CHAPTER 1 COUNTING

SECTION 1.1 THE MULTIPLICATION RULE

the multiplication rule
Suppose you have 3 shirts (blue, red, green) and 2 pairs of pants (checked, striped). The problem is to count the total number of outfits.

The tree diagram (Fig 1) shows all possibilities: there are \(2 \times 3 = 6\) outfits.

Instead of drawing the tree which takes a lot of space, think of filling a pants slot and a shirt slot (Fig 2). The pants slot can be filled in 2 ways, the shirt slot in 3 ways and the total number of outfits is the product \(2 \times 3\).

If an event takes place in successive stages (slots), decide in how many ways each slot can be filled and then multiply to get the total number of outcomes.

example 1
You take a test with five questions. Each question can be answered True, False or left Blank. For instance some responses are

TTFBF
TFBBF
FTBBF

How many responses are there?

solution
There are five slots to fill (the five questions). Each can be filled in 3 ways (with T, F or B). So the number of responses is \(3 \times 3 \times 3 \times 3 \times 3 = 3^5\).

example 2
The total number of 4-letter words is \(26 \times 26 \times 26 \times 26\) (each spot in the word can be filled in 26 ways).
example 3
The total number of 4-letter words that can be formed from 26 different scrabble chips is $26 \times 25 \times 24 \times 23$ (the first spot can be filled in 26 ways, the second in only 25 ways since you have only 25 chips left, etc.).

$7 \times 7 \times 7$ versus $7 \times 6 \times 5$
The answer $7 \times 7 \times 7$ is the number of ways of filling 3 slots from a pool of 7 objects where each object can be used over and over again; this is called sampling with replacement.
The answer $7 \times 6 \times 5$ is the number of ways of filling 3 slots from a pool of 7 objects where an object chosen for one slot cannot be used again for another slot; this is called sampling without replacement.

example 4
How many 4-letter words do not contain the letter Z.

solution
There are 4 slots and each can be filled in 25 ways. Answer is $25^4$.

example 5
How many 4-letter words begin with TH.

solution
The first two letters are determined so there are only two slots to fill, the last two letters in the word. Each can be filled in 26 ways so the answer is $26 \times 26$.

counting the list "all, none, any combination"
Suppose you buy a car and are offered the options of Air conditioning, Power windows, Tinted glass, FM radio, Shoulder harnesses. Here are some of the possibilities:
1. none of the options
2. all of the options
3. just A
4. just P
5. just T
6. A and T
7. A, FM and T etc.

The problem is to count the total number of possibilities.
Each of the five options is a slot which can be filled in two ways (yes or no).
For example, buying A and T goes with

<table>
<thead>
<tr>
<th>yes</th>
<th>no</th>
<th>yes</th>
<th>no</th>
</tr>
</thead>
<tbody>
<tr>
<td>A slot</td>
<td>P slot</td>
<td>T slot</td>
<td>FM slot</td>
</tr>
<tr>
<td>S slot</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The answer is $2 \times 2 \times 2 \times 2 \times 2 = 2^5$.

(2) Given n objects, there are $2^n$ ways to choose all, none, any combination

example 6
There are six books $B_1, \ldots, B_6$ you are considering taking on your vacation. You can take them all; you can take none of them; you can take all except $B_1$ etc. How many possibilities are there.

solution
The number of possibilities is $2^6$. 
the number of subsets of a given set
If \( Q = \{a, b, c\} \) then \( Q \) has 8 subsets:

\[ \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\} \text{ and the null set } \emptyset \]

Look at the set \( P = \{a_1, \ldots, a_6\} \).
To count the subsets think of 6 slots named \( a_1, \ldots, a_6 \), each to be filled with IN or OUT. For example the subset \( \{a_2, a_3, a_5\} \) corresponds to

\[ \begin{array}{cccc}
\text{OUT} & \text{IN} & \text{OUT} & \text{IN} \\
\end{array} \]

The null subset \( \emptyset \) corresponds to

\[ \begin{array}{cccccccc}
\text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} \\
\end{array} \]

So there are \( 2^6 \) subsets of \( P \). In general:

\[(3) \quad \text{A set with } n \text{ members has } 2^n \text{ subsets}\]

choosing the best slots
Seven people \( P_1, \ldots, P_7 \) arrive in a city and they can stay at any of three hotels \( H_1, H_2, H_3 \). For instance, here are some of the possibilities.

1. all people in \( H_1 \)
2. \( P_2 \) in \( H_1 \); others in \( H_2 \)
3. \( P_1, P_5 \) in \( H_1 \); \( P_6 \) in \( H_2 \); others in \( H_3 \)

I want to find the number of possibilities.

false start Try the hotels as slots, starting with \( H_1 \).
One possibility is that everyone stays at \( H_1 \).
Another possibility is that no one stays at \( H_1 \).
In fact any combination of the seven people could stay at \( H_1 \).
So by (2), the \( H_1 \) slot can be filled in \( 2^7 \) ways.
But then the \( H_2 \) slot depends on how the \( H_1 \) slot was filled.
If everyone stays at \( H_1 \) then there is only one possibility for the \( H_2 \) slot (no one stays there).
If no one stays at \( H_1 \) then by (2) again, there are \( 2^7 \) possibilities for \( H_2 \).
If \( P_1, \ldots, P_6 \) stay at \( H_1 \) then there are two possibilities for \( H_2 \) (\( P_7 \) or no one).
So there is no way to fill in a single value for the second slot.

good start Use the people as slots (since each person must get a hotel but not vice versa).
Each person-slot can be filled in 3 ways (with one of the 3 hotels).
The answer is \( 3^7 \).
example 7
An agency has 10 available foster families $F_1, \ldots, F_{10}$ and 6 children $C_1, \ldots, C_6$ to place. In how many ways can they do this if
(a) no family can get more than one child
(b) a family can get more than one child

solution
(a) Use the children as slots (since each child must get a family but not every family must get a child). Answer is $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$.
(b) Use the children as slots. Answer is $10^6$.

example 8
Look at all 6-digit numbers which do not repeat a digit (e.g., 897563, 345678) Don't count 045678 which is really only a 5-digit number; i.e., leading 0's are not allowed.

How many are there?

solution
Fill six slots from the 10 available digits. But don't let the first slot be 0. Answer is $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$.

using total – opposite
Suppose you want to count 4-letter words which contain repeated letters (e.g., ABZA, AACA, ZOGO, EEEE). It's easier to count the opposite (complementary) event, that is to count words without repeats:

$$\text{words with repeats} = \text{total words} - \text{words without repeats}$$
$$= 26^4 - 26 \cdot 25 \cdot 24 \cdot 23$$

PROBLEMS FOR SECTION 1.1 (see the back of the notes for the solutions)

1. The library has 4 golf books, 3 tennis books and 6 swimming books. In how many ways can you bring home three books, one in each sport.

2. How many 6 digit numbers begin and end with 8.

3. There are 50 states and 2 U.S. Senators for each state. How many committees can be formed consisting of one Senator from each of the 50 states.

4. Ten people share a house. They need a cook for each day of the coming week. In how many ways can this be done if
(a) a person can be assigned to cook on more than one day
(b) a person can't get stuck with the job for more than one day

5. In how many ways can 7 people ride a long toboggan if only John and Mary are skilled enough to steer (the one who steers is the one up front).

6. Seven people fill out application forms. The company will hire those who turn out to be qualified (e.g., they might end up hiring John and Mary and rejecting the rest). How many possibilities are there.

7. In how many ways can the symbols $A, B, C, D, E, E, E, E$ be arranged in a line so that no $E$ is next to another $E$.

8. Four students are going to enroll in college and there are 11 school available. One possibility is that John, Mary, and Bill go to $S_5$ while Betty goes to $S_{11}$.
(a) How many possibilities are there.
(b) How many possibilities are there if the four students must go to four different schools.
9. In how many ways can the letters A, B, C, d, e, f, g be lined up if the capitals must come first.

10. (a) Three hundred people \(P_1, \ldots, P_{300}\) make plane reservations but don't necessarily show up.
    One possibility is the \(P_1\) and \(P_2\) show but the others are no-shows.
    Another possibility is that \(P_1\) and \(P_{300}\) show but the others are no-shows.
    How many possibilities are there.
    (b) Suppose the airline doesn't care about the names of the people who show up but just how many do. In that case the two possibilities listed in part (a) don't count as distinct since they both amount to 2 shows and 298 no-shows.
    From this point of view how many possibilities are there.

11. Given ten boxes \(B_1, \ldots, B_{10}\) and 17 balls \(b_1, \ldots, b_{17}\). Toss the balls into the boxes.
    One possibility for instance is
    \[B_1 \text{ gets } b_1 - b_{16}, \quad B_2 \text{ gets } b_{17} \quad \text{(other boxes are empty)}\]
    How many possibilities are there.

12. (a) A hotel has 7 vacant single rooms \(R_1, \ldots, R_7\). If five travelers \(T_1, \ldots, T_5\) arrive, in how many ways can the desk clerk assign rooms.
    (b) Repeat part (a) but with nine travelers (so that 2 will be turned away).

13. How many 4-letter words
    (a) begin with Z
    (b) begin and end with Z
    (c) begin with Z and then contain no more Z's

14. The problem is to count six-digit even numbers with no repeated digits (e.g., 123456, 948712).
    The number can't have a leading zero (can't be 045678) since then it really is only a 5-digit number, and it must end with an even digit to be an even number.
    (a) Look at this attempt.
    The first digit can be picked in 9 ways (any of the 10 digits excluding 0).
    The next 4 digits can be picked in \(9 \cdot 8 \cdot 7 \cdot 6\) ways.
    So far so good but what's the problem in picking the last digit?
    (b) Try again!
    The last digit can be picked in 5 ways (there are 5 even digits).
    The middle 4 digits can be picked in \(9 \cdot 8 \cdot 7 \cdot 6\) ways.
    So far so good but what's the difficulty in picking the first digit.
    (c) Try again by breaking the original problem into cases to avoid the difficulties that turned up in (a) and (b).

15. In a certain programming language a name may consist of a single letter or a letter followed by up to 6 symbols which may be letters or digits (e.g., X, Z, R2d2, XX1234). How many names are there.
SECTION 1.2 PERMUTATIONS AND COMBINATIONS

review of factorials

\[ n! = n(n-1)(n-2)\ldots 1 \] if \( n \) is a positive integer (definition)

0! = 1 (definition)

For example

\[ 6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720 \]

\[ \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 10\cdot 9\cdot 8 = 720 \]

\[ 5! \times 6 = 6! \]

a deck of cards

A deck of cards contains 13 faces (values), namely Ace, King, Queen, Jack, 10, 9, ..., 2 and four suits, namely Spades, Diamonds, Hearts, Clubs.

The pictures are Ace, King, Queen, Jack.

A poker hand contains 5 cards.

A bridge hand contains 13 cards.

permutations (lineups)

Consider 5 objects \( A_1, \ldots, A_5 \).

To count all possible lineups (permutations) such as \( A_1A_5A_4A_3A_2 \), \( A_5A_3A_1A_2A_4 \) etc., think of filling 5 slots, one for each position in the line, and note that once an object has been picked, it can't be picked again. The total number of lineups is \( 5\cdot 4\cdot 3\cdot 2\cdot 1 \) or \( 5! \).

In general:

\[ n \text{ objects can be permuted in } n! \text{ ways} \]

Suppose you want to find the number of permutations of size 5 chosen from the 7 items \( A_1, \ldots, A_7 \), e.g., \( A_1A_4A_2A_6A_3 \), \( A_1A_6A_7A_2A_3 \).

There are 5 places to fill so the answer is \( 7\cdot 6\cdot 5\cdot 4\cdot 3 \), or, in more compact notation, \( 7!/2! \).

example 1

Ten people can be lined up (permuted) in \( 10! \) ways.

A permutation of 4 out of the 10 can be chosen in \( 10\cdot 9\cdot 8\cdot 7 \) ways.

matchups of same-size groups

Given 4 adults \( A_1, \ldots, A_4 \) and 4 children \( C_1, \ldots, C_4 \). Here are some possible matchups.

1. \( A_1 & C_4, \ A_2 & C_2, \ A_3 & C_3, \ A_4 & C_1 \)
2. \( A_1 & C_3, \ A_2 & C_1, \ A_3 & C_4, \ A_4 & C_2 \)

To count the total number of matchups take either the adults or the children to be the slots. If the adults are the slots then \( A_1 \) can be filled with any of 4 children, \( A_2 \) with any of the 3 remaining children etc. Answer is \( 4\cdot 3\cdot 2\cdot 1 \).

The number of ways to match two groups of size \( n \) is \( n! \).
combinations (committees)

Order doesn't count in a committee; it does count in a permutation.

\[ A_1, A_{17}, A_2, A_{12} \] is the same committee as \[ A_{12}, A_1, A_{17}, A_2 \] but \[ A_1 A_{17} A_2 A_{12} \] and \[ A_{12} A_1 A_{17} A_2 \] are different permutations.

The symbols \( \binom{n}{r} \), called a binomial coefficient, stands for the number of committees of size \( r \) that can be chosen from a population of size \( n \), or equivalently, the number of combinations of \( n \) things taken \( r \) at a time. Its value is given by

\[
\binom{n}{r} = \frac{n!}{r! (n-r)!}
\]

For example the number of 4-person committees that can be formed from a group of 10 is

\[
\binom{10}{4} = \frac{10!}{4! 6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210
\]

notation \( \binom{n}{r} \) (pronounced \( n \) over \( r \)) is also written as \( C(n,r) \).

why (1) works

I'll do it for a committee of size 4 from a population of \( P_1, \ldots, P_{17} \).

The number of lineups of size 4 from \( P_1, \ldots, P_{17} \) is \( 17 \cdot 16 \cdot 15 \cdot 14 \). To get the committee answer, compare lineups with committees.

\[
\begin{align*}
\text{list of committees} & \quad \text{list of lineups} \\
(a) \ P_1, P_7, P_8, P_9 & \quad (a_1) \ P_1 P_7 P_8 P_9 \\
(a_2) \ P_7 P_1 P_8 P_9 & \\
(a_{24}) \ P_9 P_9 P_1 P_8 & \\
(b) \ P_3, P_4, P_{12}, P_6 & \quad (b_1) \ P_3 P_4 P_{12} P_6 \\
(b_2) \ P_4 P_{12} P_3 P_6 & \\
(b_{24}) \ P_{12} P_3 P_4 P_6 & \\
\text{etc.} & \\
\end{align*}
\]

Each committee gives rise to \( 4! \) lineups so

\[
\frac{\text{number of committees} \times 4!}{4！} = \text{number of lineups}
\]

\[
\frac{\text{number of committees}}{4！} = \frac{17 \cdot 16 \cdot 15 \cdot 14}{4！} = \frac{17！}{4！ 13！} \quad \text{QED}
\]

example 2

A library has 100 books. In how many ways can you check out 3 of them.

solution

Each threesome is a committee of 3 books out of 100. The answer is

\[
\binom{100}{3} = \frac{100！}{3！ 97！} = \frac{100 \cdot 99 \cdot 98}{3 \cdot 2 \cdot 1} = 161700
\]
some properties of \( \binom{n}{r} \)

(2) \( \binom{n}{r} = \binom{n}{n-r} \)

For example \( \binom{17}{4} = \binom{17}{13} \).

This holds because picking a committee of size 4 automatically leaves a committee of 13 leftovers and vice versa so the list of committees of size 4 and the list of committees of size 13 are the same length.

In particular \( \binom{17}{4} \) and \( \binom{17}{13} \) are each \( \frac{17!}{4!13!} \).

(3) \( \binom{n}{1} = \binom{n}{n-1} = n \)

There are \( n \) committees of size 1 from a population of \( n \).

(4) \( \binom{n}{n} = 1 \)

There is one committee of size \( n \) from a population of \( n \).

(5) \( \binom{n}{0} = 1 \)

\( \binom{0}{0} = \frac{n!}{n! \cdot 0!} = 1 \) because 0! is defined as 1.

example 3

A poker hand is a committee of 5 chosen from 52 so there are \( \binom{52}{5} \) poker hands.

eexample 4

How many poker hands contain the queen of spades.

solution

Since the hand must contain the spade queen, the problem amounts to picking the rest of the hand, a committee of size 4 from the 51 cards left in the deck.

Answer is \( \binom{51}{4} \).

example 5

How many poker hands do not contain the queen of hearts.

solution

method 1 Choose 5 cards from the 51 non-queen-of-hearts. Answer is \( \binom{51}{5} \).

method 2

number of hands without queen of hearts

= total number of hands - number with queen of hearts

= \( \binom{52}{5} \) - \( \binom{51}{4} \) (from examples 3 and 4)

example 6

How many poker hands contain only hearts.

solution

Choose a committee of 5 hearts from the 13 hearts in the deck. Answer is \( \binom{13}{5} \).

example 7

How many poker hands contain only hearts and include the king of hearts.

solution

Choose 4 more hearts from the remaining 12 to go with the king. Answer is \( \binom{12}{4} \).
example 8
How many poker hands contain only pictures but don't include the queen of spades.

solution
Choose 5 cards from the 15 non-spade-queen pictures
Answer is \( \binom{15}{5} \).

example 9
How many poker hands contain 4 spades and the 2 of hearts.

solution
Choose 4 spades from 13. Answer is \( \binom{13}{4} \).

9 × 8 × 7 versus \( \binom{9}{3} \)
Both count the number of ways in which 3 things can be chosen from a pool of 9. But 9 × 8 × 7 corresponds to choosing the 3 things to fill slots (such as president, vice-president, secretary) while \( \binom{9}{3} \) corresponds to the case where the 3 things are not given names or distinguished from one another in any way (such as 3 co-chairs).

From a committee/lineup point of view, 9 × 8 × 7 counts lineups of 3 from a pool of 9 while \( \binom{9}{3} \) counts committees.

notation
The symbol \( P(9,3) \) is often used for the number of permutations of 3 out of 9. So
\[ P(9,3) = 9 \cdot 8 \cdot 7 = \frac{9!}{6!} \]
The symbols \( C(9,3) \) and \( \text{Binomial}[9,3] \) are often used for the number of committees of 3 out of 9. So
\[ C(9,3) = \binom{9}{3} = \frac{9!}{3! \cdot 6!} \]

example 10
Given 3 girls and 7 boys. How many ways can 3 marriages be arranged.
For example, one possibility is
\[ G_1 \text{ & } B_2 , \quad G_2 \text{ & } B_7 , \quad G_3 \text{ & } B_4 \]

solution
method 1 Take the girls as slots. Answer is 7 · 6 · 5.

method 2 First pick 3 of the 7 boys. Then match the 3 chosen boys with the 3 girls. Answer is \( \binom{7}{3} \cdot 3! \).

The two answers agree since \( \binom{7}{3} \cdot 3! = \frac{7!}{3! \cdot 4!} \cdot 3! = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 \)

PROBLEMS FOR SECTION 1.2
1. An exam has 20 questions and you are supposed to choose any 10.
   (a) In how many ways can you make your choice.
   (b) In how many ways can you choose if you have to answer the first four questions as part of the 10.

2. How many poker hands contain
   (a) only spades
   (b) only pictures
   (c) no hearts
   (d) the ace of spades and the ace of diamonds (other aces allowed too)
   (e) the ace of spades and the ace of diamonds and no other aces
3. Four men check their hats and the hats are later returned to the men at random. One possibility is

\[ M_1 \text{ gets } H_2, \ M_2 \text{ gets } H_3, \ M_3 \text{ gets } H_1, \ M_4 \text{ gets } H_4 \]

(a) How many possibilities are there.
(b) How many possibilities end with some dissatisfaction, i.e., some mismatch.

4. From a population of 10 men and 7 women how many ways are there to
(a) pick a King and Queen
(b) pick a president and vice-president
(c) pick two delegates
(d) award the good fellowship medal and the achievement medal

5. Compute
(a) \( \frac{7!}{5!} \) (b) \( \binom{7}{5} \) (c) \( \frac{8!}{5!2!} \) (d) \( \binom{12345}{1} \) (e) \( \binom{12345}{0} \) (f) \( \binom{12344}{1} \) (g) \( \binom{12345}{12345} \)

6. Show that \( \binom{10}{3} \binom{7}{3} \) is the same as \( \frac{5!}{4!3!} \frac{4!}{3!8} \).

7. Simplify \( \binom{n+m-1}{n-1} \frac{n+m}{n} \).

8. In how many ways can an employment agency match up 10 families and 6 nannies (with 4 families left over).

9. Consider committees of size 6 from the population \( A_1, \ldots, A_{10} \).
(a) How many contain both \( A_1 \) and \( A_7 \).
(b) How many do not contain both \( A_1 \) and \( A_7 \).
SECTION 1.3 PERMUTATIONS AND COMBINATIONS CONTINUED

example 1
Find the number of poker hands with 2 Jacks.

solution
Pick 2 Jacks of the 4 Jacks
Pick the 3 remaining cards from the 48 non-Jacks
Answer is \( \binom{4}{2} \binom{48}{3} \).

example 2
Find the number of poker hands with 2 Jacks and one Ace.

solution
Pick 2 of the 4 Jacks
Pick 1 of the 4 Aces
Pick the remaining 2 cards from the 44 non-Ace-non-Jacks
Answer is \( \binom{4}{2} \cdot 4 \cdot \binom{44}{2} \).

example 3
Find the number of committees of size 10 with 5 Men, 3 Women, 2 Children that can be picked from a population of 20 Men, 25 Women, 30 Children.

solution
Pick 5 from the 20 Men.
Pick 3 from the 25 Women.
Pick 2 from the 30 Children.
Answer is \( \binom{20}{5} \binom{25}{3} \binom{30}{2} \).

example 4
How many 8 digit strings contain exactly three 2's (e.g., 02322366).

solution
Pick 3 positions in the string for the 2's.
Fill the remaining 5 positions from the other 9 digits.
Answer is \( \binom{8}{3} \cdot 9^5 \).

example 5
How many poker hands contain only one suit.

solution
Can pick the suit in 4 ways. Can pick the 5 cards from that suit in \( \binom{13}{5} \) ways.
Answer is \( 4 \binom{13}{5} \).

example 6
Look at strings of digits and letters of length 7.
How many contain 2 digits, 4 consonants and 1 vowel if
(a) repetition is not allowed (e.g., z2ctaf3 is OK but z2zzaf3 is no good)
(b) repetition is allowed (e.g., 22zzyya is OK)

solution
(a) method 1 Pick 2 digits, 4 consonants, 1 vowel and then line them up
Answer is \( \binom{10}{2} \binom{21}{4} \cdot 5 \cdot 7! \).

method 2 (perhaps better because it carries over into the case where repetition is allowed)
Pick 2 positions in the string for the digits. Can be done in \( \binom{7}{2} \) ways.
Pick 4 positions for the consonants, leaving one for the vowel. \( \binom{5}{4} \) ways.
Then fill the spots. Digit spots can be filled in 10 \cdot 9 ways, consonant spots can be filled in 21 \cdot 20 \cdot 19 \cdot 18 ways, vowel spot can be filled in 5 ways,
Answer is \( \binom{7}{2} \binom{5}{4} \cdot 10 \cdot 9 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 5 \).
(b) The first method from part (a) doesn't work here.

First of all you don't have a rule yet for counting how many ways you can pick a committee of digits and letters when a digit or letter can be picked more than once.

Secondly, if the picks were say 2,7,Z,Z,Z,C,E you don't have a rule yet for permuting them because of the three identical Z's. And even if you had such a rule you would need cases because the picks might include repetition (e.g., 2,7,Z,Z,Z,C,E) or might not (e.g., A,B,C,5,D,3,E).

Here's a method that will work.

Pick two positions for the digits, four for the consonants, one for the vowel. Then fill the spots.

Answer is \( \binom{7}{2} \binom{5}{4} \cdot 10^2 \cdot 214 \cdot 5 \).

**permutations with some objects kept together**

Given ten people P₁,..., P₁₀. In how many ways can they can be lined up so that P₂, P₆, P₉ are together; e.g., one possibility is

\[
P_1 \ P_3 \ P_5 \ P_4 \ P_2 \ P_9 \ P_6 \ P_{10} \ P_8 \ P_7
\]

Temporarily think of the trio as a single item so that you have 7 loners and one trio. These 8 things can be lined up in 8! ways. Then rearrange the 3 within the trio in 3! ways.

Answer is 8! 3!.

**permutations with some objects kept apart**

Suppose you want to line up 7 girls and 3 boys so that no two boys are together. Start by lining up the 7 girls. Can be done in 7! ways (Fig 1).

There are 8 spaces in between and at the ends in Fig 1. To keep the boys apart put the 3 boys in 3 of the 8 spaces. This can be done in 8·7·6 ways (each boy is a slot) or alternatively in \( \binom{8}{3} \cdot 3! \) ways (choose 3 spaces and then match them with the 3 boys).

Answer is 7!·8·7·6 or equivalently 7!(\binom{8}{3})·3!.

To keep enemies apart, first line up the others and then select between/end places for the enemies.

**example 7**

How many permutations of the 26 letters of the alphabet have the vowels (A,E,I,O,U) together, have P,Q,R together and have X,Y,Z apart (i.e., no two of them together).

**solution**

First permute the 23 letters A–W with vowels together and P,Q,R together. To count these perms, line up a vowel clump, a PQR clump and the other 15 letters. That's a total of 17 things. Then permute within the vowel clump and within the PQR clump.

Can be done in 17! 5! 3! ways.

Now there are 18 between/ends where X,Y,Z could go. Pick places for them. Can be done in 18·17·16 ways.

Final answer is 17! 5! 3! 18·17·16.
**double counting**

A poker hand with 2 pairs has say 2 Jacks, 2 Kings and then a fifth card which is not a Jack and not a King. I’ll show you an incorrect way to count the number of poker hands with two pairs. The important thing is to understand why it’s wrong.

(1) **WRONG**

- **step 1** Pick a face value and then pick 2 in that face.
- **step 2** Pick another face value and then pick 2 cards in that face.
- **step 3** Pick a 5th card from the 44 not of the two chosen faces (to avoid a full house).

Answer is $13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 44$.

This is wrong because it counts the following as different outcomes when they are really the same hand:

- **outcome 1** Pick Queen, hearts and spades.
  - Pick Jack, hearts and clubs.
  - Pick Ace of clubs.

- **outcome 2** Pick Jack, hearts and clubs.
  - Pick Queen, hearts and spades.
  - Pick Ace of clubs.

This incorrect method uses "first face value" and "second face value" as slots but the faces can’t be designated "first" and "second" so they can’t be used as two slots.

The wrong answer in (1) counts every outcome twice.

Here’s a correct version.

- **step 1** Pick a committee of 2 face values for the pairs.
- **step 2** Pick 2 cards from each face value.
- **step 3** Pick a 5th card from the 44 not of either face.

Answer is $\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44$.

Another correct method is to take the wrong answer from (1) and convert it to a right answer by taking half, i.e., the answer is $\frac{1}{2} \cdot 13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 44$.

In general, an "answer" which counts some outcomes more than once is referred to as a **double count** (the double count in (1) happens to count every outcome exactly twice). Double counts can be hard to resist.

**warning**

A common mistake is to use say 7…6 instead of $\binom{7}{2}$, i.e., to implicitly fill slots named first thing and second thing when you should just be picking a committee of two things.

**PROBLEMS FOR SECTION 1.3** (Make sure you look at the last problem.)

1. An exam offers 20 questions and you have to choose 2 of the first 5 and 8 from the next 15. In how many ways can you choose the 10 questions you’ll answer.

2. In how many ways can you choose a committee of 2 men and 3 women from a group of 7 men and 8 women.
3. How many poker hands contain
   (a) 2 spades (meaning exactly 2 spades) one of which is the ace
   (b) 3 diamonds and 2 hearts
   (c) 4 black and 1 red
   (d) 2 aces (meaning exactly 2 aces)
   (e) 4 aces and no kings

4. How many license plates are there if a license plate contains
   (a) 3 digits and 2 letters in any order (repetition allowed)
   (b) 3 digits and 2 letters in any order, repetition not allowed
   (c) 2 letters followed by 3 digits (repetition allowed)

5. Five people will be chosen from a group of 20 to take a trip.
   (a) In how many ways can it be done.
   (b) In how many ways can it be done if A and B refuse to travel together.
   (c) In how many ways can it be done if A and B refuse to travel without each other.

6. In how many ways can 20 books $B_1, ..., B_{20}$ be distributed among 4 children so that
   (a) each child gets 5 books
   (b) the two oldest get 7 books each and the two youngest get 3 each

7. Start with 3 men, $M_1, M_2, M_3$, 7 women, $W_1, ..., W_7$ and 8 children, $C_1, ..., C_8$.
   How many committees of size 5 can be formed containing
   (a) no men
   (b) $M_3$
   (c) $M_3$ and no other men
   (d) $M_3$ and two women
   (e) $M_3, W_1, W_7$
   (f) one man

   (a) How many committees of size 6 contain 3 men.
   (b) How many lineups of size 6 contain 3 men.

9. There are 50 states and 2 senators from each state. The President invites 25
   senators to the White House. In how many ways can this be done so that 25 different
   states are represented.

10. In how many ways can you select 3 people from a group of 4 married couples
    $C_1, C_2, C_3, C_4$ so as to not include a pair of spouses.

11. (a) How many strings of 12 digits and/or letters contain 3 even digits
    (e.g., b57xyzz[4]4[c][2]3, 0429A8).
    (b) Repeat part (a) if repetition is not allowed.

12. A child asks for a bicycle, a doll, a book and candy for her birthday. Assuming
    she'll get something but not necessarily everything she's asked for, how many
    possibilities are there.

13. A word is a palindrome if it reads the same backwards as forwards.
    (a) How many palindromes are there of length 5 (e.g., xzyyx, kkkkk)? length 6?
    (b) In a palindrome, some letters naturally must appear twice. If we insist that
        letters cannot appear more than twice how many palindromes are there of length 5? of
        length 6?
14. Start with 15 people. In how many ways can you
(a) invite 4 for dinner
(b) choose 4 to be president, vice-president, secretary and treasurer of your club
(c) lend them 4 books titled A, B, C, D so that no person gets more than one book
(d) lend them 4 books A, B, C, D allowing people to get more than one book
(e) pick 5 to be on a basketball team and 4 to cheer for them.
(f) pick 5 to play basketball in the winter and 4 to play football in the fall
(g) divide them into 9 to play baseball, 2 to umpire and 4 to cheer

15. How many poker hands contain one ace.

16. How many permutations of A, A, A, A, B, B, B have the A's in ascending order and
the B's in ascending order (e.g., A, A, B, A, B, A, B, A, B).

17. There are 5 women W₁,..., W₅ and 9 men M₁,..., M₉ at a dance.
(a) In how many ways can the women choose dance partners (e.g., one possibility is
W₁, W₂, W₃, W₄, W₅ with wallflowers M₁, M₂, M₃, M₄, M₅).
(b) The answer to part (a) was not 5⋅9. What counting problem is 5⋅9 the answer to.

18. How many permutations of the letters A, B, C, D, E, F have B directly between A and C
(e.g., D, C, B, A, E, F).

19. You can order a hamburger with ketchup, mustard, pickle, relish, onion. How many
possible hamburger orders are there.

20. How many poker hands contain
(a) four aces
(b) four of a kind (e.g., 4 Jacks)

21. A room has a row of 7 seats S₁,..., S₇. Four people P₁,..., P₄ sit down.
(a) In how many ways can they be seated.
(b) In how many ways can they be seated consecutively (i.e., with no empty seats
in between).

22. How many committees of size 5 including a designated chairperson can be formed
from a population of size 15.

23. 10 boys and 6 girls will sit in a row.
(a) In how many ways can it be done if the 6 girls sit together.
(b) In how many ways can it be done if no two girls sit together.
(c) Why can't you do (b) by taking total – answer to (a).

24. Three Democrats, 5 Republicans, 6 Socialists and 7 uncommitteds will stage a
(single-file) protest march. In how many ways can it be done if
(a) each political group (but not the uncommitteds) sticks together
(b) the Dems are first, Republicans next, Socialists next and uncommitteds last

25. How many permutations of the alphabet have
(a) A, B, C together
(b) D, E not together
(c) A, B, C together and D, E together
(d) A, B, C together and D, E not together

26. John and Mary play 5 sets of tennis. How many outcomes are there if
(a) an outcome is a report of who wins what set (e.g., John wins 1st and 3rd and
Mary wins the others)
(b) an outcome is a report of who wins how many sets (e.g., Mary wins 3 sets and
John wins 2 sets)

27. How many permutations of the 26 letters of the alphabet contain the string
mother (for example, the usual order abcde...xyz does not contain mother).

28. How many 3-card hands contain 3 of a kind (i.e., 3 of the same face) (e.g., Queen of hearts, spades, clubs; 3 of diamonds, clubs, spades).

29. (a) How many 4-card hands contain 2 pairs.
(b) How many 5-card hands contain a full house (3 of a kind and a pair).

30. A club with 50 members is going to form a finance committee.
(a) If the size hasn't been determined yet except for the fact that someone will be on it and not everyone will be on it, how many possibilities are there.
(b) How many possibilities are there if the committee will contain anywhere from 2 to 6 members.

31. Given 23 points in the plane.
(a) How many lines do they determine if no three of the points are collinear.
(b) What happens if the collinearity hypothesis is removed from part (a).

32. Look at samples of size 3 chosen from A₁,..., A₇.
(a) How many samples are there if the samples are chosen with replacement (after an item is selected, it's replaced before the next draw so that A₁ for instance can be drawn more than once) and order counts (e.g., A₂A₁A₁ is different from A₁A₂A₁)
(b) How many samples are there if the drawing is without replacement and order counts.
(c) How many samples are there if the drawing is w/o replacement and order doesn't count.
(d) What's the 4th type of sample (counting here is tricky———coming up soon).

33. Here are some counting problems with proposed answers that double count. Explain how they double count (produce specific outcomes which are counted as if they are distinct but are really the same) and then get correct answers.
(a) To count the number of poker hands with 3 of a kind:
   step 1 Pick a face value and 3 cards from that value.
   step 2 Pick one of the remaining 48 cards not of that value (to avoid 4 of a kind).
   step 3 Pick one of the remaining 44 not of the first or second value (to avoid 4 of a kind and a full house).
   Answer is 13... C₄ ...48...44. WRONG

(b) To count the number of 3-letter words containing the letter z (e.g., zzz, zab, baz, zzc):
   step 1 Pick one place in the word for the letter z (to be sure it appears).
   step 2 Fill the other 2 places with any of the 26 letters (allowing more z's).
   Answer is 3...26...26. WRONG

(c) To count 7-letter words with 3A's:
   step 1 Pick a spot for the first A.
   step 2 Pick a spot for the second A.
   step 3 Pick a spot for the third A.
   step 4 Fill each of the remaining 4 places with any of the non-A's.
   Answer is 7...6...5...25⁴. WRONG

(d) To count 2-card hands not containing a pair:
   step 1 Pick any first card.
   step 2 Pick a second card from the 48 not of the first face value.
   Answer is 52...48. WRONG

(e) To count the number of ways to form 3 coed couples from 10 men and 8 women:
   step 1 Pick a man and woman for the first couple.
   step 2 Pick a man and woman for the second couple.
   step 3 Pick a man and woman for the third couple.
   Answer is 10...8...9...7...8...6. WRONG
SECTION 1.4 PERMUTATIONS OF NOT-ALL-DISTINCT OBJECTS

permutations if not all objects are distinct (anagrams)

The number of permutations of

3 X's, 4 Y's, 6 Z's, and 1 each of A, B, C, D, E (18 letters total)

is \( \frac{18!}{3! \cdot 4! \cdot 6!} \)

More generally, if among \( n \) objects there are

- \( n_1 \) identical ones of one type,
- \( n_2 \) identical ones of a second type
- \( \ldots \)
- \( n_5 \) identical ones of a fifth type

and perhaps some other unique objects then the number of permutations (anagrams) of the \( n \) objects is

\[
\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot n_4! \cdot n_5!}
\]

why (1) works

version 1

Look at permutations say of 3 A's, 2 B's, 2 C's D, E, F (10 letters total).

There are 10 spots on the line to fill.

Pick 3 of the spots for the A's. Can be done in \( \binom{10}{3} \) ways.

Pick 2 spots for the B's from the remaining 7 places. Can be done in \( \binom{7}{2} \) ways.

Pick 2 spots for the C's from the remaining 5 places. Can be done in \( \binom{5}{2} \) ways.

The other 3 spots on the line and the 3 distinct letters can be matched in 3! ways.

Number of perms = \( \binom{10}{3} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot 3! = \frac{10!}{3! \cdot 7! \cdot 2! \cdot 5! \cdot 3!} = \frac{10!}{3! \cdot 2! \cdot 2!} \)

version 2

There are 10! perms of the 10 distinct letters \( A_1, A_2, A_3, B_1, B_2, C_1, C_2, D, E, F \).

To get the number of permutations of the original 10 not-all-distinct letters, compare the following two lists.

perms of non-distincts

(a) AAABBFDECC
(b) ABABCCADEF

perms of distincts

(a1) \( A_1 A_2 A_3 B_1 B_2 F D E C_1 C_2 \)
(b1) \( A_1 B_1 A_2 B_2 C_1 C_2 A_3 D E F \)

(a2) \( A_2 A_1 A_3 B_1 B_2 F D E C_1 C_2 \)
(b2) \( A_2 B_1 A_1 B_2 C_1 C_2 A_3 D E F \)

(a3) \( A_2 A_1 A_3 B_2 B_1 F D E C_1 C_2 \)
(b3) \( A_2 B_2 A_1 B_1 C_3 A_2 D E F \)

\[ \vdots \]

\[ \vdots \]

there are \( 3! \cdot 2! \cdot 2! \) of these

there are \( 3! \cdot 2! \cdot 2! \) of these
Each perm on the left gives rise to $3!2!2!$ perms on the right. So

\[
\text{Number of perms of non-distincts} \times 3!2!2! = \text{number of perms of distincts}.
\]

\[
\text{Number of perms of non-distincts} = \frac{10!}{3!2!2!} \quad \text{QED}
\]

**example 1**

How many permutations are there of the word APPLETREE.

**solution**

The 9-letter word contains 2 P's, 3 E's, A, L, R, T. Answer is $\frac{9!}{2!3!}$.

**example 1 continued**

In how many of those permutations are the 3 E's together.

**solution**

Imagine that you have 7 objects, namely 2 P's, A, L, R, T and a clump of E's.

By (1) they can be permuted in $\frac{7!}{2!}$ ways.

**example 1 continued again**

In how many of those permutations are the letters A, L, R together.

**solution**

Imagine that you have 7 objects, namely 2 P's, 3 E's, T and an ALR clump.

By (1) they can be permuted in $\frac{7!}{2!3!}$ ways. And then the ALR clump can be rearranged in 3! ways.

Answer is $\frac{7!}{2!3!} \cdot 3! = \frac{7!}{2!}$.

**example 2**

In how many ways can the complete works of Shakespeare in 10 volumes and 6 copies of Tom Sawyer be arranged on a shelf.

**solution**

This is a permutation of 16 objects, 6 of which are identical. Answer is $\frac{16!}{6!}$.

**PROBLEMS FOR SECTION 1.4**

1. How many permutations can be made using 3 indistinguishable white balls, 7 indistinguishable black balls and 5 indistinguishable green balls.

2. How many permutations are there of (a) HOCKEY (b) FOOTBALL

3. Look at all permutations of SOCIOLOGICAL
   (a) How many have the vowels adjacent
   (b) How many have the vowels in alphabetical order (but not necessarily adjacent); e.g., S A I C I L O C O G L O
   (c) How many have the vowels adjacent and in alphabetical order.
4. Suppose you want to count all 4-letter words containing 2 pairs (e.g., ADAD, QQSS, DAAD)
(a) What's wrong with the following attempt.
   step 1 Pick a letter for the first pair. Can be done in 26 ways.
   step 2 Pick a letter for the second pair. Can be done in 25 ways.
   step 3 Arrange the 4 letters (2 of one kind and 2 of another kind).
      Can be done in \( \frac{4!}{2!*2!} \).
      Answer is 26 \cdot 25 \cdot \frac{4!}{2!*2!} . \text{WRONG}
(b) What's the right answer.

5. How many permutations of 4 plus signs, 4 minuses and 3 crosses have a plus in the middle (e.g., ++-×- + ×-×-).

6. Consider strings of length 10 from the alphabet A,B,C.
   (a) How many are there.
   (b) How many have 3 A's, 4 B's and 3 C's.
   (c) How many have 3 A's.

7. Consider permutations of APPLETREE.
   (a) How many have no adjacent E's.
   (b) How many have none of A, L, R adjacent.
   (c) How many have A, L, T together and no consecutive E's.
SECTIO N1.5  COMMITTEES WITH REPEATED MEMBERS
(INDISTINGUISHABLE BALLS INTO DISTINGUISHABLE BOXES)

the stars and bars formula
Consider committees of size 3 chosen from $A_1,\ldots, A_7$ with repetition allowed.

For example, some possibilities are
$A_2, A_2, A_3$; i.e., 2 $A_2$'s and an $A_3$
$A_1, A_5, A_7$; i.e., one each of $A_5, A_1, A_5, A_7$

Equivalently, consider unordered samples of size 3 chosen with replacement from a population of size 7.
I'll show that there are $\binom{7+3-1}{3} = \binom{9}{3}$ such committees.

The key idea (Fig 1) is that choosing a committee is like tossing 3 indistinguishable balls into 7 distinguishable boxes. For example, the committee $A_5, A_5, A_6$ corresponds to 2 balls in box $A_5$ and 1 ball in box $A_6$.

Furthermore, the ball-in-boxes problem can be thought of as lining up 3 stars and 6 bars (the inside walls of the boxes). By (1) in the preceding section this can be done in $\frac{9!}{3! \cdot 6!} = \binom{9}{6}$ ways.

<table>
<thead>
<tr>
<th>sample</th>
<th>ball-in-box version</th>
<th>stars and bars version</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5, A_5, A_6$</td>
<td>A1 A2 A3 A4 A5 A6 A7</td>
<td></td>
</tr>
<tr>
<td>$A_1, A_2, A_7$</td>
<td>* *</td>
<td>* *</td>
</tr>
<tr>
<td>$A_4, A_4, A_4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FIG 1

Here's the general result.

The following are the same:
The number of committees of size $k$ chosen from a population of size $n$ with repetition allowed.

(1) The number of ways of tossing $k$ indistinguishable balls into $n$ distinguishable boxes.

And that number is $\binom{n+k-1}{k}$

Note that since repetition is allowed on the committee, it's possible for $k$, the size of the committee, to be larger than $n$, the size of the population.
**example 1**

A school district encompasses four schools $S_1, ..., S_4$.

The district receives 9 blackboards.

In how many ways can the (identical) blackboards be distributed to the schools.

**solution**

The 4 schools are distinguishable boxes into which the 9 blackboards/balls are tossed. Answer is $\binom{4+9-1}{9} = \binom{12}{9}$.

**warning**

1. In example 1 the answer is *not* $\binom{4+9-1}{4}$. Of the two numbers 4 and 9, the one chosen for the lower spot in the formula is the one corresponding to the number of indistinguishable balls (i.e., to the size of the committee) and is not necessarily the smaller of the two numbers.
2. In example 1, the blackboards can't be used as slots since they are indistinguishable.

**Indistinguishable objects can't serve as slots.**

**example 2**

A child has a repertoire of 4 piano pieces $P_1, ..., P_4$.

Her parents insist that she practice 10 pieces a day. For example one practice session might consist of $P_3$ played 7 times (tra la la) and $P_1$ played 3 times.

(a) How many practice sessions are there.
(b) How many sessions are there if she must play $P_2$ at least 3 times.
(c) How many sessions are there if she must play $P_2$ exactly 3 times.
(d) How many sessions are there if she has to play each piece at least once.

**solution**

(a) Each session is a committee of size 10 chosen from the repertoire of size 4 with repetition allowed. Answer is $\binom{10+4-1}{10} = \binom{13}{10}$.

(b) After $P_2$ is played 3 times, the rest of the session is a committee of size 7 chosen from 4 (allow $P_2$ to be chosen again). Answer is $\binom{7+4-1}{7}$.

(c) After $P_2$ is played 3 times, the rest of the session is a committee of size 7 chosen from 3 (don't allow $P_2$ to be chosen again). Answer is $\binom{7+3-1}{7}$.

(d) Play each piece once. Then the rest of the session is a committee of 6 chosen from 4. Answer is $\binom{6+4-1}{6}$.

**example 3**

A bakery sells chocolate chip cookies, peanut butter cookies, sugar cookies and oatmeal cookies. In how many ways can you buy 7 cookies.

**solution**

You're choosing a committee of size 7 from a population of size 4 with repetition allowed. Answer is $\binom{7+4-1}{7}$.

**example 4**

A bakery has 27 chocolate chip cookies, 28 peanut butter cookies, 29 sugar cookies and 65 oatmeal cookies. In how many ways can you buy 7 cookies.

**solution**

This is the same as example 3. The numbers 27, 28, 29, 65 are irrelevant as long as they are $\geq 7$ so that you can buy as many of each kind of cookie as you like.

See "at mosts" later in this section for what happens when they aren't irrelevant.
non-negative integer solutions to an equation of the form \( x_1 + \ldots + x_n = k \)

Look at the equation

\[ x + y + z = 12 \]

where \( x, y, z \) are non-negative integers. For example one solution is \( x=2, y=3, z=7 \); another solutions is \( x=0, y=0, z=12 \).

To count the number of solutions think of \( x, y, z \) as distinguishable boxes into which 12 indistinguishable balls are tossed. (The solution \( x=2, y=3, z=7 \) corresponds to 2 balls in the \( x \) box, 3 in the \( y \) box, 7 in the \( z \) box.) The answer is \( \binom{3+12-1}{12} \).

at least

Here are three problems that are all the same:

How many non-negative integer solutions to \( x_1 + \ldots + x_4 = 18 \) have \( x_2 \geq 7 \).

If a bakery sells 4 kinds of cookies in how many ways can you choose 18 including at least 7 oatmeal.

In how many ways can 18 indistinguishable balls be tossed into 4 distinguishable boxes if the second box must get at least 7 balls.

For the first version, start \( x_2 \) at 7 and then count solutions to \( x_1 + \ldots + x_4 = 11 \).

For the second version buy 7 cookies and choose the remaining 11 from the 4 types of cookies.

For the third version put 7 balls into box 2 and then toss the 11 others into the 4 boxes.

The answer in each case is \( \binom{11+4-1}{11} \).

at leasts continued

In how many ways can you toss 17 indistinguishable balls into 5 distinguishable boxes so that each box gets at least two balls.

Put two ball into each box (there's just one way to do this since the balls don't have names). Toss the remaining 7 balls into the 5 boxes.

Answer is \( \binom{7+5-1}{7} \).

warning

This (easy) way of doing "at leasts" won't work with distinguishable balls or cookies. See Section 1.7 for what happens in that type of problem.

at mosts

A bakery sells chocolate cookies, peanut butter cookies, sugar cookies and oatmeal cookies.

You want to buy 18 cookies. In how many ways can it be done if the bakery has only 3 chocolates left (and an unlimited supply of the other 3 kinds).

This amounts to choosing a committee of size 18 from a population of size 4 with the restriction that the committee contain 3 or fewer chocolates, i.e., at most 3 chocs.

method 1

at most 3 chocs = 3 or fewer chocs = total - 4 or more chocs

The total is \( \binom{18+4-1}{18} \).

To count "4 or more chocs" (i.e., at least 4 chocs), reserve 4 places on the committee for chocs and then pick 14 more from the 4 kinds (allowing more chocs); can be done in \( \binom{14+4-1}{14} \) ways.

Answer is \( \binom{18+4-1}{18} - \binom{14+4-1}{14} \).
method 2
\[ N(\text{at most 3 chocs}) = N(\text{no chocs}) + N(\text{1 choc}) + N(\text{2 chocs}) + N(\text{3 chocs}) \]
\[ = \binom{18+3-1}{18} + \binom{17+3-1}{17} + \binom{16+3-1}{16} + \binom{15+3-1}{15} \]

betweens
Toss 70 indistinguishable balls into 5 distinguishable boxes. In how many ways can it be done so that box 1 gets between 30 and 50 balls.

method 1
Between 30 and 50 in box 1 = 30 balls into box 1 + 31 balls into box 1 + \ldots + 50 balls into box 1
For "30 balls into box 1", put 30 into box 1 and toss the other 40 into the remaining four boxes. Can be done in \( \binom{40+4-1}{40} \) ways.

For "50 balls into box 1", put 50 into box 1 and toss the other 20 into the remaining four boxes. Can be done in \( \binom{20+4-1}{20} \) ways.

Answer is \( \sum_{n=20}^{40} \binom{n+4-1}{n} \)

method 2 (better)
Look at the Venn diagram to see that
between 30 and 50 in box 1 = 30 or more in box 1 - 51 or more in box 1
= at least 30 in box 1 - at least 51 in box 1
For "30 or more", put 30 into box 1 and toss the other 40 into the five boxes. Can be done in \( \binom{40+5-1}{40} \) ways.
Similarly for "51 or more".
Answer is \( \binom{40+5-1}{40} - \binom{19+5-1}{19} \)

Check with Mathematica:
\[
\text{Binomial}[40+5-1,40] - \text{Binomial}[19+5-1,19]
\]
126896
\[
\text{Sum}[\text{Binomial}[n+4-1,n],[n,20,40]]
\]
126896

PROBLEMS FOR SECTION 1.5

1. A concert promoter has 1000 unreserved grandstand seats to distribute (free) among Alumni, Students, Faculty, Public. For example one possibility is to give half to the Students and half to the Faculty (Public and Alumni be damned). How many possibilities are there.
2. Toss a die 19 times. For each toss the die can come up 1, 2, 3, 4, 5 or 6. A histogram records how many times each outcome turns up. One possibility is shown in the diagram. How many possible histograms are there.

Problem 2

3. (a) In how many ways can 20 coins be picked from 4 large boxes filled respectively with pennies, nickels, dimes, and quarters.
   (b) Why did part (a) say "large" boxes and how large is "large".

4. In how many ways can 12 balls be tossed into 5 distinct boxes if
   (a) the balls are indistinguishable
   (b) the balls are all different colors
   (c) 8 of the balls are red (indistinguishable from one another) and 4 are white (indistinguishable from one another)

5. In how many ways can one quarter, one dime, one nickel and 25 pennies be distributed among 5 people.

6. A box contains balls $B_1, B_2, \ldots, B_{10}$. Draw 6. I don't care about the order in which they are drawn. How many possibilities are there if the drawing is
   (a) without replacement
   (b) with replacement

7. You're taking six courses, $C_1, \ldots, C_6$.
   For each course you will get a grade of A, B, C, D or E.
   How many possible report cards can you get if
   (a) the report lists the grade in each course so that one possibility is A's in $C_1$ and $C_4$, B in $C_2$, C's in the rest.
   (b) the report is less detailed and only lists the overall results so that one possibility is 2 A's, 1 B, 3 C's

8. A car dealer has 5 models in stock: 7 $M_1$'s, 8 $M_2$'s, 9 $M_3$'s, 10 $M_4$'s and 12 $M_5$'s.
   Your company is going to buy 6 cars. For instance one possibility is 3 $M_1$'s, an $M_2$ and 2 $M_5$'s.
   (a) How many possible ways are there to buy the 6 cars.
   (b) How many possibilities include exactly 2 different models (e.g., 4 $M_2$'s, 2 $M_5$'s).
   (c) How many possibilities include as many different models as possible.

9. You have 6 (identical) bats and 7 (identical) baseballs to give away to a group of 20 children.
   (a) In how many ways can it be done.
   (b) In how many ways can it be done if no child gets two balls and no child gets two bats.
   (c) In how many ways can it be done if no child gets two gifts.

10. A message consists of all the 12 symbols $S_1, \ldots, S_{12}$ (in any order) plus 50 blanks distributed between the symbols (not before and not after, just between) so that there are at least 3 blanks between successive symbols. How many messages are there.
11. A child has a repertoire of 4 piano pieces and must practice 10 pieces at each session.
   (a) How many different sessions are there.
   (b) How many different sessions are there if she has to play each piece at least twice.
   (c) Write out all the possibilities in (b) to confirm your answer.
   (d) How many sessions are there if she has to play $P_3$ four times (exactly).
   (e) How many sessions are there if she can play $P_2$ at most 3 times.

12. Toss 12 identical balls into boxes $B_1, \ldots, B_5$.
   (a) In how many ways can it be done so that no box is empty.
   (b) In how many ways can it be done so that $B_2$ gets an odd number of balls.
   (c) Suppose you want to count the number of ways it can be done so that 3 boxes get all the balls; i.e., there are (exactly) 2 empties.
      First explain what's wrong with this attempt.
      Step 1 Pick 3 of the 5 boxes to get the balls
      Step 2 Toss the 12 balls into those 3 boxes.
      Answer is $\binom{5}{3} \binom{12+3-1}{12}$. WRONG
      Then get a right answer.

13. A 5-sided die has sides named A, B, C, D, E. Toss it 100 times.
    For instance some possible outcomes are
    95 A's and 5 C's
    1 A, 1 B, 98 D's
    20 of each
    etc.
    How many of these outcomes have between 10 and 30 A's.
    Do it in a way that doesn't involve a long sum.

    In how many ways can you order a dozen cones if
    (a) you want all different flavors
    (b) exactly half must be chocolate
    (c) you want exactly 7 different flavors
    (d) the store has only enough strawberry for 2 cones

15. Consider non-negative integer solutions to $x_1 + x_2 + x_3 = 13$.
    (a) How many are there.
    (b) How many have $x_1 \geq 4$ and $x_2 \geq 2$.
    (c) How many have $2 \leq x_3 \leq 5$.
    (d) How many have $x_1 \geq 4$, $x_2 \geq 2$ and $2 \leq x_3 \leq 5$.
      Suggestion. Take care of $x_1 \geq 4$, $x_2 \geq 2$, $x_3 \geq 2$ and then worry about $x_3 \leq 5$. 
SECTION 1.6    ORS

OR versus XOR

A OR B means A or B or both, called an inclusive or.

Similarly, A OR B OR C means one or more of A, B, C (i.e., exactly one of A, B, C or any two of A, B, C or all three of A, B, C).

On the other hand, A XOR B means A or B but not both, an exclusive or.

In these notes, "or" will always mean the inclusive OR unless specified otherwise.

In the real world you'll have to decide for yourself which kind is intended. If a lottery announces that any number containing a 6 or a 7 wins then you win with a 6 or 7 or both, i.e., 6 OR 7 wins (inclusive or). But if you order a coke or a 7-up you really mean a coke or a 7-up but not both, i.e., coke XOR 7-up.

OR rule (principle of inclusion and exclusion)

Let N(A) stand for the number of ways in which event A can happen.

(1) \[ N(A \text{ or } B) = N(A) + N(B) - N(A \text{ and } B) \]

(2) \[
N(A \text{ or } B \text{ or } C) = N(A) + N(B) + N(C) - \left[ N(A \& B) + N(A \& C) + N(B \& C) \right] + N(A \& B \& C)
\]

(3) \[
N(A \text{ or } B \text{ or } C \text{ or } D) = N(A) + N(B) + N(C) + N(D) - \left[ N(A \& B) + N(A \& C) + N(A \& D) + N(B \& C) + N(B \& D) + N(C \& D) \right] + \left[ N(A \& B \& C) + N(A \& B \& D) + N(B \& C \& D) + N(A \& C \& D) \right] - N(A \& B \& C \& D)
\]

why (1) works

Suppose A can occur in 6 ways (the 4 x's and 2 y's in Fig 1). And suppose B can happen in 5 ways (the 2 y's and 3 z's in Fig 1). Then

\[ N(A \text{ or } B) = 4 \text{ x's} + 2 \text{ y's} + 3 \text{ z's} = 9 \]

On the other hand

\[ N(A) + N(B) = 4 \text{ x's} + 2 \text{ y's} + 2 \text{ y's} + 3 \text{ z's} = 11 \]

This is not the same as N(A or B) because it counts the y's twice. You do want to count them since this is an inclusive or, but we don't want to count them twice. So to get N(A or B), start with N(A) + N(B) and then subtract the number of outcomes in the intersection of A and B, i.e., subtract N(A & B) as in (1).

event A     event B
\[
\begin{array}{c}
\times \times \\
\times \times
\end{array}
\quad
\begin{array}{c}
y \quad \quad \quad \quad \quad \quad \quad z \\
y \quad \quad z \quad \quad \quad \quad z
\end{array}
\]

FIG 1
warning

The "or" in rule (1) is inclusive; it means A or B or both. You subtract away N(A & B) not because we want to throw away the boths but because you don't want to count them twice. In other words,

\[ N(A \text{ or } B) = N(A \text{ or } B \text{ or both}) = N(A) + N(B) - N(A & B) \]

why (2) works

Suppose A can occur in 7 ways, B in 9 ways and C in 9 ways as shown in Fig 2.

![Diagram](image)

Then

\[ N(A \text{ or } B \text{ or } C) = 3 \text{ x}'s + 1 \text{ y} + 1 \text{ z} + 2 \text{ u}'s + 3 \text{ w}'s + 3 \text{ v}'s + 4 \text{ k}'s = 17 \]

But \( N(A) + N(B) + N(C) \) isn't 17 because it counts the y, u's and v's twice each and it counts z three times.

Try again with

\[ N(A) + N(B) + N(C) - \left[ N(A & B) + N(A & C) + N(B & C) \right] \]

Now the u's, v's, and y will be counted once each, not twice, but z isn't counted at all.

So formula (2) adds \( N(A & B & C) \) back in to include z again.

example 1

Find the number of bridge hands with 4 aces or 4 kings.

solution

By the OR rule,

\[ N(4 \text{ aces or 4 kings}) = N(4 \text{ aces}) + N(4 \text{ kings}) - N(4 \text{ aces and 4 kings}). \]

When you count the number of ways of getting 4 aces don't think about kings at all (the hand may or may not include 4 kings --- you don't care). The other 9 cards can be picked from the 48 non-aces so there are \( \binom{48}{9} \) hand with 4 aces.

Similarly there are \( \binom{48}{9} \) hands with 4 kings.

For 4K & 4A pick the other 5 cards from the 44 remaining cards.

Can be done in \( \binom{44}{5} \) ways.

So \( N(4 \text{ aces or 4 kings}) = \binom{48}{9} + \binom{48}{9} - \binom{44}{5} \).
mutually exclusive (disjoint) events

Suppose A, B, C, D are mutually exclusive meaning that no two can happen simultaneously (i.e., A, B, C, D have no outcomes in common). Then all the "and" terms in (1), (2), (3) drop out and you get

\[(1') \quad N(A \text{ or } B) = N(A) + N(B)\]
\[(2') \quad N(A \text{ or } B \text{ or } C) = N(A) + N(B) + N(C)\]
\[(3') \quad N(A_1 \text{ or } \ldots \text{ or } A_n) = N(A_1) + \ldots + N(A_n)\]

example 2

Count poker hands containing all spades or all hearts.

solution

The events "all spades" and "all hearts" are mutually exclusive since they can't happen simultaneously. So by (1'),

\[N(\text{all spades or all hearts}) = N(\text{all spades}) + N(\text{all hearts}) = \binom{13}{5} + \binom{13}{5}\]

warning

\[N(A \text{ or } B) \text{ is not } N(A) + N(B) \] unless A and B are mutually exclusive. If they aren't, don't forget to subtract \(N(A \& B)\).

PROBLEMS FOR SECTION 1.6

1. How many poker hands have 2 aces or 3 kings.

2. How many poker hands have the ace of spades or the king of spades.

3. How many 7-digit strings have two 3's or two 6's.

4. Suppose you're computing \(N(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_8)\).
   (a) Eventually you have to add in 2-at-a-time terms like \(N(A_1 \& A_5), N(A_3 \& A_5)\) etc. How many of these terms are there.

   (b) Eventually you have to add in 3-at-a-time terms like \(N(A_1 \& A_4 \& A_7), N(A_3 \& A_7 \& A_8)\) etc. How many of these terms are there.

5. How many poker hands have (a) 2 aces or 2 kings (b) 3 aces or 3 kings

6. How many poker hands have no spades or no hearts.

7. Toss a die 7 times. Order doesn't count so that one outcome for instance is 1 two, 4 fives, 2 sixes.
   (a) How many outcomes are there.
   (b) How many outcomes have 1 one or 2 twos or 3 threes.

8. A group consists of 6 children, 7 teenagers, 8 adults. How many committees of size 3 contain only one age group.

9. How many poker hands contain the Jack of Spades XOR the Queen of Spades (i.e., Jack or Queen but not both).
SECTION 1.7 AT LEASTS

at least 4 chocolates in a paper bag (i.e., at leasts for committees with repeated members allowed) *(i.e., at leasts for indistinguishable balls into distinguishable boxes) (Section 1.5)*

A store sells 5 ice cream flavors. Here's how to find the number of ways to buy 12 cones so as to get at least one chocolate.

Include one chocolate and then choose the rest of the committee, 11 cones, from the 5 flavors (this allows more chocs). Answer is \( \binom{11+5-1}{11} \).

WRONG WAY to do at leasts for ordinary committees (no repeated members)

The method that just worked with the choc cones (put one choc on the committee to be sure) does **not** work here.

Here's an example to show why not.

Suppose you want to count poker hands with at least one ace (all cards are distinguishable from one another). Try the "pick one to be sure" method.

step 1 Pick one ace to be sure of getting at least one. Can be done in 4 ways.

step 2 Pick 4 more cards from the remaining 51. Can be done in \( \binom{51}{4} \) ways.

(The 51 allows more aces. If you use 50 you are finding **exactly** one ace)

"Answer" is \( 4 \binom{51}{4} \).

The answer is wrong because it counts the following as different outcomes when they are really the same.

outcome 1 step 1 Pick the spade ace as the sure ace.

step 2 Pick the ace, king, queen, jack of clubs.

outcome 2 step 1 Pick the club ace as the sure ace.

step 2 Pick the spade ace and the king, queen, jack of clubs.

So this attempt **double counts** (i.e., counts some outcomes more than once).

Some RIGHT WAYS to do at leasts for ordinary committees

Here are some correct methods for finding \( N(\text{at least one ace}) \).

method 1

\[
N(\text{at least one ace}) = \text{total} - N(\text{no aces}) = \binom{52}{5} - \binom{48}{5}
\]

method 2

\[
N(\text{at least one ace}) = N(1A \text{ or } 2A \text{ or } 3A \text{ or } 4A)
\]

The events 1A (meaning exactly one ace), 2A, 3A, 4A are mutually exclusive so you can use the abbreviated OR rule.

\[
N(\text{at least one ace}) = 1A + 2A + 3A + 4A = 4 \binom{48}{4} + \binom{4}{2} \binom{48}{3} + \binom{4}{3} \binom{48}{2} + 48
\]

method 3

\[
N(\text{at least one ace}) = N(A_S \text{ or } A_H \text{ or } A_C \text{ or } A_D)
\]

\[
= \left[ N(A_{S}) + N(A_{H}) + N(A_{C}) + N(A_{D}) \right] - \left[ N(A_{H} \& A_{C}) + \text{other 2-at-a-time terms} \right] + \left[ N(A_{C} \& A_{H} \& A_{D}) + \text{other 3-at-a-time terms} \right] - N(A_{S} \& A_{H} \& A_{C} \& A_{D})
\]

The first bracket contains 4 terms all having the value \( \binom{51}{4} \).
The second bracket contains \( \binom{4}{2} \) terms each of which has value \( \binom{50}{3} \).

The third bracket contains \( \binom{4}{3} \) terms each of which has value \( \binom{49}{2} \).

So \( N(\text{at least one ace}) = 4 \binom{51}{4} - \binom{4}{2} \binom{50}{3} + \binom{4}{3} \binom{49}{2} - 48 \)

In this example, method 1 was best but you'll see examples favoring each of the other methods (cf. problem 5)

**footnote**

The "pick one to be sure" method that works for committees with repetition allowed does not work for ordinary committees. But methods 1 and 2 that worked for the ordinary committees will also work for committees with repetition.

**example 1**

In how many ways can 10 sandwiches be given to 4 people so that Mary gets at least 3 sandwiches if

(a) the sandwiches are all tuna (indistinguishable)
(b) the sandwiches are tuna, ham, jelly,... (all different from one another)

**solution** (a) This is tossing 10 indistinguishable balls into 4 distinguishable boxes (same as committees with repetition allowed).

Give Mary 3 tunas. Then toss the remaining 7 tunas into the 4 distinguishable people. Answer is \( \binom{7+4-1}{7} \).

(b) total \(-\ N(\text{Mary gets none or one or two})

\[ = \text{total} - \left[ N(\text{Mary gets none}) + N(\text{Mary gets 1}) + N(\text{Mary gets 2}) \right] \]

To find the total, think of each sandwich as a slot which can be filled in 4 ways. To find \( N(\text{Mary gets none}) \), fill each sandwich slot in 3 ways.

For \( N(\text{Mary gets 1}) \), pick a sandwich for Mary (can be done in 10 ways) and then fill each of the remaining 9 sandwich slots in 3 ways (can be done in \( 3^9 \) ways).

For \( N(\text{Mary gets 2}) \), pick 2 sandwiches for Mary (can be done in \( \binom{10}{2} \) ways and then fill each of the remaining sandwich slots in 3 ways (can be done in \( 3^8 \) ways).

Answer is \( 4^{10} - \left[ 3^{10} + 10 \cdot 3^9 + \binom{10}{2} \cdot 3^8 \right] \).

**example 2**

How many 7-digit strings have at least five 3's (e.g., 3376333, 3133333).

**solution** \( N(\text{at least five 3's}) = N(\text{five 3's}) + N(\text{six 3's}) + N(\text{seven 3's}) \)

To find \( N(\text{five 3's}) \) pick five places for the 3's and fill the other 2 places with non-3's.
To find \( N(\text{six 3's}) \) pick six places for the 3's and fill the one remaining place with a non-3. So

\[ N(\text{at least five 3's}) = \binom{7}{5} 9^2 + \binom{7}{6} 9 + 1 \]

**wrong way to do example 2**

If you pick 5 places in the string for 3's (to be sure) and then fill the other places with any of 10 digits you get "answer" \( \binom{7}{5} 10^2 \) and it's WRONG.

It's wrong because it counts the following as different outcomes when they are really the same (they are both the string 3333339):

outcome 1 step 1 Pick positions 1-5 for the sure 3's
step 2 Fill position 6 with a 3 and position 7 with a 9.

outcome 1 step 1 Pick positions 1-4, 6 for the sure 3's
step 2 Fill position 5 with a 3 and position 7 with a 9
Except for committees with repeated members (i.e., tossing indistinguishable balls into distinguishable boxes) you can't do "at least n of them" by starting with n "sure" ones and then picking the rest.

**at mosts**

Here's how to find the number of bridge hands with at most 10 spades.

\[ N(\text{at most 10 spades}) = \text{total} - N(11 \text{ spades or 12 spades or 13 spades}) \]
\[ = \text{total} - N(11 \text{ spades}) - N(12 \text{ spades}) - N(13 \text{ spades}) \]
\[ = \binom{52}{13} - \binom{13}{11} \binom{39}{2} - \binom{13}{12} 39 - 1 \]

**exactlys combined with at leasts**

I'll find the number of poker hands with (exactly) 2 spades and at least 1 heart.

**method 1**

Within the 2-spade world, some hands have at least one heart and the rest have no hearts.

The number of poker hands with 2 spades and at least 1 heart is region II in Fig 1. You can find it by taking the entire 2-Spade region and subtracting region I. So

\[ N(2S \text{ and at least 1H}) = N(2S) - N(2S \text{ and no H}) \]
\[ = \binom{13}{2} \binom{39}{3} \text{ pick 2 spades and 3 others} - \binom{13}{2} \binom{26}{3} \text{ pick 2 spades and 3 non-hearts} \]

**warning**

\[ N(2 \text{ spades at at least 1 H}) \] is not \[ N(2 \text{ spades}) - N(\text{no H}). \]

It is \[ N(2 \text{ spades}) - N(2 \text{ spades and no H}) \]

**method 2**

\[ N(2S \text{ and at least 1H}) = N(2S \& 1H) + N(2S \& 2H) + N(2S \& 3H) \]
\[ = \binom{13}{2} \cdot 13 \cdot \binom{26}{2} + \binom{13}{2} \binom{13}{2} \cdot 26 + \binom{13}{2} \binom{13}{3} \]

**at least one of each**

Choose committees of 6 from a population of 10 Americans, 7 Italians, 5 Germans. (Remember that people are always distinguishable, i.e., the Americans are A1, ...A10 etc.)

\[ N(\text{no European nationality left out}) \]
\[ = N(\text{at least one of each European nationality}) \]
\[ = N(\text{at least one I and at least one G}) \]
\[ = \text{total} - N(\text{no I or no G}) \]
\[ = \text{total} - \left[ N(\text{no I}) + N(\text{no G}) - N(\text{no I} \& \text{no G}) \right] \]
\[ = \binom{22}{6} - \left[ \binom{15}{6} + \binom{17}{6} - \binom{10}{6} \right] \]

**N(no nationality left out)**
\[ = N(\text{at least one of each nationality}) \]
\[ = N(\text{at least one A and at least one I and at least one G}) \]
\[ = \text{total} - N(\text{no A or no I or no G}) \]
total - \[ \begin{bmatrix} N(\text{no A}) + N(\text{no I}) + N(\text{no G}) \\ N(\text{no A} \& \text{no I}) + N(\text{no A} \& \text{no G}) + N(\text{no I} \& \text{no G}) \\ N(\text{no A} \& \text{no G} \& \text{no I}) \end{bmatrix} \]

= \binom{22}{6} - \binom{12}{6} + \binom{15}{6} + \binom{17}{6} - \left( \binom{0}{6} + \binom{7}{6} + \binom{10}{6} \right) + 0

\textbf{warning}

1. Don't try to do "at least one of each nationality" by picking a person from each nationality to be sure. That will double count.

2. The opposite of "at least one of each European nationality" is not "no Europeans"; i.e., the opposite is \textit{not} "no I and no G". The correct opposite is "no I \text{ OR } no G".

\textbf{some opposite (complementary) events}

<table>
<thead>
<tr>
<th>event</th>
<th>opposite</th>
</tr>
</thead>
<tbody>
<tr>
<td>A or B</td>
<td>not A and not B</td>
</tr>
<tr>
<td>A and B</td>
<td>not A or not B</td>
</tr>
<tr>
<td>at least one King</td>
<td>no Kings</td>
</tr>
<tr>
<td>at least one King and at least one Queen</td>
<td>no Kings or no Queens</td>
</tr>
<tr>
<td>at least one King or at least one Queen</td>
<td>no Kings and no Queens</td>
</tr>
<tr>
<td>at least one of each suit</td>
<td>no S or no H or no C or no D</td>
</tr>
<tr>
<td>all reds in poker</td>
<td>at least one black</td>
</tr>
<tr>
<td>at most 3 Blues and at most 4 Greens</td>
<td>at least 4 B or at least 5C</td>
</tr>
</tbody>
</table>

\textbf{warning}

The opposite of "at least one K and at least one Q" is \textit{not} "no K and no Q". The correct opposite is "no K \text{ or } no Q".

\textbf{example 3 (tricky)}

A box contains

100 Reds $R_1, \ldots, R_{100}$
100 Whites $W_1, \ldots, W_{100}$
100 Blacks $B_1, \ldots, B_{100}$
100 Greens $G_1, \ldots, G_{100}$
100 Yellows $Y_1, \ldots, Y_{100}$

Draw 30 without replacement.
The number of samples containing (exactly) 3 colors.

\textbf{wrong way}

\textbf{step 1} Pick which 3 colors to have. Can be done in \( \binom{5}{3} \) ways.

Say you pick R, W, B.
step 2 Pick 30 from the 300 R, W, B
Can be done in \( \binom{300}{30} \)
"Answer" is \( \binom{5}{3} \binom{300}{30} \) WRONG
It's wrong because it allows fewer than 3 colors. Some of the outcomes counted are
(1) 30 Reds (just one color)
(2) 10 R, 20 B (just two colors)
right way
step 1 (same as above) Pick which 3 colors to have. Can be done in \( \binom{5}{3} \) ways.
Say you pick R, W, B.
step 2 Pick 30 from the 300 R, W, B so as to get at least one of each.
Use total - opp.
Total is \( \binom{300}{30} \)
Opp = N(no R or no W or no B )
\[
= N(\text{no } R) + N(\text{no } W) + N(\text{no } B) \quad \text{(there are three 1-at-a-time terms)}
\]
\[
+ \left[ N(\text{no } R \text{ & no } B) + \text{other 2-at-a-time terms} \right] \quad \text{(there are } \binom{3}{2} \text{ 2-at-a-time terms)}
\]
\[
= 3 \left( \binom{200}{30} \right) - \binom{3}{2} \left( \binom{100}{30} \right)
\]
Final answer is \( \binom{5}{3} \left[ \binom{300}{30} - 3 \binom{200}{30} + \binom{3}{2} \binom{100}{30} \right] \)

PROBLEMS FOR SECTION 1.7

1. How many poker hands have at least (a) 1 spade (b) 2 spades (c) 4 spades
2. How many bridge hands have at most 2 spade.
3. A hundred people including the Smith family (John, Mary, Bill, Henry) buy a lottery ticket apiece. Three winning tickets will be drawn without replacement. The Smith family will be happy if someone in their family wins. Find the number of ways in which the family can end up happy. (For practice, see if you can find three methods.)
4. Is this a correct way to count the number of 6-letter words containing two or more Z's.
   Pick 2 spots for the Z's.
   Fill each of the remaining 4 spots with any of 26 letters (i.e., allow more Z's).
   Answer is \( \binom{6}{2} \cdot 26^4 \).
5. (a) How many bridge hands have a void, i.e., are missing at least one suit.
   (b) How many bridge hands have at least one royal flush (AKQJ10 in same suit).
6. Pick 11 letters from the alphabet. The order of the draw doesn't matter. How many possibilities have at most 3 vowels if the drawing is
(a) without replacement
(b) with replacement
Note. A committee with 5A and 6Z is considered to have 5 vowels.
A committee with 2A, 3E and 5Z has 5 vowels.

7. In how many ways can 10 cookies be given to 4 people so that no person gets left out if
(a) the cookies are all oatmeal
(b) the cookies are all different

8. (a) Look at strings of length 10 using the letters a, b, c. How many contain
(i) at least one b
(ii) at least one of each letter
(b) Repeat part (a) but with committees of size 10 (necessarily with repeated members) instead of strings.

9. There are 50 states and 2 senators from each state. How many committees of 15 senators can be formed containing at least one from each of Hawaii, Massachusetts and Pennsylvania.

10. How many permutations of the 26 letters of the alphabet contain neither cat nor dog.

11. How many poker hands contain 4 pictures including at least one ace.

12. How many 6-letter words have
(a) A and B appearing once each
(b) A and B appearing at least once each
(c) at least one from the list A, B; i.e., at least one A or at least one B
(d) at least one A and exactly one B
(e) two A's and at least two B's

13. (the game of rencontre—the matching game)
   Look at all the ways in which seven husbands H₁,..., H₇ and their wives W₁,..., W₇ can be matched up to form seven coed couples. For example, one match is H₁W₃, H₂W₁, H₃W₇, H₄W₄, H₅W₅, H₆W₆, H₇W₂
   (a) Find the number of matches in which H₃ is paired with his own wife.
   (b) Find the number of matches in which H₂ and H₅ are paired with their own wives.
   (c) Find the number of matches in which at least one husband is paired with his wife and simplify to get a pretty answer.
   Suggestion: The only feasible method is to use
   \[ N(\text{at least one match}) = N(H₁ \text{ gets his own wife or } H₂ \text{ gets his wife or } \ldots \text{ or } H₇ \text{ gets his wife}) \]
   (d) Find the number of matches in which no husband is paired with his wife.

14. (like example 3 but this time with 100 identical red balls, 100 identical whites etc)
   A box contains
   100 Reds (identical)
   100 Whites
   100 Blacks
   100 Greens
   100 Yellows
   Draw 30 without replacement.
   Find the number of samples containing (exactly) 3 colors.
SECTION 1.8 REVIEW PROBLEMS

1. Given a population of 20 people. In how many ways can you
   (1) choose a leader, a senior assistant and a junior assistant
   (2) choose a leader and two assistants
   (3) line them up for a photo if the group includes a set of identical triplets
   (4) choose a committee of 6 which must contain either Tom or Mary
   (5) choose a committee of 6 not containing Sam
   (6) give out 5 different books if a person is allowed to get more than one book
   (7) give out 5 different books if no one can get more than one book
   (8) give out 5 copies of the same book if no one can get more than one copy
   (9) give out 5 copies of the same book if a person can get more than one copy

2. How many poker hands have
   (a) 3 kings
   (b) at least 3 kings
   (c) no clubs
   (d) only clubs
   (e) all one suit
   (f) no suit missing

3. Rewrite the product \(67 \cdot 66 \cdot 65 \cdot \ldots \cdot 35 \cdot 34\) more compactly using factorial notation.

4. Start with 5 Men and 12 Women. How many permutations are there
   (a) with the men together
   (b) where the two end positions are men
   (c) with no two men adjacent

5. Given 6 balls all of different colors and 4 boxes \(B_1, \ldots, B_4\).
   In how many ways can the balls be distributed among the boxes
   (a) if there are no restrictions
   (b) so that \(B_2\) is empty
   (c) so that \(B_2\) gets at least one ball
   (d) so that no box is empty

6. A jury pool consists of 25 women and 17 men. Among the men, 2 are Hispanic and among the women, 3 are Hispanic.
   A jury of 12 people will be chosen from the pool.
   (a) A jury is called unrepresentative if it contains no women.
   It is also called unrepresentative if it contains no Hispanics.
   How many unrepresentative juries are there.
   (b) A jury is very unrepresentative if it contains neither women nor Hispanics.
   How many very unrepresentative juries are there.

7. A fast food stand sells hamburgers, pizza, tacos, chicken. If a car with 12 people drives up and each person orders one item, how many different orders could the waitress get (e.g., one possibility is 11 pizzas and a taco).

8. In how many ways can the symbols A, B, C, D, 2, 3, 4, 5 be permuted if the numbers and letters must alternate.

9. How many poker hands have
   (a) the ace of spades or KQJ of spades
   (b) all hearts or no hearts
10. (a) John, Mary and Tim sit in an empty row of 10 theater seats. For instance, some possibilities are
   \[ J----M---T \]
   \[ J--T M----- \]
   \[ J T M-------- \]

   (a) In how many ways can it be done.
   (b) In how many ways can it be done so that they sit together.
   (c) In how many ways can it be done so that none of them are together.

11. In how many ways can 10 people be assigned to 3 identical jobs in a typing pool, 4 identical file clerk jobs and 3 non-identical administrative jobs \( A_1, A_2, A_3 \).

12. Start with 10 men, 20 women, 30 boys, 40 girls. How many committees are there
   (a) of size 4 containing one of each type
   (b) of size 7 with 2 men, 4 women and 1 boy
   (c) of size 12 with at least 3 women
   (d) of size 12 with at most 1 woman

13. Grades will be given in a class of 29 students, \( S_1, \ldots, S_{29} \).
    The allowable grades are A, B, C, D, F.
    (a) The teacher has to hand in a list of students and their grades.
    How many possible lists are there
    (b) One possible distribution of grades is 28 A's and 1 B.
    Another distribution is 2 A's, 1 B, 23 C's and 3 D's.
    How many distributions are there.
    (c) Suppose you insist on giving 5 A's, 6 B's, 7 C's, 6 D's and 5 E's. In how many ways can the grades be assigned to the students.

14. The pizza place offers mushroom, anchovies, sausage and onion toppings. You can be as greedy or abstemious as you like. In how many ways can you order your pizza.

15. You want to move from A to B along the streets of the town in the diagram. You are allowed to move only east and south and can change directions at any intersection.
   For example one possibility is to walk 1 block E, 5 blocks S and then 3 blocks E.
   How many possible routes are there from A to B.

16. Compute (a) \( \binom{48}{2}/\binom{48}{3} \) (b) \( \frac{100}{98} \)

17. Two 5-digit strings (leading zeroes allowed) are considered equivalent if one is a permutation of the other. For example, 12033, 01332, 20331 are equivalent.
    How many "distinct" (i.e., non-equivalent) strings are there.

18. You have 26 scrabble chips, one for each letter. How many 5 letter words can be made containing the B chip.
19. A school cafeteria has ten dishes in its repertoire: 3 beef, B_1, B_2, B_3 and 7 vegetarian, V_1, ..., V_7.

Each day they offer three of the dishes (e.g., B_2, V_4, V_6; B_2, V_4, V_7). (a) How many days are there in a cycle, before they have to start repeating menus. For example, B_2, V_6, V_7 and B_3, V_6, V_7 is not a menu-repeat but B_2, V_6, V_7 and then B_2, V_6, V_7 again is a repeat.

(b) How many times does B_3 appear on the menu in one cycle.

(c) Suppose you only eat beef. On how many days do you get something to eat.

(d) You like to order two lunches, one beef and one veggie. On how many days in a cycle is this possible.

20. In how many ways can four numbers be selected from -5, -4, -3, -2, -1, 1, 2, 3, 4 so that their product is positive if the selection is

(a) without (b) with replacement.

21. Look at permutations of UNUSUAL.

(a) How many are there.

(b) How many have the 3 U's together.

(c) How many have all 4 vowels together.

(d) How many have no consecutive U's.

(e) How many have no consecutive consonants.

22. A hundred messages are sent through 4 available channels. For instance, here are some possibilities

(1) 73 go through C_1, 27 through C_2 and none through C_3 and C_4

(2) 97 go through C_2 and 1 each through C_1, C_3, C_4

etc

(a) How many possibilities are there.

(b) How many possibilities are there in which messages actually go through C_1, C_2 and C_3 but not C_4.

23. A trip is oversubscribed: 15 men, 16 women and 17 teenagers sign up but only 8 can go. In how many ways can the 8 be chosen so that

(a) the group has at least one adult and the same number of men as women.

(b) the group has 2 men and at least 2 women


For example, one possibility is 6 A's; another is 2 A's and 4 Q's.

(a) How many possibilities are there.

And does it matter whether the drawing is with or without replacement.

(b) How many of the possibilities in (a) have 3 pairs. For instance 2A, 2Q, 2Z is one possibility. Note that 4B and 2J does not count as 3 pairs.
SECTION 1.9 BINOMIAL AND MULTINOMIAL EXPANSIONS

the binomial expansion

When \((x + y)^7\) is multiplied out you can write the result like this:

\[
(x + y)^7 = \binom{7}{0}x^7 + \binom{7}{1}x^6y + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4 + \binom{7}{5}x^2y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7
\]

In general:

\[
(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n
\]

For example, when \((x + y)^9\) is multiplied out, the coefficient of the term \(x^3y^6\) is

\[
\binom{9}{6} = \frac{9!}{6!3!} = 84
\]

why (1) works

Remember that when you expand something like

\((a + b + c)(d + e)(f + g)\)

each term in the answer comes from multiplying together one term from each of the parentheses.

For instance one term in the expansion is \(adf\) (take \(a\) from the 1st paren, \(d\) from the 2nd, \(f\) from the 3rd).

Another term is \(adg\) (take \(a\) from the 1st paren, \(d\) from the 2nd, \(g\) from the 3rd).

A non-term is \(abd\) since \(a\) and \(b\) come from the same paren.

Now look at

\[(x + y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)\]

I'll show that there's a term \(x^3y^4\) in the expansion and its coeff is \(\binom{7}{4}\).

One of the terms in the product is \(x\ x\ x\ y\ y\ y\ y\), i.e.,

- \(x\) from first paren
- \(x\) from second paren
- \(x\) from third paren
- \(y\) from fourth paren
- \(y\) from fifth paren
- \(y\) from sixth paren
- \(y\) from seventh paren.

Another is \(x\ y\ x\ y\ y\ x\), i.e.,

- \(x\) from first paren
- \(y\) from second paren
- \(x\) from third paren
- \(y\) from fourth paren
- \(y\) from fifth paren
- \(y\) from sixth paren
- \(x\) from seventh paren.

Each of these is \(x^3y^4\) and there are as many of them as there are ways of permuting 3 \(x\)'s and 4 \(y\)'s, namely \(\frac{7!}{3!4!}\).

So there is an \(x^3y^4\) term with coeff \(\frac{7!}{3!4!}\), or equivalently \(\binom{7}{4}\).
Pascal's triangle and Pascal's identity

Fig 1 shows the coeffs in the expansion of \((x + y)^n\) lined up in rows to form Pascal's triangle.

Each line of coeffs can be gotten from the preceding line with the indicated addition process.

Fig 2 shows the same triangle in combinatorial notation.

\[
\begin{array}{cccccc}
& 1 & 2 & 1 &  & \\
1 & 3 & 3 & 1 &  & \\
1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]

\[
\begin{array}{cccccc}
& \binom{2}{0} & \binom{2}{1} & \binom{2}{2} &  & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} &  & \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & \\
\end{array}
\]

In the notation of Fig 2, the addition rule from Fig 1 looks like this:

\[
\binom{4}{0} + \binom{4}{1} = \binom{5}{1}, \quad \binom{4}{1} + \binom{4}{2} = \binom{5}{2}
\]

In general:

\[
\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \quad \text{(Pascal's identity)}
\]

why (2) works

Think of committees of size \(k\) chosen from \(n+1\) people \(P_1, \ldots, P_{n+1}\). Then

\[
\text{total committees} = \text{committees with } P_1 + \text{committees without } P_1
\]

The total number of committees is \(\binom{n+1}{k}\).

There are \(\binom{n}{k-1}\) committees containing \(P_1\) and \(\binom{n}{k}\) committees not containing \(P_1\) so

\[
\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \quad \text{QED}
\]

example 1

The next line in the triangle in Fig 1 is

\[
1 \ 6 \ 15 \ 20 \ 15 \ 6 \ 1
\]

so

\[(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6\]

example 2

The coeff of \(x^{26}y^{31}\) in the expansion of \((x + y)^{57}\) is \(\binom{57}{26}\) or equivalently \(\binom{57}{31}\)
the multinomial expansion

Here's an illustration of the general idea. If

\[(a + b + c + d)^{17}\]

is multiplied out, there are terms of the form

\[ab^5cd^{10}, \quad b^{16}d, \quad a^9b^2c^3d^3 \text{ etc.}\]

(In each term, the exponents add up to 17.)

The coefficient of \(a^8b^2c^3d^4\) in the expansion of \((a+b+c+d)^{17}\) is

\[\frac{17!}{8! \cdot 2! \cdot 3! \cdot 4!},\]

called a multinomial coefficient.

why (3) works

When \((a + b + c + d)^{17}\) is multiplied out, each term is created by taking one letter from each of the 17 parentheses.

One of the terms in the product is

\[a \ a \ a \ a \ a \ a \ a \ b \ b \ c \ c \ c \ d \ d \ d \ d,\]
i.e.,

- a's from first eight parens
- b's from next two parens
- c's from next three parens
- d's from next four parens

Another is

\[a \ b \ a \ a \ a \ a \ a \ a \ b \ c \ c \ c \ d \ d \ d \ d.\]

Each of these is \(a^8b^2c^3d^4\) and there are as many of them as there are ways of permuting 8 a's, 2 b's, 3 c's, 4 d's. So there is an \(a^8b^2c^3d^4\) term with coeff \(\frac{17!}{8! \cdot 2! \cdot 3! \cdot 4!}\) \((1) \text{ in Section 1.4)\)

example 3

In the expansion of \((x + y + z)^{15}\)

the coeff of \(x^4y^5z^6\) is \(\frac{15!}{4! \cdot 5! \cdot 6!}\)

the coeff of \(x^5z^{10}\) is \(\frac{15!}{5! \cdot 0! \cdot 10!}\) \((\text{you can leave out the 0! since it's 1})\)

eetc.

PROBLEMS FOR SECTION 1.9

1. Find the coeff of \(x^{43}y^6\) in the expansion of \((x + y)^{49}\).

2. Expand \((x + y)^7\) using Pascal's triangle and check some of the coeffs with (1).

3. Find the coeff of (a) \(a^2bc^4d^6\) (b) \(abcd^{10}\) in the expansion of \((a+b+c+d)^{13}\).
4. Find the coeff of $x_3^5 x_4^8$ in the expansion of $(x_1 + x_2 + x_3 + x_4)^{13}$.

5. How many terms are there in the expansion of $(x + y + z + w)^{17}$ (after you combine like terms).

Suggestion: A typical term is of the form $x_i^a y^b z^c w^d$ where the sum of the boxes is 17.

6. The last problem in Section 1.6 was solved twice. One method gave the answer $\binom{50}{4} + \binom{50}{4}$ and a second method gave the answer $\binom{51}{4} + \binom{51}{4} - 2\binom{50}{3}$.

Use Pascal to reconcile the two answers without resorting to any numerical computation.
SECTION 1.10 PARTITIONS

dividing people into distinguishable groups

Here's an example to illustrate the general idea. Suppose a group of 30 people is to be divided (partitioned) into a
- 4-person school board
- 4-person park board
- 4-person sewer board
- 5-person street board
- 5-person tree board
- 7-person housing board
- 1-person transportation board

To find the number of possibilities, use each board as a slot:

\[
\text{Answer} = \binom{30}{4} \binom{26}{4} \binom{22}{4} \binom{18}{5} \binom{13}{5} \binom{8}{7}
\]

\textit{footnote}

The expression in (1) is

\[
\frac{30!}{4!26!} \frac{26!}{4!22!} \frac{22!}{4!18!} \frac{18!}{5!13!} \frac{13!}{5!8!} \frac{8!}{7!1!}
\]

which cancels down to the more compact version

\[
\frac{30!}{(4!)^3 (5!)^2 7! 1!}
\]

In this section I mostly stick with the longer form.

\textit{more footnote}

Here's another method which jumps right to the compact form in (2). Print up the following 30 labels
- 4 school board labels
- 4 park board labels
- 4 sewer board labels
- 5 street board labels
- 5 tree board labels
- 7 housing board labels
- 1 transportation board label

Permute the labels. Can be done in \[
\frac{30!}{4!4!4!5!5!7!1!}
\] ways by (1) in §1.4 Now, that the labels are permuted, pin them respectively on the people P1,..., P30. (one way to do it). So the final answer is the one in (2).

dividing people into indistinguishable (nameless) groups

Take the same 30 people again but this time find the number of ways to divide (partition) them into seven groups (cells) of sizes 4,4,5,5,7,1.

For example, the list of outcomes looks like this:

\begin{enumerate}
  \item \text{outcome 1}
    \begin{itemize}
      \item P1, P2, P3, P4
      \item P5, P6, P7, P8
      \item P9, P10, P11, P30
      \item P12-P16
      \item P17-P21
      \item P22-P28
      \item P29
    \end{itemize}
\end{enumerate}
To count the number of outcomes here, you can't use slots because there is no such thing as the first group of size 4, the second group of size 4, the third group of size 4 and similarly the two groups of size 5 aren't distinguished from one another. So this problem is different from the one above. Here's how to get an answer using the connection between this problem and the previous one.

Each possibility in this problem gives rise to 3! 2! possibilities in the preceding example because once you have divided the 30 people into groups of sizes 4,4,4,5,5,7,1 you can give the three groups of size 4 the labels school, park, sewer in 3! ways; and you can give the two groups of size 5 the labels street, tree in 2! ways. For example, item 1 above gives rise to the following possibilities in the preceding example:

outcome 1a

\[
\begin{align*}
\text{P1, P2, P3, P4} & \quad \text{school} \\
\text{P5, P6, P7, P8} & \quad \text{park} \\
\text{P9, P10, P11, P30} & \quad \text{sewer} \\
\text{P12-P16} & \quad \text{street} \\
\text{P17-P21} & \quad \text{tree} \\
\text{P22-P28} & \quad \text{housing} \\
\text{P29} & \quad \text{trans}
\end{align*}
\]

outcome 1b

\[
\begin{align*}
\text{P1, P2, P3, P4} & \quad \text{park} \\
\text{P5, P6, P7, P8} & \quad \text{school} \\
\text{P9, P10, P11, P30} & \quad \text{sewer} \\
\text{P12-P16} & \quad \text{street} \\
\text{P17-P21} & \quad \text{tree} \\
\text{P22-P28} & \quad \text{housing} \\
\text{P29} & \quad \text{trans}
\end{align*}
\]

etc.

So

\[
\text{answer to this problem } \times 3! 2! = \text{answer to the preceding problem}
\]

\[
\text{answer to this problem } = \frac{\text{answer to preceding problem}}{3! 2!}
\]

The number of ways to divide 30 people into groups of sizes 4,4,4,5,5,7,1 is

\[
\frac{30}{4} \cdot \frac{26}{4} \cdot \frac{22}{4} \cdot \frac{18}{5} \cdot \frac{13}{5} \cdot \frac{8}{7} \cdot \frac{3! 2!}{3! 2!} \quad \leftarrow \text{extra factors}
\]
example 1
Start with 12 people.
In how many ways can they be divided into four trios.

solution
I'm going to do another problem first: Divide the 12 people into trios named T1, T2, T3, T4 (say to cook, serve, wash, dry). For this new problem you can use slots:

\[ \text{answer to new problem} = \frac{12 \cdot 9 \cdot 6}{3 \cdot 3 \cdot 3} \cdot 4! \]

In the original problem, the trios do not have names.

Answer to original problem \( \times 4! \) ways to name the trios = answer to new problem.

So the answer to the old problem = \( \frac{\text{answer to new problem}}{4!} \)

\[ \frac{12 \cdot 9 \cdot 6}{3 \cdot 3 \cdot 3} \cdot 4! \]

clarification
The answer to example 1 is not \( \frac{12 \cdot 9 \cdot 6}{3 \cdot 3 \cdot 3} \). This "answer" double counts (by a factor of 4!). For example, it counts the following outcomes as different when they are really the same.

\[ \text{outcome 1} \quad P_1, P_2, P_3; \quad P_4, P_5, P_6; \quad P_7, P_8, P_9; \quad P_{10}, P_{11}, P_{12} \\
\text{outcome 2} \quad P_4, P_5, P_6; \quad P_1, P_2, P_3; \quad P_7, P_8, P_9; \quad P_{10}, P_{11}, P_{12} \]

example 2
(a) In how many ways can 76 people be divided into 9 cells of sizes 7, 7, 13, 13, 13, 13, 6, 3, 1.

(b) In how many ways can 76 people be divided among 9 hotels \( H_1, \ldots, H_8 \) if

\( H_1 \) and \( H_2 \) have 7 vacancies each
\( H_3, H_4, H_5, \) and \( H_9 \) have 13 vacancies each
\( H_6 \) has 6 vacancies
\( H_7 \) has 3 vacancies
\( H_8 \) has 1 vacancy

solution
(a) The two groups of size 7 are not distinguishable from one another and neither are the four groups of size 13 so if you use 9 slots you'll have to divide by the extra factors 2!4! to compensate. The answer is

\[ \frac{76 \cdot 69 \cdot 62 \cdot 49 \cdot 36 \cdot 23 \cdot 10 \cdot 4}{7 \cdot 7 \cdot 13 \cdot 13 \cdot 13 \cdot 6 \cdot 3} \]

(b) Use the hotels as the slots:

\[ \frac{76 \cdot 69 \cdot 62 \cdot 49 \cdot 36 \cdot 23 \cdot 10 \cdot 4}{7 \cdot 7 \cdot 13 \cdot 13 \cdot 13 \cdot 6 \cdot 3} \]

footnote (another point of view)
The problem in (b) can be thought of as tossing 76 distinguishable balls into 8 distinguishable boxes \( B_1, \ldots, B_9 \) so that \( B_1 \) and \( B_2 \) get 7 balls each, \( B_3, B_4, B_5, B_9 \) get 13 balls each, \( B_6 \) gets 6 balls, \( B_7 \) gets 3 balls and \( B_8 \) gets 1 ball.

The problem in part (a) is to toss 76 distinguishable balls into 9 indistinguishable boxes so that 2 of the boxes get 7 balls each, 4 boxes get 13 balls each, one box gets 6 balls, one box gets 3 balls and one box gets 1 ball.
example 3 (when the two types coincide)
(a) In how many ways can 12 people be divided into groups of sizes 3, 4, 5.
(b) In how many ways can 12 people be divided into a 3-person school board, a 4-person park board and a 5-person sewer board.

solution
These are the same problem. In part (a), the groups are named "group of 3", "group of 4" and "group of 5". In each case the answer is
\[ \binom{12}{3} \binom{9}{4} \]
For part (b) the slots are school, park, sewer.
For part (a) the slots are "group of 3", "group of 4", "group of 5".

example 4
In how many ways can you divide 30 people into two trios, five quartets and two duos named D1 and D2.

solution
First do this problem. Divide 30 people into two trios T1, T2, five quartets Q1, ... Q5 and the duos D1, D2. Use the T's, Q's and D's as slots. Can be done in
\[ \binom{30}{3} \binom{27}{3} \binom{24}{4} \binom{20}{4} \binom{16}{4} \binom{12}{4} \binom{8}{4} \binom{4}{2} \]
In the original problem the trios were indistinguishable and the quartets were indistinguishable. Each outcome in the original problem gives rise to 2!5! outcomes in the new problem (the trios can be named in 2! ways, the quartets in 5! ways). So
\[ \text{answer to original problem} \times 2!5! = \text{answer to new problem} \]
\[ \text{answer to original problem} = \frac{\binom{30}{3} \binom{27}{3} \binom{24}{4} \binom{20}{4} \binom{16}{4} \binom{12}{4} \binom{8}{4} \binom{4}{2}}{2!5!} \]

PROBLEMS FOR SECTION 1.10

1. Four people will divide up 16 books B1,..., B16 so that Tom and Dick get 6 apiece, Harry gets 1 and Mary gets 3. In how many ways can it be done.

2. (a) Twelve people at a party divide into four conversational groups of 3 each. In how many ways can it be done.

(b) Twelve office workers arrange their vacations so that 3 go in each of the months of June, July, Aug, Sept. How many vacation schedules are there.

3. There are 100 children in the fifth grade.
(a) In how many ways can they be split into 4 sections of 25 each.
(b) In how many ways can 25 be assigned to Ms. X as their teacher, 25 to Mr. Y, 25 to Z and 25 to W.
(c) In how many ways can they be split into 4 sections of 25 each so that John and Mary are in different sections
(d) In how many ways can they be split into 4 sections of 25 each so that John and Mary are in the same section.

4. Twenty-one people show up for a tennis tournament. In how many ways can the 10 matches and one bye be arranged. Cancel to get a short answer.
5. In how many ways can 18 players be divided into 
   (a) two teams of 9 each 
   (b) a home team of size 9 and an away team of size 9 

6. In how many ways can 170 people be divided into 
   2 groups of size 5 
   3 groups of size 20 
   4 groups of size 10 
   a red team of size 30 
   a blue team of size 30 

   Cancel to get a compact answer. 

7. 350 people register for a course. 
   (a) In how many ways can they be split into sections of sizes 50, 100, 200. 
   (b) In how many ways can they be split into 50 for Section A, 100 for Section B and 200 for Section C. 
   (c) In how many ways can they be split into sections of sizes 50, 100, 200 so that John and Mary are in the same section. 

8. A company makes 20 different cereals. They decide to market them in a 10-pack and two 5-packs. They have to decide how to divide up the cereals among the packs. For instance should Tweeties go in the 10-pack and if so with what other 9 cereals or should Tweeties go in a 5-pack and if so with what 4 other cereals. How many possibilities are there. 

9. Here are two problems with proposed solutions. Are the solutions correct. If a solution is not correct, explain why not and fix it. 
   (a) To count the ways to divide 100 people into two groups of size 50: 
      Pick 50 of the 100 people. Can be done in \( \binom{100}{50} \) ways. 
      This automatically leaves a second group of 50. 
      Answer is \( \binom{100}{50} \). 
   
   (b) To count the ways to divide 100 people into groups of size 25 and 75: 
      Pick 25 of the 100 people. Can be done in \( \binom{100}{25} \) ways. 
      This automatically leaves a group of size 75. 
      Answer is \( \binom{100}{25} \). 

REVIEW PROBLEMS FOR CHAPTER 1

1. Show that \( \binom{r-1}{r-n} = \binom{r-1}{n-1} \)

2. Consider strings of digits of length 7 (e.g., 0000000, 2345678).
   (a) How many are there.
   (b) How many contain no 5's.
   (c) How many have at least one 2.
   (d) How many have an at least one 2 or 3.
   (e) How many have at least one 2 and at least one 3.
   (f) How many have one 3 and two 4's.
   (g) How many have one 3 and at least two 4's.

3. From a pool of 50 people, choose 5 for a basketball game, 4 for a hockey game and 9 for a baseball game. In how many ways can it be done if
   (a) the 3 games are played simultaneously
   (b) the games are played on different days

4. Consider 6-letter words using only the three letters P, Q, R.
   (a) How many have 4 of one kind, one of a second kind and one of a third kind (e.g., RRRRPQ, PPQPRP).
   (b) How many have 3 of a kind and a pair (e.g., PPPRRQ, QRQRQP).

5. There are 6! ways to arrange A₁, ..., A₆ on a line. In how many ways can they be arranged on a circle.
   Note that circles I and II below are different but I and III are the same (spinning doesn't change the circle).

6. (a) Find the coeff of \( x^9 y^{11} \) in the expansion of \((x + y)^{20}\).
    (b) Find the coeff of \( a^2 b^3 c^4 \) in the expansion of \((a + 5b + c)^9\).

7. You want to count the ways to divide 10 people into 5 pairs.
   (a) Explain why these proposed solutions are wrong.
      (i) There are \( \binom{10}{2} = 45 \) pairs. Pick 5 of them. Answer is \( \binom{45}{5} \) WRONG
      (ii) Pick 5 people. Then match these 5 with the remaining 5.
            Answer is \( \binom{10}{5} \cdot 5! \) WRONG
      (iii) Keep picking twosomes: \( \binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} \) WRONG
   (b) Find the right answer.

8. Four companies C₁, ..., C₄ bid for eleven different government grants. In how many ways can the grants be awarded if C₂ must get between 2 and 5 grants.

9. How many permutations of the 26 letters of the alphabet have exactly three letters between A and B.
10. Show that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$
   
   (a) by expanding $(1 + 1)^n$
   
   (b) with a combinatorial proof by making up a counting problem and solving it two different ways.

11. You have 10 hours to study for History, Math, Spanish. For example, two possibilities are
   
   - 9 hours on H, 1 hour on S
   - 3 hours on H, 3 hours on M, 4 hours on S
   
   How many possibilities are there.

12. You are required to take
   
   - one course from $A_1$, $A_2$, $A_3$
   - two courses from $B_1$, $B_2$, $B_3$, $B_4$, $B_5$
   - one course from $C_2$, $C_2$, $C_3$, $C_4$

   In how many ways can you fill the requirements.

13. In how many ways can 20 offices be linked by intercoms if there is equipment for three 2-office hookups, one 4-office hookup and two 5-offices hookups.

14. Use Pascal's identity to combine $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$ into a single binomial coefficient.

15. You're spending 7 days at a resort and there are four activities A, B, C, D available.
   
   (a) Suppose you want to do one activity each day so as to eventually try each one.
   
   For example, one schedule might be
   
   Monday B
   Tues B
   Wed A
   Thurs D
   Fri C
   Sat C
   Sun B

   How many of these schedules are there.

   (b) Suppose you are going to rest for four days and do a different activity on each of the other 3 days. For example, one schedule might be
   
   Mon rest
   Tues rest
   Wed B
   Thurs rest
   Fri A
   Sat rest
   Sun C

   How many of these schedules are there.

16. Use algebra to simplify $\binom{2n+3}{2n+1} - 3\binom{n+2}{n}$.

17. A store has 7 A's, 4 B's and 2 C's in stock. You come in to buy eight items. But if you ordered 1 A, 5 B's and 2 C's the store couldn't fill your order because it doesn't have enough B's. How many unfillable orders are there.

18. How many strings of 0's and 1's of length 7 have three 1's.
19. Five people $P_1, \ldots, P_5$ sit down in a 12 chair row. For example here's one possibility:

$$\begin{array}{ccccccc}
    & P_5 & & P_1 & P_2 & & P_4 & \ & P_3 \\
\end{array}$$

Find the number of possibilities using the following methods.
(a) Fill slots.
(b) Permute 5 people and 7 empty chairs.
(c) Pick chairs for the people and then fill them.
(d) (excruciatingly clever) Line up the people and then toss 7 identical chairs into the 6 in-betweens and ends.

20. The guidebook to Rome recommends 13 churches and 4 museums. One possibility is that you visit $C_1$, $C_5$, $C_{12}$, $M_2$. Another possibility is that you visit only $C_2$.

How many possibilities are there.

21. How many 4 letter words have no repeats.
For example, ABAC is no good; PQBC is OK.

22. In how many ways can 10 marbles be put into 4 distinct containers if
(a) all the marbles look the same
(b) the marbles are all different colors
(c) the marbles are all the same and each box must get at least one marble
(d) the marbles are all different colors and each box must get at least one marble

23. How many permutations of the alphabet contain
(a) MOTHER but not PA.
(b) neither MOTHER nor PA

24. Out of 100 students, 20 will be picked for academic honors and 12 for athletic honors (overlap allowed, i.e., athletic scholars are possible).
(a) In how many ways can it be done.
(b) How many of the possibilities from part (a) have 5 overlaps, i.e., how many of the possibilities give 5 people both athletic and academic honors.

25. A function $f(x,y)$ has the following 2nd order partial derivatives:

$$
\begin{align*}
    f_{xx} & \quad \text{(differentiate twice w.r.t. x)} \\
    f_{yy} & \quad \text{(differentiate twice w.r.t. y)} \\
    f_{xy} & \quad \text{(differentiate first w.r.t. x and then w.r.t. y)} \\
    f_{yx} & \quad \text{(differentiate first w.r.t. y and then w.r.t. x)} \\
\end{align*}
$$

Similarly, the 10th order partials of $f(x,y,z,w)$ are $f_{xxxxxxyyyyyyzzzz}$ etc.
(a) How many 10th order partials does $f(x,y,z,w)$ have.
(b) It's a theorem in calculus that $f_{xy} = f_{yx}$. Similarly $f_{xyzz} = f_{yzzx}$. All that counts is how often the differentiation is done for each variable; the order in which it's done doesn't matter.

With this in mind, how many truly different 10th order partials does $f(x,y,z,w)$ have.

26. (a) How many 15-letter words have five 3-of-a-kinds
(e.g., AAAAABBCCDQQZZZZ, ABCQ2ABC2ABCQ2)

(b) How many 15 card hands have five 3-of-a-kinds
(e.g., $A_{11}$, $A_{3}$, $A_{C}$, $Q_{1}$, $Q_{D}$, $Q_{S}$, $3_{B}$, $3_{C}$, $3_{S}$, $9_{D}$, $9_{C}$, $9_{S}$, $7_{H}$, $7_{C}$, $7_{S}$)