CHAPTER 4 RECURRENCE RELATIONS

SECTION 4.1 SETTING UP RECURRENCE RELATIONS TO DO COUNTING PROBLEMS

example 1

Look at binary strings (strings of 0's and 1's).
For example, 10111 is a binary string of length 6.
The problem is to count the number of binary strings of length n with no consecutive 0's.
For convenience let's call them "good" strings.
Let $a_n$ be the number of good strings of length n (good n-strings).

Every good n-string (where $n \geq 3$) is exactly one of these two types:
(a) 1 followed by a good (n-1)-string
(b) 01 followed by a good (n-2)-string

For example, 11011101 is a good 8-string and fits into category (a) because it's a 1 followed by the good 7-string 1011101. And 0111011 is a good 8-string of type (b) because it's 01 followed by the good 6-string 110110.

There are $a_{n-1}$ strings of type (a).
There are $a_{n-2}$ strings of type (b).

Since the total number of good n-strings is number of (a)'s + number of (b)'s,

(1) $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$

And by inspection there are two good 1-strings (1 and 0) and three good 2-strings (11, 10, 01). So

(2) $a_1 = 2$ and $a_2 = 3$

You can put (1) and (2) together to grind out successive values of $a_n$:

$a_3 = a_2 + a_1 = 3 + 2 = 5$
$a_4 = a_3 + a_2 = 5 + 3 = 8$
$a_5 = a_4 + a_3 = 8 + 5 = 13$

In other words there are 5 good strings of length 3, 8 good strings of length 4, 13 good strings of length 5 and you can keep going as far as you like.

recurrence relations and initial conditions

The equation in (1) is called a recurrence relation or a recursion relation (rr) or a difference equation ($\Delta E$).
The specific values in (2) are called initial conditions (IC).
The rr in (1) is second order because it tells you how to get a term from the preceding term and the pre-preceding term; i.e., it's second order because you need two IC to get rolling.

For example, the recurrence relation

$y_n = 3y_{n-1} + n$

is a first order rr because it tells you how to get a term from the preceding term; you only need one IC to get started.

Here are some examples of third order recurrence relations.

$h_n = h_{n-3}$
$h_n = 6h_{n-1} + nh_{n-3}$
$h_n = 9h_{n-1} + 2h_{n-2} - 5h_{n-3}$

In each case you'd have to know $b_1$, $b_2$ and $b_3$ (or $b_0$, $b_1$, $b_2$ or $b_8$, $b_9$, $b_{10}$ etc.) before you could use the relation to keep finding successive $b_n$ values.
example 1 continued

Suppose the strings can use any of the 10 digits, not just 0 and 1. How many strings of length \( n \) have no consecutive 0's.

**solution** Now every good \( n \)-string must be exactly one of these two types.

(a) \( \overline{0} \) (nonzero) followed by a good \((n-1)\)-string
(b) \( 00 \) followed by a good \((n-2)\)-string

There are \( 9a_{n-1} \) strings of type (a) because the \( \overline{0} \) can be picked in any of 9 ways and the good \((n-1)\)-string can be picked in \( a_{n-1} \) ways.

There are \( 9a_{n-2} \) strings of type (b). So

\[
a_n = 9a_{n-1} + 9a_{n-2} \text{ for } n \geq 3
\]

This is a second order rr so I need two IC:

\[
a_1 = 10 \quad (\text{every string of length 1 is good})
\]

\[
a_2 = 99 \quad (\text{there are 100 two digit strings and they're all good except 00})
\]

Then

\[
a_3 = 9a_2 + 9a_1 = 891 + 90 = 981
\]

\[
a_4 = 9a_3 + 9a_2 = 8829 + 891 = 9720 \text{ and so on}
\]

example 2

Let \( u_n \) be the number of words of length \( n \) that contain 3 or more A's in a row (good words). Find a recursion relation and IC for \( u_n \).

**solution** A good word of length \( n \) must be exactly one of the following types:

(a) \( \overline{A} \) followed by a good word of length \( n-1 \)

**warning 1**

Don't call this \( \overline{A} u_{n-1} \).

That confuses a description (good word of length \( n-1 \)) with how many there are with that description (the number \( u_{n-1} \)).

(b) \( AA \) followed by a good word of length \( n-2 \)
(c) \( AAA \) followed by a good word of length \( n-3 \)
(d) \( AAA \) followed by any word of length \( n-3 \)

**warning 2**

This type is \( AAA \) followed by any \((n-3)\)-word, not \( AAA \) followed by a good \((n-3)\)-word; and the number of words of type (d) is \( 26^{n-3} \), not \( u_{n-3} \).

For example, \( CZAAAAB \) is type (a), \( ABXYAAAZAAAA \) is type (b), \( AAXYAAABZ \) is type (c), \( AAAAA \) and \( AAACXYZ \) are type (d).

There are \( 25u_{n-1} \) words of type (a).

There are \( 25u_{n-2} \) words of type (b).

There are \( 25u_{n-3} \) words of type (c).

There are \( 26^{n-3} \) words of type (d).

So

\[
u_n = 25u_{n-1} + 25u_{n-2} + 25u_{n-3} + 26^{n-3} \text{ for } n \geq 4
\]

This is a 3rd order rr and needs three IC:

\[
u_1 = 0, u_2 = 0, u_3 = 1.
\]

Now you can start more computing values of \( u_n \).
\[ u_4 = 25u_3 + 25u_2 + 25u_1 + 26^1 = 25 + 0 + 0 + 26 = 51 \]
\[ u_5 = 25u_4 + 25u_3 + 25u_2 + 26^2 = 1275 + 25 + 0 + 676 = 1976 \]

**Warning 3**

When you claim that "a good n-word must be one of the following types" make sure that your types don't overlap (or else you'll overcount) and that they include all possibilities, i.e., are exhaustive (or else you'll undercount).

**PROBLEMS FOR SECTION 4.1**

1. You can go up a flight of stairs taking either one or two steps at a time.
   For example, you can climb a 6-step staircase with the patterns
   
   1221, 2211, 1122, 111111, 222, etc
   
   Let \( y_n \) be the number of ways you can climb a staircase with \( n \) steps.
   
   (a) Find a recurrence relation and initial condition for \( y_n \).
   (b) Use part (a) to find the number of ways to climb a 6-step staircase.

2. The problem is to count the number of words of length \( n \) with an even number of A's. For example, ZABBXA is a good word of length 6.
   
   Find a recursion relation and initial condition.

3. A legal string consists of digits and the three operators +, *, - all arranged so that they make arithmetic sense.
   
   The length of a string is the number of symbols in the string.
   
   Here are some examples of legal strings:
   
   234+81*0+12345-29 (length is 17)
   00014+8-3 (length is 9)
   
   It's considered legal to have leading 0's so 000014 is OK.
   Can't have two operations in a row so 2+*3 is illegal.
   Every legal string must begin with a digit and end with a digit.
   
   Let \( u_n \) be the number of legal strings of length \( n \).
   
   Every legal string with length 2 or more begins either with DD (two digits) or D op (a digit followed by an operation).
   
   In particular, a legal \( n \)-string, \( n \geq 3 \), must be one of these two types:
   
   (i) D followed by a legal \((n-1)\)-string
   (ii) D op followed by a legal \((n-2)\)-string
   
   (Stop and see if you could have figured this out for yourself.)
   
   For example, the 9-string 346+098-2 is the digit 3 followed by the legal 8-string 46+098-2. The 9-string 3+456*1+0 is 3+ followed by the legal 7-string 456*1+0.
   
   (a) Find a recursion relation and initial condition for \( u_n \).
   (b) Use the recursion to find the number of legal strings of length 4.
4. A code uses the three words a, ab, bc. A message is a sequence of words. For example, one message is a ab ab bc a which we will write without spaces as aababbca.

Spaces aren't necessary between words in a message because this happens to be a uniquely decipherable code. For instance the message aababbca can be decoded in only one way, namely as a ab ab bc a.

Some other messages for instance are a, aaa, ab, aab.

(a) Find a recursion relation and IC for the number of messages of length n (i.e., with n symbols).

(b) Use part (a) to find the number of messages of length 9.

5. A board comes with two rows of squares.

The diagram shows the two types of tiles which can be used to cover it. (Tiles of the first type can be turned sideways.)

![Board and tiles diagram]

For example here are some ways to tile a board of length 3

Let \( y_n \) be the number of ways of tiling a board of length n (an n-board).

Find a recurrence relation and IC for \( y_n \).

6. Sentences are composed from among the 26 letters A–Z and the symbol # representing a blank. A sentence can't begin or end with a blank or have two blanks in a row. For example, AA#BZZG#A is a 9-symbol sentence, HOW#ARE#YOU is an 11-symbol sentence.

Set up a recurrence relation with IC to find the number of n-symbol sentences.

7. Look at strings of 0's, 1's, 2's, 3's. A good string is one where a 3 never occurs past a 0; i.e., once a 0 appears the rest of the string can contain only 0's, 1's, 2's.

For example the string 123023 is not a good string because there is a 3 to the right of a 0.

Let \( a_n \) be the number of good n-strings (strings of length n). Find a recurrence relation and IC for \( a_n \). Then use them to find the number of good strings of length 4.

8. Let \( b_n \) be the number of binary strings of length n with two or more consecutive 0's (e.g., 1000001, 0010).

(a) Look at this attempt.

The good strings of length n have to be one of these types

(i) 00 followed by any (n-2)-string

(ii) any digit followed by a good (n-1)-string

Derive the rr that goes with this attempt.

And then explain what's wrong with the attempt.

(b) Look at this attempt.

The good strings of length n have to be one of these types

(i) 00 followed by any (n-2)-string

(ii) 1 followed by a good (n-1)-string

Derive the rr that goes with this attempt.

And then explain what's wrong with the attempt.

(c) Get a rr, with IC, that works. Then find \( b_5 \).
9. Call a string of A's, B's, C's, D's good if it has no consecutive identical letters. For example, ABCACAD and ABABABAB are good strings. The string BCCCAD is no good because of the CCC.

Let \( y_n \) be the number of good \( n \)-strings (i.e., number of good strings of length \( n \)).

(a) Find \( y_n \) directly using the methods of Chapter 1.

(b) Find a recurrence relation and IC for \( y_n \).

(c) Check that your answer in (b) does satisfy the rr and IC from (a).

10. Let \( S_n = \{1,2,3,4,...,n\} \).

Let \( y_n \) be the number of subsets of \( S_n \) that contain no consecutive integers (call them good subsets).

For example some good subsets of \( S_{10} \) are the null set
\[
\phi \quad \text{(the null set)}
\]
\[
\{1,3,5\}
\]
\[
\{2,4,7,10\}
\]
\[
\{5\}.
\]

Some not-good subsets are \( \{3,4,7\} \), \( \{1,5,6\} \).

Find a recurrence relation and IC for \( y_n \). Then use them to find \( y_8 \).

Suggestion: Divide the good subsets of \( S_n \) into these two types: "contains \( n \)" and "doesn't contain \( n \)."

11. Your bank pays 6% interest a year, compounded monthly.

This means that the bank pays you 1/2% interest every month. The interest is added to your account so that the next month you get interest on your original principal plus the accumulated interest.

For example if you start with $400 in the bank then at the end of a month you get \( \frac{0.005 \times 400}{2} \approx 2 \) interest. At the end of the next month you'd get interest on $402.

You deposit $1000 initially and then starting with the next month, you deposit $200 a month.

Let \( u_n \) be the amount of money you have after \( n \) months.

Find a recurrence relation and IC for \( u_n \).

12. Let \( a_n \) be the number of ways in which \( 2n \) people (an even number of people) can be divided into \( n \) pairs. With this notation, \( a_8 \) is the number of ways of pairing up sixteen (not 8) people, \( a_7 \) is the number of ways of pairing up fourteen people

(a) Write down a formula for \( a_n \) immediately (using Chapter 1).

(b) What's wrong with the following way of getting a recurrence relation for \( a_n \).

Pick a pair. Can be done in \( \binom{2n}{2} \) ways. Now there are \( 2n-2 \) people left. They can be paired up in \( a_{n-1} \) ways. So \( a_n = \binom{2n}{2} a_{n-1} \).

(c) Get a recurrence relation and IC for \( a_n \) correctly.

(d) Check that the answer in (a) does satisfy the recurrence relation in (c).
13. You have \( n \) switches, in order from 1st to \( n \)-th, each of which can be ON or OFF. Here are the flipping rules.

The first switch can be flipped (ON to OFF or OFF to ON) any time.

But except for the first one, a switch can't be flipped (either way) until the preceding switch is ON and all the ones before that are OFF (e.g., you can't change the 10-th switch until the 9-th is ON and the first 8 are OFF).

For example, if you have 3 switches it takes 7 flips, as follows, to go from OFF OFF OFF to OFF OFF ON:

```
OFF OFF OFF
ON OFF OFF
ON ON OFF
OFF ON OFF
OFF ON ON
ON ON ON
ON OFF ON
OFF OFF ON
```

You start with \( n \) switches OFF and want to end up with the \( n \)-th switch ON and the others OFF. Let \( u_n \) be the number of flips it takes. Find a RR and IC for \( u_n \).
SECTION 4.2 SOLVING HOMOGENEOUS RECURRENCE RELATIONS

equivalent forms of a recurrence relation
The three equations
\[ y_n = 3y_{n-1} + 5y_{n-2} \]
\[ y_{n+2} = 3y_{n+1} + 5y_n \]
\[ y_{n+10} = 3y_{n+9} + 5y_{n+8} \]
are the same (second order) recurrence relation. Each says
\[ \text{term} = 3 \cdot \text{preceding term} + 5 \cdot \text{pre-preceding term} \]
Similarly, these two recursion relations with IC are the same:
\[ y_n = ny_{n-1} + 2^{n+3} \text{ for } n \geq 3 \text{ with IC } y_1 = 7, y_2 = 4 \]
\[ y_{n+1} = (n+1)y_n + 2^{n+4} \text{ for } n \geq 2 \text{ with IC } y_1 = 7, y_2 = 4 \]

warning
In the second version, \(2^{n+3}\) becomes \(2^{n+4}\).
It doesn't stay \(2^{n+3}\).
But the IC stay the same.

Each says
\[ a \text{ term} = \text{its term number} \times \text{preceding term} + 2^{3+\text{term number}} \]

iteration versus recursion

Given
\[ (1) \quad y_n = 4y_{n-1} - 4y_{n-2} \text{ for } n \geq 2 \text{ with IC } y_0 = 0, y_1 = 2 \]
you can find \(y_4\) (and similarly, any other \(y\) value) by successively finding \(y_2, y_3,\) and finally \(y_4\). The process is called iteration:
\[ y_2 = 4y_1 - 4y_0 = 4 \cdot 2 - 4 \cdot 0 = 8 \]
\[ y_3 = 4y_2 - 4y_1 = 4 \cdot 8 - 4 \cdot 2 = 24 \]
\[ y_4 = 4y_3 - 4y_2 = 4 \cdot 24 - 4 \cdot 8 = 64 \]
(That's what I did in examples 1 and 2 in the preceding section.)

You can also find \(y_4\) by going backwards until everything is expressed in terms of \(y_1\) and \(y_0\). That process is called recursion:
\[
\begin{align*}
y_4 &= 4y_3 - 4y_2 \\
     &= 4(4y_2 - 4y_1) - 4(4y_1 - 4y_0) \\
     &= 4 \left(4(4y_1 - 4y_0) - 4y_1 \right) - 4(4y_1 - 4y_0) \\
     &= 4 \left(4(8-0) - 4 \cdot 2 \right) - 4(8-0) = 64
\end{align*}
\]
Here's an iterative program (in Mathematica) to compute values of $y_n$:

```mathematica
y[n_] := Module[{y0 = 0, y1 = 2},
    Do[{y0, y1} = {y1, 4 y1 - 4 y0}, {i, 2, n}]; y1
]
y[4]
64
```

Here's a recursive program (the program calls itself) to compute values of $y_n$:

```mathematica
y[0] = 0;
y[1] = 2;
y[n_] := 4 y[n-1] - 4 y[n-2]
y[4]
64
```

The recursive program was much faster.

In this section and the next you'll learn how to solve certain kinds of recurrence relations and get a formula for $y_n$ so that you can jump to $y_4$ (or $y_{1000}$) immediately.

It will turn out that the solution to (1) is

$$y_n = n 2^n$$

Once you have this solution you can get $y_4$ directly:

$$y_4 = 4 \cdot 2^4 = 64$$

**Linear recurrence relations with constant coefficients**

Let $a,b,c,d$ be fixed constants. Let $f_n$ be a function of $n$, like $n^2$ or $2^n$ or 3.

Recurrence relations of the form

$$a y_{n+1} + b y_n = f_n \quad (1\text{st order})$$
$$a y_{n+2} + b y_{n+1} + c y_n = f_n \quad (2\text{nd order})$$
$$a y_{n+3} + b y_{n+2} + c y_{n+1} + d y_n = f_n \quad (3\text{rd order})$$

and so on are called *linear* recurrence relation with *constant coefficients*.

The function $f_n$ is called the *forcing function*.

If the forcing function is 0 then the rr is called *homogeneous*.

Here are some rr that fit the pattern:

- $3y_{n+4} + y_n = 0$ (4-th order, homog)
- $y_n = 3y_{n-1} - 2y_{n-2} + 3n$ (2nd order, non-homog (forcing function is 3n))
- $y_{n+10} = y_{n+5} - y_{n+4}$ (6-th order (not 10-th order), homog)

(alias $y_{n+6} - y_{n+1} + y_n = 0$)

Here are some rr that don't fit the pattern:

- $y_{n+2} - y_{n+1} + n y_n = 0$ (the coeffs aren't all constants -- the coeff of $y_n$ is $n$)
- $y_n + y_{n/2} = 4$ (the subscripts are off)

From now on, *I'll only consider linear recurrence relations with constant coefficients*. The methods for solving given here don't work otherwise.

Until IC are specified, a rr has many solutions (since it can start any way you like). A *general solution* to an n-th order rr is a solution containing n arbitrary constants (to ultimately be determined by n IC). It can be shown that a general solution includes all possible solutions.
superposition rule

If \( u_n \) is a solution of \( ay_n + by_{n+1} + cy_n = f_n \) and \( v_n \) is a solution of \( ay_n + by_{n+1} + cy_n = g_n \) then

\[ u_n + v_n \]
\[ k u_n \]

is a solution of \( ay_n + by_{n+1} + cy_n = f_n + g_n \)
\[ k u_n \]

proof of the \( u_n + v_n \) part of the superposition rule

Assume that \( u_n \) produces \( f_n \) and \( v_n \) produces \( g_n \) when substituted into \( ay_n + by_{n+1} + cy_n \).

Substitute \( u_n + v_n \) to see what happens:

\[
a(u_{n+2} + v_{n+2}) + b(u_{n+1} + v_{n+1}) + c(u_n + v_n) = f_n + g_n
\]

QED

special case of superposition for homog rr

If \( u_n \) is a solution of \( ay_n + by_{n+1} + cy_n = 0 \) and \( v_n \) is a solution of \( ay_n + by_{n+1} + cy_n = 0 \) then

\[ u_n + v_n \]
\[ ku_n \]

is a solution of \( ay_n + by_{n+1} + cy_n = 0 + 0 \)
\[ k u_n \]

In other words, a constant multiple of a solution to a homog rr is also a solution, and the sum of sols to a homog rr is also a solution.

finding the general sol to a second order homog rr

Look at this example to see the idea behind the solving process:

\[ y_{n+2} - 5y_{n+1} + 6y_{n} = 0 \]

for \( n \geq 2 \) with IC \( y_0 = 1, y_1 = 4 \).

As a guess, try a solution of the form

\[ y_n = \lambda^n \]

Substitute into the rr to see what \( \lambda \), if any, will make it work:

\[ \lambda^{n+2} - 5\lambda^{n+1} + 6\lambda^n = 0 \] (substitute)
\[ \lambda^2 - 5\lambda + 6 = 0 \] (cancel \( \lambda^n \)'s)
\[ (\lambda-3)(\lambda-2) = 0 \]
\[ \lambda = 3, \lambda = 2 \]

Now you know that \( 3^n \) and \( 2^n \) satisfy the rr. By the superposition principle for homogeneous rr, the general solution is

\[ y_n = A3^n + B2^n \]
Now plug the IC into (3) and determine A and B.

\[
\begin{align*}
1 &= A + B & \text{(set } n = 0, \ y_n = 1) \\
4 &= 3A + 2B & \text{(set } n = 1, \ y_n = 4)
\end{align*}
\]

The solution is \(A = 2, \ B = -1\). So the final answer is

\[
y_n = 2 \cdot 3^n - 2^n
\]

Here's the rule.

To solve the second order homogeneous recurrence relation

\[
ay_{n+2} + by_{n+1} + cy_n = 0 \quad \text{plus IC}
\]

or equivalently to solve

\[
ay_n + by_{n-1} + cy_{n-2} = 0 \quad \text{plus IC}
\]

first get a general solution (a solution with two arbitrary constants) as follows.

Find the roots of the equation

\[
a\lambda^2 + b\lambda + c = 0 \quad \text{(called the characteristic equation)}
\]

If \(\lambda = \lambda_1, \lambda_2\) (real unequal roots) then the general solution is

\[
y_n = A \lambda_1^n + B \lambda_2^n
\]

If \(\lambda = \lambda_1, \lambda_1\) (real equal roots) then the general solution is

\[
y_n = A \lambda_1^n + Bn \lambda_1^n \quad \text{(step up by } n) \quad \text{(proof omitted)}
\]

If the \(\lambda\)'s are not real, forget it. I'm skipping this case.

For example,

if \(\lambda = -2, \ 5\) then a general solution is \(y_n = A(-2)^n + B 5^n\)
if \(\lambda = 2, \ 2\) then a general solution is \(y_n = A2^n + Bn2^n\)
if \(\lambda = 1, \ 1\) then a general solution is \(y_n = A1^n + Bn1^n = A + Bn\)

Once you get the general solution, plug in the IC to determine the two constants.

example 1

To solve

\[
y_{n+2} + 3y_{n+1} + 2y_n = 0 \quad \text{with IC } \ y_1 = 0, \ y_2 = 6
\]

first solve the characteristic equation:

\[
\lambda^2 + 3\lambda + 2 = 0 \\
(\lambda + 2)(\lambda + 1) = 0 \\
\lambda = -2, -1
\]

(5) gen \(y_n = A(-2)^n + B(-1)^n \quad \text{warning} \quad \text{Don't write } -2^n \text{ when it should be } (-2)^n
\]

Now plug in the IC.
\[0 = A \cdot -2 + B \cdot -1\]
\[6 = A (-2)^2 + B (-1)^2\]
\[-2A - B = 0\]
\[4A + B = 6\]

The solution is \( A = 3, \ B = -6 \). So the final answer is

\[(6) \quad y_n = 3(-2)^n - 6(-1)^n\]

And here’s a partial check. By iteration you have

\[y_3 = -3y_2 - 2y_1 = -18\]
\[y_4 = -3y_3 - 2y_2 = 54 - 12 = 42\]
\[y_5 = -3y_4 - 2y_3 = -126 + 36 = -90\]

Using the solution in (6) you can get \( y_5 \) immediately:

\[y_5 = 3(-2)^5 - 6(-1)^5 = 3 \cdot -32 - 6 \cdot -1 = -90\]

solving higher and lower order homogeneous \( rr \)

Here’s how to generalize the procedure that worked for second order homogeneous \( rr \).

The \( 4 \)-th order recurrence relation equation

\[ay_{n+4} + by_{n+3} + cy_{n+2} + dy_{n+1} + ey_n = 0\]

has characteristic equation \( a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e = 0 \). If the roots are \( \lambda = 2, 3, -4, 6 \) then the general sol is

\[y_n = A2^n + B3^n + C(-4)^n + D6^n\]

If you have a 5-th order homogeneous \( rr \) and \( \lambda = 2, 2, 2, 2, 5 \) then a gen sol is

\[y_n = A2^n + Bn2^n + Cn^22^n + Dn^32^n + E5^n \quad \text{(keep stepping up by n)}\]

If you have a 5-th order homogeneous \( rr \) and \( \lambda = 1, 1, 1, 2, 2 \) then a gen solution is

\[y_n = A1^n + Bn1^n + Cn^21^n + D2^n + E2^n\]

The same idea works for first order recurrence relations. The \( rr \) \( ay_{n+1} + by_n = 0 \)

has characteristic equation \( a\lambda + b = 0 \). If \( \lambda = 4 \) then a gen sol is \( y_n = A4^n \).

**PROBLEMS FOR SECTION 4.2**

1. Find the general solution

(a) \( y_{n+2} - 3y_{n+1} - 10y_n = 0 \)  
(b) \( y_{n+2} + 3y_{n+1} - 4y_n = 0 \)

(c) \( 2y_{n+2} + 2y_{n+1} - y_n = 0 \)  
(d) \( y_n + 3y_{n-1} - 4y_{n-2} = 0 \)
2. Solve $y_{n+2} + 2y_{n+1} - 15y_n = 0$ with IC $y_0 = 0$, $y_1 = 1$

3. Given $y_{n+2} - y_{n+1} - 6y_n = 0$ with $y_0 = 1$, $y_1 = 0$
   (a) Before doing any solving, find $y_3$.
   (b) Now solve and find a formula for $y_n$.
   (c) Use the formula from part (b) to find $y_3$ again, as a check.

4. The Fibonacci sequence begins with $y_0 = 0$, $y_1 = 1$ and from then on each term is the sum of the two preceding terms. Find a formula for $y_n$.

5. Suppose $y_1 = 5$, $y_2 = 7$ and thereafter each term is the average of the two surrounding terms.
   (a) Write out some terms and see if you can find a formula for $y_n$ by guessing.
   (b) Find a formula for $y_n$ by solving a rr.

6. The $\lambda$s for $y_{n+2} - 6y_{n+1} + 9y_n = 0$ are 3 and 3. So $3^n$ and $n3^n$ are supposed to be solutions.
   Check that $n3^n$ really is a solution.

7. Solve $y_n = 4y_{n-1} - 4y_{n-2}$ with IC $y_0 = 0$, $y_1 = 2$.

8. Find a general solution to $y_{n+7} = 2y_{n+5}$

9. If the characteristic equation of a homog rr has the following roots, find a general sol.
   (a) $-3, 4, 4$
   (b) $5, 5, 5, 5, 2$
   (c) $1, 1, 1, 6, -1$

10. Go backwards and find a rr with the general sol $y_n = A + Bn + C2^n$.

11. Solve $y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3} = 0$ with $y_1 = 0$, $y_2 = 1$, $y_3 = 0$.

12. Solve by inspection and then solve again (overkill) with the methods of this section.
   (a) $y_{n+1} - y_n = 0$ with $y_1 = 4$
   (b) $ay_{n+2} + by_{n+1} + cy_n = 0$ with $y_0 = 0$, $y_1 = 0$

13. (a) Find the general solution to the rr $y_n = 5y_{n-1} - 6y_{n-2}$
   (b) Check that your solution really satisfies the rr (substitute it into the rr and see that it works).

14. Look at $y_n - 4y_{n+1} + 4y_n = 0$ with IC $y_0 = 7$, $y_1 = 5$.
   $\lambda = 2, 2$ so you should step and get the general solution $y_n = A2^n + Bn2^n$.
   What happens if you mistakenly think the general solution is $y_n = A2^n + B2^n$ (unstepped up) and plug in the IC
Section 4.3 Solving Nonhomogeneous Recurrence Relations

Solving a nonhomogeneous recurrence relation involves the idea behind the solving process:

I want to solve

\[ y_{n+2} - 5y_{n+1} + 6y_n = 10n + 3 \text{ for } n \geq 2 \]  

with IC

\[ y_0 = 1, \quad y_1 = 2 \]

It's a non-homogeneous recurrence because of the $10n + 3$ on the right-hand side.

As a guess, try a solution of the form

\[ p_n = Cn + D \]

Then

\[ p_{n+1} = C(n+1) + D \]
\[ p_{n+2} = C(n+2) + D \]

Substitute into (1) to see what values of $C$ and $D$ (if any) make the guess work:

\[ C(n+2) + D - 5(C(n+1) + D) + 6(Cn + D) = 10n + 3 \]

\[ 2Cn + 2D - 3C = 10n + 3 \]

We want this to be true for all $n$.

To get it, match the coeffs of corresponding terms on each side.

The $n$ coeff on the LHS of (4) is $2C$ and on the RHS it's $10$ so

\[ 2C = 10 \]

The constant term on the LHS of (4) is $2D - 3C$ and on the RHS it's $3$ so

\[ 2D - 3C = 3 \]

Solve to get $C = 5, \quad D = 9$.

Plug into (3) to get the particular solution

\[ p_n = 5n + 9 \]

But $p_n$ doesn't satisfy the IC. Here's how to turn it into a general solution, one with the required two arbitrary constants so that you can make it satisfy the IC.

First find the general homogeneous solution $h_n$, the solution to

\[ y_{n+2} - 5y_{n+1} + 6y_n = 0 \]

The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$, so $\lambda = 2, 3$ and

\[ h_n = A2^n + B3^n \]

The general solution $y_n$ to the nonhomogeneous recurrence relation in (1) is

\[ y_n = h_n + p_n \]
\[ = (6) + (5) \]
\[ = A2^n + B3^n + 5n + 9 \]

It's general because it contains two arbitrary constants. And, by superposition, it's still a solution to (1): $h_n$ produces $0$ and $p_n$ produces $f_n$ so the sum produces $0 + f_n$ which is $f_n$. 
Now that you have the general solution, plug in the IC.
To get $y_0 = 1$ set $n = 0$, $y_n = 1$ in (7) to get
$$1 = A + B + 9$$
To get $y_1 = 2$ set $n = 1$, $y_n = 2$ in (7) to get
$$2 = 2A + 3B + 5 + 9$$
The solution is $A = -12$, $B = 4$ so the final answer is
$$y_n = -12 \cdot 2^n + 4 \cdot 3^n + 5n + 9$$
Here's the general rule.

Here's how to solve
\[(8A)\quad ay_{n+2} + by_{n+1} + cy_n = f_n \text{ plus IC.}\]

Let $h_n$ be the general homog solution (the sol to $ay_{n+2} + by_{n+1} + cy_n = 0$).
Find it with the method of the preceding section.

Let $p_n$ be a particular nonhomog sol (a sol, with no constants, to the given nonhomog rr). This section will show you how to find one. Then
\[(8B)\quad y_n = h_n + p_n \text{ is a general nonhomog sol}\]
To get the specific solution to (8A), plug the IC into the general solution in (8B) to determine the constants.

finding a particular nonhomog solution for three types of $f_n$'s

Look at
$$ay_{n+2} + by_{n+1} + cy_n = f_n$$
Here's how to find a particular solution $p_n$ in three cases.
The letters $A,B,C,D$ stand for constants.

case 1  $f_n$ is a constant (i.e., there's a plain number on the righthand side)
Try $p_n = A$.
Substitute the trial $p_n$ into the rr to determine the constant $A$.

case 2  $f_n$ is a polynomial
Suppose $f_n = 7n^3 + 2n$ (a cubic). Try $p_n = An^3 + Bn^2 + Cn + D$ (a cubic not missing any terms even though $f_n$ was missing a few).
Substitute the trial $p_n$ into the rr to determine the constants $A$, $B$, $C$, $D$.
Similarly if $f_n = 3n^2 + 4n + 1$ (quadratic) then try $p_n = An^2 + Bn + C$ etc.

case 3  $f_n$ is an exponential
Suppose $f_n = 9 \cdot 2^n$. Then try $p_n = A2^n$.
Substitute the trial $p_n$ into the rr to determine the constant $A$. 
example 1
Look at
\[ y_{n+2} - 5y_{n+1} + 6y_n = 4^n \] for \( n \geq 3 \) with IC \( y_1 = 4, \, y_2 = 0 \).

(a) Find \( y_5 \) by iteration.

(b) Solve for \( y_n \).

(c) Find \( y_5 \) again using the solution from (b).

solution

(a) In general, \( y_{n+2} = 5y_{n+1} - 6y_n + 4^n \) so
\[
y_3 = 5y_2 - 6y_1 + 4 = 0 - 24 + 4 = -20 \quad \text{warning The exponent on 4 is 1 not 3.}
y_4 = 5y_3 - 6y_2 + 4^2 = -100 - 0 + 16 = -84
\]
y_5 = 5y_4 - 6y_3 + 4^3 = -420 + 120 + 64 = -236

(b) First find \( h_n \).
\[
\lambda^2 - 5\lambda + 6 = 0
\]
\[
(\lambda-2) (\lambda-3) = 0
\]
\[
\lambda = 2, 3
\]
\[
h_n = B 2^n + C 3^n
\]

Then look for a particular solution. Try
\[
p_n = A 4^n
\]
Substitute into the rr to see what value of A will make it work. You need
\[
A 4^{n+2} - 5A 4^{n+1} + 6A 4^n = 4^n
\]
Rewrite the left side to display all the \( 4^n \) terms:
\[
A 4^2 4^n + 5A 4^n + 6A 4^n = 4^n
\]
\[
16A 4^n - 20A 4^n + 6A 4^n = 4^n
\]
\[
2A 4^n = 4^n
\]
The coeff of the \( 4^n \) term on the LHS is 2A and the coeff of \( 4^n \) term on the RHS is 1 so
\[
2A = 1, \quad A = \frac{1}{2}
\]
\[
p_n = \frac{1}{2} 4^n
\]
(8) \quad \text{gen } y_n = h_n + p_n = B 2^n + C 3^n + \frac{1}{2} 4^n

Now plug in the IC.
To get \( y_1 = 4 \) you need \( 4 = 2B + 3C + 2 \)
To get \( y_2 = 0 \) you need \( 0 = 4B + 9C + 8 \)
The solution is \( B = 7, \, C = -4 \) so the final answer is
(9) \quad y_n = 7 \cdot 2^n - 4 \cdot 3^n + \frac{1}{2} 4^n.

warning The \( y_n \) in (9) is the specific \( y_n \) that satisfies the rr and IC.
The \( y_n \) in (8) is the general solution to the rr so you should label it as "gen" to distinguish between the two.

(c) \[ y_5 = 7 \cdot 2^5 - 4 \cdot 3^5 + \frac{1}{2} 4^5 = 224 - 972 + 512 = -236 \]
warning
1. If the forcing function is \( n \) or \( 5n \) of \(-6n\) try \( p_n = An + B \), not just plain \( An \).

Similarly if the forcing function is \( 3n^2 \) or \( n^2 + 3 \) or \( 9n^2 + n \), try \( p_n = An^2 + Bn + C \), a quadratic not missing any terms.

2. Determine the various constants at the appropriate stage.

For a nonhomog rr with IC, first find \( h_n \) (containing constants). Then find \( p_n \) (the trial \( p_n \) contains constants but they must be immediately determined to get the genuine \( p_n \)).

The general solution is \( y_n = h_n + p_n \) (contains constants in the \( h_n \) part).

Use the IC to determine the constants in the gen sol.

Don't use the IC on \( h_n \) alone in the middle of the problem.

stepping up \( p_n \)

Look at \( ay_{n+2} + by_{n+1} + cy_n = f_n \)

Here are some exceptions to the rules in cases 1-3 above

case 1
Suppose \( f_n = 6 \). Ordinarily you try \( p_n = A \).

But if \( A \) is already a homog sol (i.e., if one of the \( \lambda \)'s was 1) then try \( p_n = An \) (step up).

If \( A \) and \( Bn \) are both homog sols (i.e., \( \lambda = 1, 1 \)) then try \( p_n = An^2 \) (step up some more).

If the rr was 3rd order and \( A, Bn, Cn^2 \) are all homog sols (i.e., \( \lambda = 1, 1, 1 \)) try \( p_n = An^3 \) etc.

case 2
Suppose \( f_n = 6n^2 + 3 \). Ordinarily you try \( p_n = An^2 + Bn + C \).

But if \( C \) is a homog sol (i.e., one of the \( \lambda \)'s is 1) then try \( p_n = n(An^2 + Bn + C) = An^3 + Bn^2 + Cn \) (step up).

If \( C \) and \( Bn \) are both homog sols (i.e., \( \lambda = 1, 1 \)) try \( p_n = n^2(An^2 + Bn + C) = An^4 + Bn^3 + Cn^2 \) etc.

case 3
Suppose \( f_n = 9 \cdot 2^n \). Ordinarily you try \( p_n = A \cdot 2^n \).

But if \( 2^n \) is a homog sol (i.e., one of the \( \lambda \)'s is 2) try \( p_n = An \cdot 2^n \) (step up).

If \( 2^n \) and \( n2^n \) are both homog sols (i.e., \( \lambda = 2, 2 \)) try \( p_n = An^2 \cdot 2^n \) etc.

If the rr was 3rd order and \( 2^n, n2^n, n^22^n \) are all homog sols (i.e., \( \lambda = 2, 2, 2 \)) try \( p_n = An^3 \cdot 2^n \) etc.

example 2

Solve \( y_{n+2} - 2y_{n+1} + y_n = 2n + 3 \) with IC \( y_0 = 1, y_1 = 3 \)

First find \( h_n \).

\[
\lambda^2 - 2\lambda + 1 = 0, \quad \lambda = 1, 1 \\
h_n = A + Bn
\]
Ordinarily you would try \( p_n = Cn + D \). But \( D \) and \( Cn \) are already homog solutions so step up and try

\[
p_n = n^2 \ (Cn + D)
\]

Substitute into the rr.

\[
\begin{align*}
(n+2)^2 \left[ C(n+2) + D \right] - 2(n+1)^2 \left[ C(n+1) + D \right] + n^2 \left[ Cn + D \right] &= 2n+3 \\
6Cn + 6C + 2D &= 2n + 3
\end{align*}
\]

When you collect terms, the \( n^3 \) terms drop out, the \( n^2 \) terms drop out and you're left with

\[
6Cn + 6C + 2D = 2n + 3
\]

To make this true for all \( n \), make the coeffs of like terms match on each side.

Equate the coeffs of \( n \): \( 6C = 2 \), \( C = \frac{1}{3} \)

Equate the constant terms: \( 6C + 2D = 3 \), \( D = \frac{1}{2} \)

So

\[
p_n = n^2 \left( \frac{1}{3}n + \frac{1}{2} \right)
\]

\[
genn y_n = A + Bn + n^2 \left( \frac{1}{3}n + \frac{1}{2} \right)
\]

Now plug in the IC.

To get \( y_0 = 1 \) you need \( 1 = A \)

To get \( y_1 = 3 \) you need \( 3 = A + B + \left( \frac{1}{3} + \frac{1}{2} \right) \)

So \( A = 1 \), \( B = \frac{7}{6} \) and the final answer is

\[
y_n = 1 + \frac{7}{6}n + n^2 \left( \frac{1}{3}n + \frac{1}{2} \right)
\]

**warning** (recurrence relations not included in Sections 4.2 and 4.3)

These solving methods are only for

(9) \( ay_{n+2} + by_{n+1} + cy_n = f_n \) where \( a,b,c \) are constants

and similar equations of higher or lower order. The methods don't work on

(10) \[
\begin{align*}
n^3 y_{n+2} + n y_{n+1} + 6y_n &= 8n^2
\end{align*}
\]

where the coefficients contain \( n \)'s instead of being constant.

And they don't work on equations like

(11) \[
\begin{align*}
y_n y_{n-1} &= y_{n-2} + 3
\end{align*}
\]

and

(12) \[
\begin{align*}
y_{n+2} - 5 y_{n+1} + 2 y_n &= n
\end{align*}
\]

which don't have the pattern in (9) at all.

For the equations in (10)-(12), you can find values of \( y_n \) by iteration or recursion once you have IC, but, in the context of this course, you can't get a formula for \( y_n \).
PROBLEMS FOR SECTION 4.3

1. Solve (a) \( y_{n+2} - y_{n+1} - 2y_n = 1 \) for \( n \geq 3 \) with IC \( y_1 = 1, \ y_2 = 3 \)
   (b) \( y_{n+2} + 2y_{n+1} - 15y_n = 6n + 10 \) for \( n \geq 2 \) with IC \( y_0 = 1, \ y_1 = -\frac{1}{2} \)

2. (a) Find a general sol to \( y_{n+2} - 3y_{n+1} + y_n = 10 \cdot 4^n \).
   (b) Rewrite the rr from part (a) so that it involves \( y_n, y_{n-1} \) and \( y_{n-2} \) and then find the general solution again.

3. Solve \( y_{n+2} - y_{n+1} - 6y_n = 18n^2 + 2 \) for \( n \geq 2 \) with \( y_0 = -1, \ y_1 = 0 \)

4. Given \( y_n = 2y_{n-1} + 6n \) for \( n \geq 2 \) with \( y_1 = 2 \).
   (a) Use iteration to find \( y_4 \).
   (b) Find a formula for \( y_n \).
   (c) Use the formula from (b) to find \( y_4 \) again as a check.

5. Find a general solution.
   (a) \( y_{n+1} + 2y_n = 4 \)  (b) \( y_{n+1} + 2y_n = 4^n \)

6. Given the following forcing functions and roots of the characteristic equ. what \( p_n \) would you try.

<table>
<thead>
<tr>
<th>forcing function ( f_n )</th>
<th>( \lambda's )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( n^4 + 2n )</td>
<td>( \pm 2 )</td>
</tr>
<tr>
<td>(b) ( n^4 + 2 )</td>
<td>( 1,1,1,1,3 )</td>
</tr>
<tr>
<td>(c) ( 6 \cdot 2^n )</td>
<td>( 1,6 )</td>
</tr>
<tr>
<td>(d) ( 6 \cdot 2^n )</td>
<td>( 3,6 )</td>
</tr>
<tr>
<td>(e) ( 3^n )</td>
<td>( 3,3 )</td>
</tr>
</tbody>
</table>

7. Solve \( 2y_{n+1} - y_n = \left( \frac{1}{2} \right)^n \) for \( n \geq 2 \) with \( y_1 = 2 \).

8. Solve \( y_{n+2} - 2y_{n+1} + y_n = 1 \) for \( n \geq 2 \) with \( y_0 = 1, \ y_1 = \frac{1}{2} \).

9. Let \( S_n \) be the sum of the first \( n \) squares.
   For example \( S_5 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55 \).
   Find a formula for \( S_n \) by solving a recurrence relation.

10. (a) For \( y_{n+2} - 3y_{n+1} + 2y_n = 6 \cdot 2^n \) you have \( h_n = A2^n + B \) so for \( p_n \) you should step up and try \( p_n = Cn2^n \). What happens if you forget to step up (you dope) and try \( p_n = A \cdot 2^n \).

    (b) For \( 2y_{n+2} + 3y_{n+1} + 4y_n = 18n \) you should try \( p_n = An + \) \( B \). What happens if you violate the warning and try \( p_n = An \).
11. To get a particular solution to $y_{n+2} - 2y_{n+1} - 3y_n = 5n^2$ you should try
$p_n = An^2 + Bn + C$.

Suppose you try $p_n = An$ instead, like this. What's wrong with it.

I need

$A(n+2) - 2A(n+1) - 3An = 5n^2$

$An + 2A - 2An - 2A - 3An = 5n^2$  \hspace{1cm} \text{(multiply out)}

$-4An = 5n^2$ \hspace{1cm} \text{(collect terms)}

$A = -\frac{5}{4}n$ \hspace{1cm} \text{(solve for A)}

So $p_n = -\frac{5}{4}n^2$ QED
REVIEW PROBLEMS FOR CHAPTER 4

1. Find a general solution to $y_{n+2} - 9y_n = 56n^2$.

2. Solve $2y_{n+1} + 4y_n = 6 \cdot 7^n$ for $n \geq 2$ with IC $y_1 = 5$.

3. Find a general solution to $y_{n+2} + 5y_{n+1} - y_n = 6$.

4. Use a difference equation to find a formula for the sum of the first $n$ integers (i.e., a formula for $1 + 2 + 3 + \ldots + n$).

5. Find a general solution to $y_{n+2} - 9y_n = 5 \cdot 3^n$.

6. Suppose you start to solve

$$y_{n+2} - 2y_{n+1} = 0$$

like this:

$$\lambda^2 - 2\lambda = 0, \quad \lambda = 0, 2$$

and the general solution is

$$y_n = A \cdot 0^n + B \cdot 2^n = B \cdot 2^n$$

and you suddenly lost one of your two constants (which you would need if you were going to satisfy two IC). You've always led a good clean life. How could something like this happen to you and what are you going to do about it?

7. (The tower of Hanoi) The game begins with $n$ rings in increasing size on peg 1. The idea is to transfer them all to peg 2 but never place a larger ring on top of a smaller ring at any stage of the game. Rings may be moved temporarily to peg 3 as they eventually go from peg 1 to peg 2.

Let $y_n$ be the minimum number of moves required in a game with $n$ rings (the minimum number of moves in which the transfer from peg 1 to peg 2 can be accomplished, including moves to and from peg 3).

(a) Write a difference equation for $y_n$ and find IC.

(b) Solve.
8. Here are some examples of binary trees of height 3 (call them 3-trees)

A binary tree begins with one node and then each node has either a left child or a right child or both or neither.

There is one binary tree of height 0, the tree consisting of one vertex and no edges.

There are three binary trees of height 1 (i.e., 1-trees):

Left Child Right Child Both Children

Let \( y_n \) be the number of binary trees of height \( n \), i.e., the number of \( n \)-trees.

(a) Write a difference equation for \( y_n \).

Suggestion: Think of an \( n \)-tree as starting with one of the 1-trees followed by an \( (n-1) \)-tree.

(b) Use the difference equation from (a) to find \( y_2 \) and \( y_3 \).

(c) Can you solve the difference equation in the context of this course.

9. A street has \( n \) parking spaces along one side (for parallel parking).

A limo takes 2 parking places and a stretch limo takes 3 parking places.

Find a difference equation and IC for the number of ways in which the \( n \) spaces can be completely filled with limos and stretch limos.

For example if the street is 5 spaces long then there are two ways:

10. You are going to take out a 3-year $10,000 car loan to be paid off in equal monthly installments of \( D \) dollars. The interest rate is 12% a year (1% monthly) The problem is to find \( D \); i.e., what should the monthly payment be so that the loan is paid off in 3 years.

Suggestion Let \( y_n \) be the unpaid balance after \( n \) months. Find a difference equation and IC for \( y_n \) and solve for \( y_n \). Then use the "end" condition \( y_{36} = 0 \) to determine \( D \).