deposit + interest on old principal

So

\[
\begin{align*}
u_n &= u_{n-1} + 200 + 0.005 u_{n-1} \\
u_n &= 1.005 u_{n-1} + 200 \text{ for } n \geq 1 \text{ (first order \( \text{rr} \)) with IC } u_0 = 1000
\end{align*}
\]

12. (a) See Section 1.10.

First do a new problem where the pairs have names, say Pair 1, Pair 2, ....., Pair \( n \).
Then the pairs can be used as the slots and the number of ways to divide the 2n
people into these pairs is \( \binom{2n}{2} \binom{2n-2}{2} \ldots \binom{4}{2} \)

In the original problem the pairs did not have names.

So answer to original problem = \( \frac{\binom{2n}{2} \binom{2n-2}{2} \ldots \binom{4}{2}}{n!} \)

Cancels down nicely to \( \frac{(2n)!}{(2!)^n n!} \)

(b) It double counts. It counts these two outcomes as different when they are
really the same. Say \( n = 8 \)

\text{outcome 1} Pick \( P_1 P_7 \) as the initial pair.

Divide the others into \( P_2 P_3, P_4 P_5, P_6 P_8 \).

\text{outcome 2} Pick \( P_2 P_3 \) as the initial pair.

Divide the others into \( P_1 P_7, P_4 P_5, P_6 P_8 \).

(c) To pair up the 2n people, first pair John with one of the 2n-1 others. This
    can be done in 2n-1 ways.

Then divide up the 2n-2 others into pairs. This can be done in \( a_{n-1} \) ways.

So \( \text{(*) } a_n = (2n-1) a_{n-1} \text{ for } n \geq 2 \).
This is a first order rr. The IC is $a_1 = 1$

(Remember that $a_1$ is the number of ways of pairing up two people and it can be done in only one way.)

(d) Let $a_n = \frac{(2n)!}{(2!)^n}$. Then $a_{n-1} = \frac{(2[n-1])!}{(2!)^{n-1} (n-1)!} = \frac{(2n-2)!}{(n-1)! 2^{n-1}}$.

Substitute into (*) and see if it works.

RHS of (*) = \frac{(2n-2)!}{(n-1)! 2^{n-1}} = \frac{(2n-1)!}{(n-1)! 2^n}

LHS of (*) = \frac{(2n)!}{(2!)^n} = \frac{2n(2n-1)!}{n! 2^n} = \frac{(2n-1)!}{(n-1)! 2^{n-1}}

LHS = RHS so it checks.

13. To get the last switch ON, you must go through these stages.

(a) OFF OFF ... OFF OFF OFF (the starting configuration)
(b) OFF OFF ... OFF ON OFF (now you're ready to flip the last switch)
(c) OFF OFF ... OFF ON ON (You finally flipped the last switch)
(d) OFF OFF ... OFF OFF ON (You turned the preceding switch OFF)

To go from (a) to (b) takes $u_{n-1}$ flips.

To go from (b) to (c) takes 1 flip.

Going from (c) to (d) takes the same number of flips as going from (a) to (b), namely $u_{n-1}$.

So $u_n = 2u_{n-1} + 1$ for $n \geq 2$ (first order rr) with IC $u_1 = 1$
SOLUTIONS Section 4.2

1. (a) \( \lambda^2 - 3\lambda - 10 = 0 \), \( \lambda = -2, 5 \), \( y_n = A(-2)^n + B5^n \)

(b) \( \lambda^2 + 3\lambda - 4 = 0 \), \( \lambda = 1, -4 \), \( y_n = A\cdot1^n + B(-4)^n = A + B(-4)^n \)

(c) \( 2\lambda^2 + 2\lambda - 1 = 0 \), \( \lambda = \frac{-1 \pm \sqrt{3}}{2} \)

\[ y_n = A \left[ \frac{-1 + \sqrt{3}}{2} \right]^n + B \left[ \frac{-1 - \sqrt{3}}{2} \right]^n \]

(d) this is the same equation as (b) so it has the same answer

2. \( \lambda^2 + 2\lambda - 15 = 0 \), \( \lambda = -5, 3 \), gen \( y_n = A(-5)^n + B3^n \)

Need \( A + B = 0 \), \(-5A + 3B = 1\). So \( A = -\frac{1}{8}, B = \frac{1}{8}\). Answer is \( y_n = -\frac{1}{8}(-5)^n + \frac{1}{8}3^n \)

3. (a) \( y_{n+2} = y_{n+1} + 6y_n \) so

\[ y_2 = y_1 + 6y_0 = 0 + 6 \cdot 1 = 6 \]

\[ y_3 = y_2 + 6y_1 = 6 + 6 \cdot 0 = 6 \]

(b) \( \lambda^2 - \lambda - 6 = 0 \), \( \lambda = 3, -2 \), gen \( y_n = A3^n + B(-2)^n \)

Need \( A + B = 0 \), \( 3A - 2B = 1 \). So \( A = \frac{2}{5}, B = \frac{3}{5}\). Sol is \( y_n = \frac{2}{5}3^n + \frac{3}{5}(-2)^n \)

(c) \( y_3 = \frac{2}{5}3^3 + \frac{3}{5}(-2)^3 = 6 \)

4. \( \lambda = \frac{1 + \sqrt{5}}{2} \), gen \( y_n = A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n \)

To get the IC you need

\[ A + B = 0, \]
\[ \frac{1}{2} A(1 + \sqrt{5}) + \frac{1}{2} B(1 - \sqrt{5}) = 1, \]

The solution is \( A = \frac{1}{\sqrt{5}}, B = -\frac{1}{\sqrt{5}} \) so the final answer is

\[ y_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

In case you don't believe that this formula actually produces integers here is a table of the first 9 values.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
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<td>13</td>
</tr>
<tr>
<td>8</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>34</td>
</tr>
</tbody>
</table>

5. (a) Sequence is 5, 7, 9, 11, 13, ... Pattern looks like \( y_n = 2n + 3 \)

(b) \( y_{n+1} = \frac{y_{n+2} + y_n}{2} \), \( y_{n+2} - 2y_{n+1} + y_n = 0 \), \( \lambda = 1, 1 \), gen \( y_n = A + Bn \).

To get the IC \( y_1 = 5, y_2 = 7 \) need \( A + B = 5, A + 2B = 7 \)

So \( A = 3, B = 2 \). Answer is \( y_n = 3 + 2n \)

6. Substitute \( n3^n \) into the rr.

\[
\text{LHS} = (n+2)3^{n+2} - 6(n+1)3^{n+1} + 9n3^n \\
= (n+2)3^n - 6n3^n + 9n3^n \\
= (9n + 18 - 18n - 18 + 9n)3^n = 0 \quad \text{So it checks.}
\]
7. \( r r \) is \( y_{n+2} - 4y_{n+1} + 4y_n = 0 \)
\[ \lambda^2 - 4\lambda + 4 = 0, \quad \lambda = 2, 2, \] gen \( y_n = A2^n + Bn2^n \)
To get \( y_0 = 0 \) need \( 0 = A \)
To get \( y_1 = 2 \) need \( 2 = 2A + 2B \), \( B = 1 \)
Answer is \( y_n = n2^n \)

8. The eqn is really \( y_{n+2} - 2y_n = 0 \) so \( \lambda^2 + 2 = 0, \quad \lambda = \pm \sqrt{2} \)
gen \( y_n = A(\sqrt{2})^n + B(-\sqrt{2})^n \)

9. (a) \( y_n = A(-3)^n + B4^n + Cn4^n \)
(b) \( y_n = A5^n + Bn5^n + Cn^2 5^n + Dn^3 5^n + E2^n \)
(c) \( y_n = A + Bn + Cn^2 + D6^n + E(-1)^n \)

10. \( \lambda = 1, 1, 2, \quad (\lambda-1)^2 (\lambda-2) = 0, \quad \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0 \)
\( rr \) is \( y_{n+3} - 4y_{n+2} + 5y_{n+1} - 2y_n = 0 \)

11. \( (\lambda - 1)^3 = 0, \quad \lambda = 1, 1, 1, \) gen \( y_n = A + Br + Cn^2 \)
Need
\[ A + B + C = 0 \]
\[ A + 2B + 4C = 1 \]
\[ A + 3B + 9C = 0 \]
Solution is \( A = -3, \quad B = 4, \quad C = -1 \)
So \( y_n = -3 + 4n - n^2 \)

12. (a) by inspection If \( y_1 = 4 \) and \( y_{n+1} = y_n \) then \( y_2 = 4, y_3 = 4 \) and the solution is \( y_n = 4 \) for every \( n \)
overkill \( \lambda = 1, \) gen \( y_n = A. \) Plug in the IC \( y_1 = 4 \) to get \( A = 4 \). Sol is \( y_n = 4 \)
(b) by inspection If \( y_0 = 0, y_1 = 0 \) and thereafter \( y_{n+2} = -\frac{b}{a} y_{n+1} - \frac{c}{a} y_n \) then every term is 0, i.e., \( y_n = 0 \) for all \( n \)
overkill Say the roots of the characteristic eqn are \( \lambda_1 \) and \( \lambda_2 \). Then the general sol is \( A \lambda_1^n + B \lambda_2^n \) (assuming \( \lambda_1 \neq \lambda_2 \) which is another story). Plug in the IC and you get \( 0 = A + B, \quad 0 = A \lambda_1 + B \lambda_2, \) The only solution for \( A \) and \( B \) is \( A = 0, \quad B = 0 \) which makes \( y_n = 0 \).
13. (a) \( \lambda = 3,2 \), gen \( y_n = A3^n + B2^n \)

(b) I'm going to compute \( 5y_{n-1} - 6y_{n-2} \) and see that it comes out to be \( y_n \).

\[
5y_{n-1} - 6y_{n-2} = 5A3^{n-1} + 5B2^{n-1} - 6A3^{n-2} - 6B2^{n-2}
\]

\[
= \left( \frac{5A}{3} \right) 3^n + \left( \frac{5B}{2} \right) 2^n - \left( \frac{6A}{3^2} \right) 3^n - \left( \frac{6B}{2^2} \right) 2^n \quad \text{(algebra)}
\]

\[
= \left( \frac{5A}{3} - \frac{6A}{3^2} \right) 3^n + \left( \frac{5B}{2} - \frac{6B}{2^2} \right) 2^n \quad \text{(more algebra)}
\]

\[
= \left( \frac{5A}{3} - \frac{6A}{3^2} \right) 3^n + \left( \frac{5B}{2} - \frac{6B}{2^2} \right) 2^n \quad \text{(arithmetic)}
\]

\( = y_n \quad \text{QED} \)

14. You would need

\[
7 = A + B
\]

\[
5 = 2A + 2B
\]

i.e.,

\[
A + B = 7
\]

\[
A + B = 5/2
\]

\( A+B \) can't be \( 7 \) and \( 5/2 \) so there is no solution. The faux general solution \( A2^n + B2^n \)

can't be made to satisfy the IC.

\( A2^n + B2^n \) is \( (A+B)2^n \). There really is only one constant, namely \( A+B \), so it isn't
general. And it wasn't enough to satisfy two IC.
SOLUTIONS Section 4.3

1. (a) \( \lambda = 2, -1, \) \( h_n = A 2^n + B (-1)^n \). Try \( p_n = c \).

Then \( p_{n+1} = c, p_{n+2} = c \). Substitute into the rr to get \( c - c - 2c = 1, c = -\frac{1}{2} \)

\( \text{gen } y_n = A 2^n + B (-1)^n - \frac{1}{2} \)

Plug in the IC: \( 1 = 2A - B - \frac{1}{2}, 3 = 4A + B - \frac{1}{2} \)

So \( A = \frac{5}{6}, B = \frac{1}{6} \). Answer is \( y_n = \frac{5}{6} 2^n + \frac{1}{6} (-1)^n - \frac{1}{2} \)

(b) \( \lambda = -5,3, \) \( h_n = A 3^n + B (-5)^n \). Try \( p_n = c n + d \)

Need \( c (n+1) + d + 2 \left[ c (n+1) + d \right] - 15 (c n + d) = 6n + 10 \)

Equate n coeffs \(-12c = 6, c = -\frac{1}{2} \)
Equate constant terms \( 4c - 12d = 10, d = -1 \)

\( p_n = -\frac{1}{2} n - 1 \)

\( \text{gen } y_n = A 3^n + B (-5)^n - \frac{1}{2} n - 1 \)

The IC make \( A = \frac{11}{8}, B = \frac{5}{8} \), Answer is \( y_n = \frac{11}{8} 3^n + \frac{5}{8} (-5)^n - \frac{1}{2} n - 1 \)

2. (a) \( \lambda^2 - 3\lambda + 1 = 0, \) \( \lambda = \frac{3 \pm \sqrt{5}}{2}, \) \( h_n = A \left( \frac{3 + \sqrt{5}}{2} \right)^n + B \left( \frac{3 - \sqrt{5}}{2} \right)^n \)

Try \( p_n = c 4^n \).

\( c 4^{n+2} - 3c 4^{n+1} + c 4^n = 10 \cdot 4^n \)

\( 16c 4^n - 12c 4^n + c 4^n = 10 \cdot 4^n \)

Equate coeffs of \( 4^n \): \( 5c = 10, c = 2 \)

\( p_n = 2 \cdot 4^n, \text{gen } y_n = A \left( \frac{3 + \sqrt{5}}{2} \right)^n + B \left( \frac{3 - \sqrt{5}}{2} \right)^n + 2 \cdot 4^n \)

(b) The new version of the rr is \( y_n - 3y_{n-1} + y_{n-2} = 10 \cdot 4^{n-2} \) warning Not \( 10 \cdot 4^n \)

Same \( \lambda \) equation, same \( h_n \) as in part (a).

\( \text{method 1 for getting } p_n \) write \( 4^{n-2} \) as \( 4^n 4^{-2} \) and use \( y_n - 3y_{n-1} + y_{n-2} = \frac{10}{16} 4^n \)

Try \( p_n = c 4^n \).

\( c 4^n - 3c 4^{n-1} + c 4^{n-2} = \frac{10}{16} 4^n \)

\( c 4^n - \frac{3}{4} c 4^n + \frac{3}{16} c 4^n = \frac{10}{16} 4^n \)

Match the coeffs of \( 4^n \): \( c - \frac{3}{4} c + \frac{3}{16} c = 2, \frac{5}{16} c = \frac{10}{16}, c = 2 \)

\( p_n = 2 \cdot 4^n, \text{same as in part (a)} \)

Same \( y_n \) as in part (a).

\( \text{method 2 for getting } p_n \) (the first method was better so you could skip this)

Leave it \( y_n - 3y_{n-1} + y_{n-2} = 10 \cdot 4^{n-2}, \text{try } p_n = c 4^{n-2} \). Need

\( c 4^{n-2} - 3c 4^{n-3} + c 4^{n-4} = 10 \cdot 4^{n-2} \)

\( c 4^{n-2} - \frac{3}{4} c 4^{n-2} + \frac{1}{16} c 4^{n-2} = 10 \cdot 4^{n-2} \)

Match the coeffs of \( 4^{n-2} \): \( \frac{5}{16} c = 10, c = 32 \)

\( p_n = 32 \cdot 4^{n-2} = 32 \cdot 4^n - 4^{2} = 2 \cdot 4^n, \text{same as before} \).
3. \(\lambda = 3, -2\), \(h_n = D3^n + E(-2)^n\). Try \(p_n = An^2 + Bn + C\). Need

\[
A(n+2)^2 + B(n+2) + C - \left( A(n+1)^2 + B(n+1) + C \right) - 6(An^2 + Bn + C) = 18n^2 + 2
\]

Match \(n^2\) coeffs, \(-6A = 18\)
Match \(n\) coeffs, \(2A - 6B = 0\)
Match constant terms, \(3A + B - 6C = 2\)
So \(A = -3\), \(B = -1\), \(C = -2\)

\[\text{gen } y_n = D3^n + E(-2)^n - 3n^2 - n - 2\]

The IC make \(D = \frac{8}{5}\), \(E = -\frac{3}{5}\) Answer is \(y_n = \frac{8}{5} \cdot 3^n - \frac{3}{5} (-2)^n - 3n^2 - n - 2\)

4. (a) \(y_1 = 2\)
\[y_2 = 2y_1 + 6 = 12 = 16\]
\[y_3 = 2y_2 + 6 \cdot 3 = 50\]
\[y_4 = 2y_3 + 6 \cdot 4 = 124\]

(b) The rr is \(y_n - 2y_{n-1} = 6n\). Or you can use \(y_{n+1} - 2y_n = 6(n+1)\). As long as you are consistent.

Either way, \(\lambda -2 = 0, \lambda = 2\), \(h_n = A2^n\).

*method 1 for getting \(p_n\)* Use \(y_n - 2y_{n-1} = 6n\)

Try \(p_n = Bn + C\). Need

\[
Bn + C - 2 \left[ B(n-1) + C \right] = 6n
\]
Equate \(n\) coeffs, \(-B = 6, B = -6\)
Equate constant terms, \(2B - C = 0\)
So \(B = -6, C = -12\), \(p_n = -6n - 12\)

Then \(\text{gen } y_n = A2^n - 6n - 12\)

The IC makes \(2 = 2A - 6 - 12, A = 10\). Answer is \(y_n = 10 \cdot 2^n - 6n - 12\)

*method 2 for getting \(p_n\)* Use \(y_{n+1} - 2y_n = 6(n+1)\)

Try \(p_n = Bn + C\). Need

\[
B(n+1) + C - 2(Bn + C) = 6n + 6
\]
Equate coeffs of \(n\): \(-2B = 6, B = -3\)
Equate constant terms: \(B + C - 2C = 6, C = -12\)
So \(p_n = -6n - 12\) again

\(c) \ y_4 = 10 \cdot 16 - 6 \cdot 4 = 124\)

5. (a) \(\lambda = 2\), \(h_n = A(-2)^n\). Try \(p_n = B\). Need

\[
B + 2B = 4, B = \frac{4}{3}
\]

So \(p_n = \frac{4}{3}\) and a general solution is \(y_n = A(-2)^n + \frac{4}{3}\)

(b) Same \(h_n\) as part (a).

Try \(p_n = C4^n\). Need

\[
c4^{n+1} + 2c4^n = 4^n
\]
\[c \cdot 4^n + 2c4^n = 4^n
\]

The coeff of \(4^n\) on the left is \(6C\), on the right it's \(1\) so set \(6C = 1\), \(C = \frac{1}{6}\), \(p_n = \frac{1}{6}4^n\)

The general solution is \(y_n = A(-2)^n + \frac{1}{6} \cdot 4^n\)
6. (a) Try \( p_n = An^4 + Bn^3 + Cn^2 + Dn + E \)
(b) Try \( p_n = n^4( An^4 + Bn^3 + Cn^2 + Dn + E) = An^8 + Bn^7 + Cn^6 + Dn^5 + En^4 \)
Step up because \( A, n, n^2 \) and \( n^3 \) are all homog solutions
(c) Try \( p_n = An^2 \) (step up)
(d) Try \( p_n = A \cdot 2^n \)
(e) Try \( p_n = An^2 \cdot 3^n \) (step up because \( 3^n \) and \( n \cdot 3^n \) are both homog sols))

7. \( \lambda = \frac{1}{2}, h_n = A \left( \frac{1}{2} \right)^n \). Try \( p_n = Bn \left( \frac{1}{2} \right)^n \) (step up) Need
\[
2B(n+1) \left( \frac{1}{2} \right)^{n+1} - Bn \left( \frac{1}{2} \right)^n = \left( \frac{1}{2} \right)^n
\]
\[
2B(n+1) \left( \frac{1}{2} \right)^n - Bn \left( \frac{1}{2} \right)^n = \left( \frac{1}{2} \right)^n
\]
\[
B \left( \frac{1}{2} \right)^n = \left( \frac{1}{2} \right)^n
\]
Equate coeffs of \( \left( \frac{1}{2} \right)^n \). So \( B = 1 \), \( p_n = \left( \frac{1}{2} \right)^n \), gen \( y_n = A \left( \frac{1}{2} \right)^n + n \left( \frac{1}{2} \right)^n \)
The IC make \( A = 3 \). Answer is \( y_n = 3 \left( \frac{1}{2} \right)^n + n \left( \frac{1}{2} \right)^n \)

8. \( \lambda = 1,1, h_n = A + Bn \). Try \( p_n = Cn^2 \) (step up twice from plain \( C \)) Need
\[
c(n+2)^2 - 2c(n+1)^2 + cn^2 = 1
\]
The \( n^2 \) terms and \( n \) terms cancel out.
Equate constant terms: \( 2c = 1, C = \frac{1}{2} \)
\[
p_n = \frac{1}{2} n^2
\]
gen \( y_n = A + Bn + \frac{1}{2} n^2 \)
The IC make \( A = 1, \frac{1}{2} = A + B + \frac{1}{2} \). So \( B = -1 \). Answer is \( y_n = 1 - n + \frac{1}{2} n^2 \)

9. \( S_{n+1} = S_n + (n+1)^2, S_{n+1} - S_n = (n+1)^2 \) with IC \( S_1 = 1 \).
\( \lambda = 1, h_n = D \). Try \( p_n = n(An^2 + Bn + C) = An^3 + Bn^2 + Cn \). (Step up because \( C \) is a homog sol.) Need
\[
A(n+1)^3 + B(n+1)^2 + C(n+1) - (An^3 + Bn^2 + Cn) = n^2 + 2n + 1
\]
The \( n^3 \) coeffs drop out.
Equate \( n^2 \) coeffs \( 3A = 1, A = \frac{1}{3} \)
Equate \( n \) coeffs \( 3A + 2B = 2, B = \frac{1}{2} \)
Equate constant terms \( A + B + C = 1, C = \frac{1}{6} \)
\[
p_n = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n
\]
Gen sol is \( S_n = D + \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \)
The IC \( S_1 = 1 \) makes \( D = 0 \).
Answer is \( S_n = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \), usually written as \( S_n = \frac{n(n+1)(n+2)}{6} \)
10. (a) Need $c_{2n+2} - 3c_{2n+1} + 2c_{2n} = 6 \cdot 2^n$, $4c_{2n} - 6c_{2n+1} + 2c_{2n+2} = 6 \cdot 2^n$.

The C's cancel out leaving $0 = 6 \cdot 2^n$ which can't be satisfied. There is no value of $C$ which makes $2^n$ work.

In general, if you don't step up when you should, some or all of your constants will cancel out and you can't make your trial $p_n$ fit the $rr$.

(b) Need $2A(n+2) + 3A(n+1) + 4An = 18n$, $9An + 7A = 18n$.

Equate $n$ coeffs $9A = 18$, $A = 2$

Equate constant terms $7A = 0$, $A = 0$

Impossible to have $A=2$ and $A = 0$. So there is no solution of the form $An$.

11. Ugh !!!

A was supposed to be a constant. When you substituted $p_n$ into the lefthand side of the $rr$ and got $A(n+2) - 2A(n+1) - 3An$, you treated $A$ as a constant. You can't change your mind now and allow $A$ to be $\frac{-5}{4} n$.

To make $-4An$ equal $5n^2$, coeffs of like terms have to match:
The $n$ term on the left has to match the $n$ term on the right.
The $n^2$ term on the left has to match the $n^2$ term on the right.

There is no way to make that happen here.

So your conclusion should be that there are no values of the constant $A$ that will make $-4An = 5n^2$ so there is no $p_n$ of the form $An$ where $A$ is a constant.

footnote (if you still don't believe me)

$-\frac{5}{4} n^2$ doesn't work in the $rr$ if you substitute it into the lefthand side you get

$-\frac{5}{4}(n+2)^2 - 2 \cdot -\frac{5}{4}(n+1)^2 - 3 \cdot -\frac{5}{4}n^2$

which comes out to be $\frac{5}{2} n^2 - \frac{5}{4}$ not $5n^2$.

warning When you determine the constants in $p_n$, they should come out to be just that, constants.

A can't have $n$'s in it.
SOLUTIONS review problems for Chapter 4

1. \(\lambda^2 - 9 = 0, \ \lambda = \pm 3, \ \ h_n = A 3^n + B (-3)^n\)

Try \(y_p = Cn^2 + Dn + E\)

\[C(n+2)^2 + D(n+2) + E - 9(Cn^2 + Dn + E) = 56n^2\]

\[\text{equate } n^2 \text{ coeffs} \quad -8C = 56, \ C = -7\]

\[\text{equate } n \text{ coeffs} \quad 4C - 8D = 0, \ D = -7/2\]

\[\text{equate constant terms} \quad 4C + 2D - 8E = 0, \ E = -35/8\]

\[p_n = -7n^2 - 7/2 n - 35/8\]

\[\text{gen } y_n = h_n + p_n = A 3^n + B (-3)^n -7n^2 - 7/2 n - 35/8\]

2. \(\lambda = -2, \ h_n = A(-2)^n. \ \text{Try } p_n = B 7^n.\)

Need \(2B 7^{n+1} + 4B 7^n = 6 \cdot 7^n\)

\[14B 7^n + 4B 7^n = 6 \cdot 7^n\]

So \(18B = 6, \ B = \frac{1}{3} \), \(p_n = \frac{1}{3} \cdot 7^n\), \text{ gen } \(y_n = A(-2)^n + \frac{1}{3} \cdot 7^n\)

To get the IC we need \(5 = -2A + \frac{7}{3}, \ A = -\frac{4}{3}.\)

Answer is \(y_n = -\frac{4}{3}(-2)^n + \frac{1}{3} \cdot 7^n\)

3. \(\lambda = \frac{-5 \pm \sqrt{29}}{2}, \ h_n = A \left[\frac{-5 + \sqrt{29}}{2}\right]^n + B \left[\frac{-5 - \sqrt{29}}{2}\right]^n\)

Try \(p_n = D. \ \text{Need } D + 5D = 6, \ D = \frac{6}{5}.\)

\[p_n = \frac{6}{5}\]

\[\text{gen } y_n = A \left[\frac{-5 + \sqrt{29}}{2}\right]^n + B \left[\frac{-5 - \sqrt{29}}{2}\right]^n + \frac{6}{5}\]

4. \(S_{n+1} = S_n + n+1, \ S_{n+1} - S_n = n+1\) \text{ with IC } \(S_1 = 1.\)

\(\lambda = 1, \ h_n = A.\)

Try \(p_n = n(Bn + C) \text{ (step up) } = Bn^2 + Cn\)

Need \(B(n+1)^2 + C(n+1) - (Bn^2 + Cn) = n+1\)

The \(n^2\) terms drop out on each side

Equate \(n\) coeffs \(2B + C - C = 1, \ B = \frac{1}{2}\)

Equate constant terms \(B + C = 1, \ C = \frac{1}{2}\)

\[p_n = \frac{1}{2} n^2 + \frac{1}{2} n\]

\[\text{gen } S_n = A + \frac{1}{2} n^2 + \frac{1}{2} n\]

To get \(S_1 = 1\) we need \(1 = A + \frac{1}{2} + \frac{1}{2}, \ A = 0.\)

Answer is \(S_n = \frac{1}{2} n^2 + \frac{1}{2} n\) \text{ usually written as } \(S_n = \frac{n(n+1)}{2}\)

5. \(\lambda = \pm 3, \ h_n = A 3^n + B (-3)^n. \ \text{Try } p_n = Dn 3^n \text{ (step up).}\)

Need \(D(n+2) 3^{n+2} - 9Dn 3^n = 5 \cdot 3^n\)

\(9D(n+2) 3^n - 9Dn 3^n = 5 \cdot 3^n\)

The \(n^3\) terms drop out.

Match the \(3^n\) coeffs: \(18D = 5, \ D = \frac{5}{18}, \ p_n = \frac{5}{18} n^3\)

\[\text{gen } y_n = A 3^n + B (-3)^n + \frac{5}{18} n^3\]
6. The recurrence can be written as $y_{n+1} - 2y_n = 0$ and it is only first order. It would come with only one IC and its general sol (namely $B2^n$) should only have one constant. So nothing is wrong.

7. (a) Note that $y_n$ is the minimum number of moves it takes to move $n$ rings from one peg to a second peg, using the third peg for intermediate storage. To get all the rings moved you have to pass through the following stages:

- Start here
- Move the top $n-1$ rings from peg 1 to peg 3 using peg 2 as storage
  - Takes $y_{n-1}$ moves
- Move the largest ring to peg 2
  - Takes one move
- Move the $n-1$ rings from peg 3 to peg 2 using peg 1 as storage
  - Takes $y_{n-1}$ moves

So $y_n = 2y_{n-1} + 1$ for $n \geq 2$. Need one IC.

It only takes one move in a 1-ring game so the IC is $y_1 = 1$.

(b) $\lambda = 2$, $h_n = A \cdot 2^n$. Try $p_n = B$.

Need $B - 2B = 1$, $B = -1$

Gen sol is $y_n = A \cdot 2^n - 1$. To get the IC we need $1 = 2A - 1$, $A = 1$.

Sol is $y_n = 2^n - 1$.

For example to move a 10-ring tower it takes a minimum of $2^{10} - 1$ moves.

8. Every $n$-tree is one of these five types (see the diagram below).

(i) A 1-tree with Left Child followed by an (n-1)-tree

(ii) A 1-tree with a Right Child followed by an (n-1)-tree

(iii) A 1-tree with Both Children, with an (n-1)-tree on the Left Child and either a 0-tree or a 1-tree or ... or an (n-2)-tree on the Right Child

(iv) A 1-tree with Both Children, with an (n-1)-tree on the Right Child and either a 0-tree or a 1-tree or ... or an (n-2)-tree on the Left Child

(v) A 1-tree with both Children, with an (n-1)-tree on each child
There are $y_{n-1}$ trees of type (i).
There are $y_{n-1}$ trees of type (ii).
There are $y_{n-1} \cdot (y_0 + y_1 + \ldots + y_{n-2})$ trees of type (iii).
(The dangle on the left can be chosen in $y_{n-1}$ ways and the dangle on the right can be chosen in $y_0 + y_1 + \ldots + y_{n-2}$ ways.)
There are $y_{n-1} \cdot (y_0 + y_1 + \ldots + y_{n-2})$ trees of type (iv).
There are $y_{n-1}^2$ trees of type (v).

So $y_n = 2y_{n-1} + 2y_{n-1} \cdot (y_0 + \ldots + y_{n-2}) + y_{n-1}^2$

(b) Start with $y_0 = 1$, $y_1 = 3$. Then

\[
y_2 = 2y_1 + 2y_1 y_0 + y_1^2 = 6 + 6 + 9 = 21
\]

\[
y_3 = 2y_2 + 2y_2 (y_0 + y_1) + y_2^2 = 42 + 42.4 + 441 = 651
\]

(c) No. It has products $y_0 y_{n-1}$, $y_1 y_{n-1}$, $\ldots$, $y_{n-2} y_{n-1}$ in it so it doesn't have the right pattern for the methods in Sections 2 and 3.

**Question** Can you say that each $n$-tree is one of these four types.

**Answer**

Only if you're careful. These types are exhaustive but not mutually exclusive. Types (iii)(a) and (iv)(a) overlap. In fact the overlap is precisely type (v) above so if you just add the number of trees in (i),(ii),(iii)(a), (iv)(a) you will have counted trees of type (v) twice. So

\[
y_n = \# \text{ of type (i)} + \# \text{ type (ii)} + \# \text{ type (iii)(a)} + \# \text{ type (iv)(a)} - \# \text{ type (v)}
\]

\[
= 2y_{n-1} + 2y_{n-1} \cdot (y_0 + y_1 + \ldots + y_{n-1}) - y_{n-1}^2
\]

which agrees algebraically with the other version.
9. Let \( y_n \) be the number of ways of filling up \( n \) spaces.

Every way must look like one of these:
(a) a limo followed by parking in \( n-2 \) spaces
(b) a stretch limo followed by parking in \( n-3 \) spaces

So \( y_n = y_{n-2} + y_{n-3} \) (3rd order rr, need 3 IC)

All in all, \( y_1 = 0 \), \( y_2 = 1 \), \( y_3 = 1 \) and \( y_n = y_{n-2} + y_{n-3} \) for \( n \geq 4 \),

10. Let \( y_n \) be the unpaid balance after \( n \) months when the interest rate is 1% a month and you make monthly payments of \( D \) dollars. Then

unpaid balance this month
\[ = \text{last month's unpaid balance} + \text{interest on that unpaid balance} - \text{your payment } D \]

So
\[ y_{n+1} = y_n + 0.01y_n - D \]
\[ y_{n+1} - 1.01y_n = -D \quad \text{(nonhomog, first order) with IC } y_0 = 10,000 \]
\[ \lambda - 1.01 = 0, \quad \lambda = 1.01, \quad h_n = A(1.01)^n \]

Try \( p_n = B \). Substitute it into the rr to determine \( B \).
\[ B - 1.01B = -D \]
\[ B = 100D \]

So \( p_n = 100D \) and \( \text{gen } y_n = A(1.01)^n + 100D \)

Plug in the IC \( y_0 = 10,000 \). Get \( A = 10,000 - 100D \)

So \( y_n = (10,000 - 100D)(1.01)^n + 100D \)

Now arrange to get \( y_{36} = 0 \):
\[ 0 = (10,000 - 100D)(1.01)^{36} + 100D \]
\[ D = \frac{100(1.01)^{36}}{(1.01)^{36} - 1} = \$332.14 \]

So your monthly payments should be $332.14