SOLUTIONS  Section 6.1

1. (a) \[ \frac{2}{L} \int_0^L K \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} K \cdot \left. \left( -\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right) \right|_0^L = -\frac{2K}{n\pi} (\cos \frac{n\pi}{2} - 1) \]

But \( \cos \frac{n\pi}{2} = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} \)

so \( -\frac{2K}{n\pi} (\cos \frac{n\pi}{2} - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4K}{n\pi} & \text{if } n \text{ is odd} \end{cases} \)

(b) \[ \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \left[ \int_0^{L/2} a \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^L b \sin \frac{n\pi x}{L} \, dx \right] \]

\[ = \frac{2}{L} \left[ -a \left. \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right|_0^{L/2} - b \left. \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right|_{L/2}^L \right] \]

\[ = \frac{2a}{n\pi} \left( 1 - \cos \frac{n\pi}{2} \right) + \frac{2b}{n\pi} (\cos \frac{n\pi}{2} - \cos n\pi) \]

If \( n \) is odd then \( \cos \frac{n\pi}{2} = 0 \) and \( \cos n\pi = -1 \) and this comes out to be \( \frac{2(a+b)}{n\pi} \)

If \( n = 2,6,10,\ldots \) then \( \cos \frac{n\pi}{2} = -1, \cos n\pi = 1 \) and this comes out to be \( \frac{4(b-a)}{n\pi} \)

If \( n = 4,8,12,\ldots \) then \( \cos \frac{n\pi}{2} = 1 \) and \( \cos n\pi = 1 \) and this comes out to be 0 QED

Now be grateful for the rest of the tables.

2. (a) Use (2) with \( L = 4 \) and multiply by 5 because of the \( 5x \).

\[ \text{integral} = \begin{cases} -\frac{40}{n\pi} & \text{if } n \text{ is even} \\ \frac{40}{n\pi} & \text{if } n \text{ is odd} \end{cases} \]

(b) Use (4b) with \( a = 5, b = 0, L = 6 \).

\[ \text{integral} = \begin{cases} \frac{10}{n\pi} & \text{if } n = 1,5,9,\ldots \\ -\frac{10}{n\pi} & \text{if } n = 3,7,11,\ldots \\ 0 & \text{if } n \text{ is even} \end{cases} \]

(c) Can't use (2c) because the upper limit is 6 but it's \( \sin \frac{n\pi x}{3} \).

\[ \text{integral} = \frac{2}{6} \left[ \int_0^3 5 \sin \frac{n\pi x}{3} \, dx + \int_3^6 0 \sin \frac{n\pi x}{3} \, dx \right] = \frac{5}{3} \int_0^3 \sin \frac{n\pi x}{3} \, dx \]

You can do this directly if you like or you can use (1) with \( L = 3 \) but multiply by \( L/2 \) because you're missing the \( 2/L \) in front.
integral = \begin{cases} 
\frac{5}{3} \frac{3}{2} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even}
\end{cases}

(d) Can't use (4a) because f(x) breaks at 2, not in the middle of $[0,6]$.

\[
\text{integral} = \frac{2}{6} \left[ \int_0^2 5 \sin \frac{n\pi x}{6} \, dx + \int_2^6 0 \sin \frac{n\pi x}{6} \, dx \right]
\]

\[
= \frac{5}{3} \int_0^2 \sin \frac{n\pi x}{6} \, dx = -\left. \frac{5}{3} \frac{6}{n\pi} \cos \frac{n\pi x}{6} \right|_0^2 = -\frac{10}{n\pi} \cos \frac{n\pi}{3} + \frac{10}{n\pi}
\]

If $n = 1$ the integral is $\frac{5}{\pi}$

If $n = 2$ the integral is $\frac{15}{2\pi}$

If $n = 3$ the integral is $\frac{20}{3\pi}$

If $n = 4$ the integral is $\frac{5}{4\pi}$

If $n = 5$ the integral is $\frac{15}{5\pi}$

If $n = 6$ the integral is $0 \text{ etc.}$

(e) The graph of $f(x)$ looks like the picture in (5) with $L = 8$, $K = 12$

\[
\text{integral} = \begin{cases} 
0 & \text{if } n \text{ is odd or if } n = 4,8,12,\ldots \\
-\frac{192}{n^2\pi^2} & \text{if } n = 2,6,10,\ldots
\end{cases}
\]

3. (a) Part I The equation was separated in Part I of example 1 so I won't repeat it.

The BC separate to $X(0) = 0$, $X(4) = 0$

$X(0) = 0$ makes $A = 0$

$X(4) = 0$ makes $B \sin 4\lambda = 0$, $4\lambda = n\pi$, $\lambda = \frac{n\pi}{4}$

So $X = B \sin \frac{n\pi x}{4}$ and $T = Ce^{-k\left(\frac{n\pi}{4}\right)^2 t}$

Part III By superposition,

\[
(*) \quad u = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{4}\right)^2 t} \sin \frac{n\pi x}{4}
\]

To get the IC you need

\[
8 = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{4} \quad \text{for } 0 \leq x \leq 4
\]
which you can get with
\[ c_n = \frac{2}{4} \int_0^4 8 \sin \frac{n\pi x}{4} \, dx = \begin{cases} 
0 & \text{if } n \text{ is even} 
\frac{32}{n\pi} & \text{if } n \text{ is odd} 
\end{cases} \] (use tables (A) with K=8)

Plug this into (*) to get final answer
\[ u = \frac{32}{\pi} e^{-\frac{k}{4t}} \left( \sin \frac{\pi x}{4} + \frac{32}{3\pi} e^{-\frac{k}{4t}} \sin \frac{3\pi x}{4} \right) 
+ \frac{32}{5\pi} e^{-\frac{k}{4t}} \sin \frac{5\pi x}{4} + \ldots \text{ for } 0 \leq x \leq 4, t \geq 0 \]

(b) This is like part (a) but to satisfy the IC you need
\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{4} \text{ for } 0 \leq x \leq 4 \text{ which you can get with} \]
\[ c_n = \frac{2}{4} \int_0^4 f(x) \sin \frac{n\pi x}{4} \, dx = \begin{cases} 
0 & \text{if } n = 4,8,12,\ldots 
\frac{24}{n\pi} & \text{if } n = 2,6,10,\ldots 
\frac{12}{n\pi} & \text{if } n \text{ is odd} 
\end{cases} \]

(Use (4a) in the tables with a = 6, b=0.)

Final solution is
\[ u = \frac{12}{\pi} e^{-\frac{k}{4t}} \left( \sin \frac{\pi x}{4} + \frac{24}{2\pi} e^{-\frac{k}{4t}} \sin \frac{2\pi x}{4} \right) 
+ \frac{12}{3\pi} e^{-\frac{k}{4t}} \sin \frac{3\pi x}{4} + \frac{12}{5\pi} e^{-\frac{k}{4t}} \sin \frac{5\pi x}{4} 
+ \frac{24}{6\pi} e^{-\frac{k}{4t}} \sin \frac{6\pi x}{4} + \ldots \text{ for } 0 \leq x \leq 4, t \geq 0 \]

(c) Continuing as in part (a), for 0 \leq x \leq 4 you need
\[ 5 \sin 2\pi x + 6 \sin 5\pi x = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{4} \]
\[ = c_1 \sin \frac{\pi x}{4} + c_2 \sin \frac{2\pi x}{4} + c_3 \sin \frac{3\pi x}{4} + \ldots \]

You don't need the fancy formulas for Fourier sine coeffs to accomplish this. By inspection what you need is c_8 = 5, c_{20} = 6, other c's = 0.

Final answer is
\[ u = 5 e^{-\frac{k}{4t}} \sin \frac{8\pi x}{4} + 6 e^{-\frac{k}{4t}} \sin \frac{20\pi x}{4} \text{ for } 0 \leq x \leq 4, t \geq 0 \]
SOLUTIONS Section 6.2

1. (a) $X'(0) = 0$  
   (b) $X'(5) = 0$  
   (c) doesn't sep  
   (d) doesn't sep  
   (e) $X(4) = 0$  
   (f) doesn't sep  
   (g) $T(0) = 0$

2. (a) I separated the equation in example 1 so I won't repeat it here. 
The BC separate to $X'(0) = 0$, $X'(6) = 0$ 
Consider the case where 
$$X = A \cos \lambda x + B \sin \lambda x, \quad T = Ce^{-k\lambda^2 t}, \quad X' = -\lambda A \sin \lambda x + \lambda B \cos \lambda x.$$ 

$X'(0) = 0$ makes $B = 0$ 
$X'(6) = 0$ makes $-\lambda A \sin 6\lambda = 0$, $6\lambda = n\pi$, $\lambda = \frac{n\pi}{6}$

Then $X = A \cos \frac{n\pi x}{6}, \quad T = Ce^{-k\left(\frac{n\pi}{6}\right)^2 t}$. 

Since the BC are of the form $X'(0) = 0$, $X'(6) = 0$ you should try the zero separation case; it produces the solution $X = A x + B$, $T = C$, $X' = A$.

The BC make $A = 0$ so $u = X(x) T(t) = BC = \psi$ 
By superposition, 

$$(*) \quad u = \psi + \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{6}\right)^2 t} \cos \frac{n\pi x}{6}$$

To get the IC you need $f(x) = \psi + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{6}$ for $x$ in $[0,6]$ 
So $\psi = \text{av value of } f = 7$

$$C_n = \frac{2}{6} \int_0^6 f(x) \cos \frac{n\pi x}{6} \, dx = \begin{cases} 
-\frac{8}{n\pi} & \text{if } n = 1,5,9,\ldots \\
\frac{8}{n\pi} & \text{if } n = 3,7,11,\ldots \\
0 & \text{if } n \text{ is even} 
\end{cases} \quad \text{(tables (4))}$$

Plug these into $(*)$ to get the final answer 

$$u = 7 - \frac{8}{\pi} e^{-k\left(\frac{\pi}{6}\right)^2 t} \cos \frac{\pi x}{6} + \frac{8}{3\pi} e^{-k\left(\frac{3\pi}{6}\right)^2 t} \cos \frac{3\pi x}{6} - \frac{8}{5\pi} e^{-k\left(\frac{5\pi}{6}\right)^2 t} \cos \frac{5\pi x}{6} + \ldots \text{ for } 0 \leq x \leq 6, \ t \geq 0$$

(b) (i) The rod is initially at $20^\circ$. The lateral surface of the rod is insulated and the ends are insulated so no calories flow out. In fact no calories flow anywhere in the rod since it is all at the same temp. So the rod stays at $20^\circ$ for all time; i.e., sol is $u(x,t) = 2$ for $t \geq 0$, $0 \leq x \leq 6$. 
(ii) Continue as in (a). To get the IC you need 

$$2 = \psi + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{6} \text{ for } x \text{ in } [0,6]$$

By inspection you can get this with $\psi = 2$, $C_n = 0$ (This is what you'd get if you do it the long way and use the formulas for the Fourier cosine coeffs.) 
Plug these into $(*)$ get final answer $u = 2$.  
3. Part I Try \( u(x,t) = X(x)T(t) \). Then \( XT' = X'T - XT \). There are two ways to continue the separation.

**method 1** \( XT' = (X'' - X)T, \quad \frac{T'}{T} = \frac{X'' - X}{X} = \lambda, \)

\[ T' - \lambda T = 0, \quad X'' - (1 + \lambda)X = 0 \]

The three cases to consider here are \( 1 + \lambda \) positive, negative, zero.

**method 2** \( X''T = X(T' + T), \quad \frac{X''}{X} = \frac{T' + T}{T} = \lambda, \)

\[ T' + (1 - \lambda)T = 0, \quad X'' - \lambda X = 0 \]

The three cases here are \( \lambda \) positive, negative, zero.

The two methods will eventually produce the same collection of solutions but the second method is simpler since it makes the \( X \) part simpler (albeit at the expense of the \( T \) part). So continue with the separation in method 2.

In the case where \( \lambda \) is negative and renamed \(-\lambda^2\) you have

\[ X'' + \lambda^2 X = 0, \quad X = A \cos \lambda x + B \sin \lambda x, \]

\[ T' + (1 + \lambda^2)T = 0, \quad T = Ce^{-(1+\lambda^2)t} \]

The BC separate to \( X(0) = 0 \) and \( X(L) = 0 \)

**Part II** (plug in BC)

\( X(0) = 0 \) makes \( A = 0 \)

\( X(L) = 0 \) makes \( B \sin \lambda L = 0, \quad \lambda = \frac{n\pi}{L} \)

**Part III** (get a gen sol and go for the IC)

The solution is

\[ u = \sum_{n=1}^{\infty} C_n e^{-(1+(\frac{n\pi}{L})^2)t} \sin \frac{n\pi x}{L} \]

To get the IC you need

\[ \frac{\partial u}{\partial x} \bigg|_{x=0} = \sum_{n=1}^{\infty} C_n \frac{n\pi}{L} e^{-(1+(\frac{n\pi}{L})^2)t} \sin \frac{n\pi x}{L} \bigg|_{x=0} = \left\{ \begin{array}{ll}
0 & \text{if } n \text{ is even} \\
\frac{32}{n\pi} & \text{if } n \text{ is odd} 
\end{array} \right. \quad \text{(tables (1))} \]

Can get this with \( C_n = \frac{2L}{\pi} \int_0^L 8 \sin \frac{n\pi x}{L} \, dx = \left\{ \begin{array}{ll}
0 & \text{if } n \text{ is even} \\
\frac{32}{n\pi} & \text{if } n \text{ is odd} 
\end{array} \right. \)

Plug these constants into \((*)\). The answer is

\[ u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(1+(\frac{n\pi}{L})^2)t} \sin \frac{n\pi x}{L} \bigg|_{x=0} = \left\{ \begin{array}{ll}
0 & \text{if } n \text{ is even} \\
\frac{32}{n\pi} & \text{if } n \text{ is odd} 
\end{array} \right. \]

\[ \frac{32}{5\pi} e^{-(1+(\frac{5\pi}{L})^2)t} \sin \frac{5\pi x}{L} + \ldots \text{for } 0 \leq x \leq L, \ t \geq 0 \]
4. (a) Plug $t = \infty$ into the solution in (5). The steady state solution is $u = 1$ because when $t \to \infty$, the exponentials all $\to 0$.

(Initially, the left half of the rod is at temp 0, the right half is at temp 2, the lateral surface and the ends are insulated, so calories flow within the rod until the temperature evens out at 1.)

(b) **physical argument**

The lateral surface of the rod is insulated, the ends are insulated, and the initial temp distribution is $f(x)$. Calories flow within the rod (they can't escape) until the temperature "evens out". The steady state temperature distribution is a constant, namely is the average value of $f(x)$.

**mathematical version**

The solution satisfying the heat equation and the BC look like (1) but with $L$ instead of $8$:

$$u = A_0 + \sum_{n=1}^{\infty} A_n \ e^{-k \left( \frac{n\pi}{L} \right)^2 t} \cos \left( \frac{n\pi x}{L} \right) \text{ for } 0 \leq x \leq L, \ t \geq 0$$

The steady state solution is the constant $A_0$ because when $t \to \infty$, the exponentials all $\to 0$.

To satisfy the IC you need

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right) \text{ for } 0 \leq x \leq L$$

so $A_0 = \frac{1}{L} \int_{0}^{L} f(x) \ dx = \text{average value of } f(x) \text{ on the interval } [0,L]$.

So the steady state solution is the average value of $f(x)$ on the interval $[0,L]$.

5. $P'(5)Q(q) = 0$ for all $q$

$P'(5) = 0$ or $Q(q) = 0$ for all $q$

But if $Q(q) = 0$ for all $q$ then $v(p,q) = 0$ for all $q$. So this is one possibility. But when it comes to using superposition to add solutions to get a good solution with lots of arbitrary constants so that some IC can be satisfied, adding in the solution $v(p,q) = 0$ will not be helpful.

So the only useful possibility is $P'(5) = 0$.

So $\frac{\partial v}{\partial p}(5,q) = 0$ for all $q$ separates to $P'(5) = 0$. 
SOLUTIONS Section 6.3

1. Use (1)-(3) with \(L = 6\), \(f(x)\) as in the picture, \(g(x) = 0\). Then \(D_n = 0\). To find \(C_n\) use (5a) on the reference page with \(L = 6\), \(K = 2\):

\[
C_n = \begin{cases} 
0 & \text{if } n \text{ is even} \\
\frac{16}{n^2 \pi^2} & \text{if } n = 1, 5, 9, \ldots \\
-\frac{16}{n^2 \pi^2} & \text{if } n = 3, 7, 11, \ldots
\end{cases}
\]

Solution is

\[
y = \frac{16}{\pi^2} \left[ \cos \frac{\pi at}{6} \sin \frac{\pi x}{6} - \frac{1}{9} \cos \frac{3\pi at}{6} \sin \frac{3\pi x}{6} + \frac{1}{25} \cos \frac{5\pi at}{6} \sin \frac{5\pi x}{6} - \ldots \right]
\]

2. Use the solution in (1)-(3) with \(f(x) = 0\), \(g(x) = \delta(x - \frac{1}{2} L)\). Then \(C_n = 0\).

\[
d_n = \frac{L}{n \pi a} \frac{2}{L} \int_0^L \delta(x - \frac{1}{2} L) \sin \frac{n \pi x}{6} \, dx
\]

\[
= \frac{L}{n \pi a} \frac{2}{L} \sin \frac{n \pi L/2}{L} \quad (\text{sifting property } \S 6.2A)
\]

\[
= \frac{2}{n \pi a} \sin \frac{n \pi}{2} = \begin{cases} 
0 & \text{if } n \text{ is even} \\
2/n \pi a & \text{if } n = 1, 5, 9, \ldots \\
-2/n \pi a & \text{if } n = 3, 7, 11, \ldots
\end{cases}
\]

Solution is

\[
y = \frac{2}{\pi a} \left[ \sin \frac{\pi at}{L} \sin \frac{\pi x}{L} - \frac{1}{3} \sin \frac{3\pi at}{L} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi at}{L} \sin \frac{5\pi x}{L} - \ldots \right]
\]

3. **Part I** Separate the variables

The equation was separated in Part I of example 1 so I won't repeat it all here. The two potentially useful cases are:

**case 3**

\[
X = A \cos \lambda x + B \sin \lambda x \\
T = C \cos \lambda at + D \sin \lambda at
\]

**case 1**

\[
X = Ax + B \\
T = Ct + D
\]

The BC separate to \(X'(0) = 0\), \(X'(L) = 0\)

**Part II** Plug in the homog BC

Begin with case 3 where the \(X\) solutions are sines and cosines. First find

\[
X' = -\lambda A \sin \lambda x + \lambda B \cos \lambda x
\]

\(X'(0) = 0\) makes \(B = 0\)

\(X'(L) = 0\) makes \(-\lambda A \sin \lambda L = 0\)
Either \( A = 0 \) (which together with \( B = 0 \) produces only the trivial solution \( y = 0 \)) or \( \lambda = 0 \) (impossible since in this case \(-\lambda^2\) represents a negative number) or

\[ \sin \lambda L = 0 \]

\[ \lambda L = n\pi, \quad \lambda = \frac{n\pi}{L} \]

\[ X = A \cos \frac{n\pi x}{L} \]

Since the BC are \( X'(0) = 0, X'(L) = 0 \), anticipate that the zero separation case will produce a solution too. If \( \lambda = 0 \) then

\[ X = Ax + B, \quad T = Ct + D, \]

The BC's \( X'(0) = 0, X'(L) = 0 \) force \( A = 0 \).

So \( X = B \) and from this case you have solutions

\[ y(x,t) = X(x)T(t) = B(Ct + D) = Pt + Q \text{ for any } P,Q. \]

**Part III** Get a general sol and plug in the IC

By superposition, a general solution is

\[
(1) \quad y(x,t) = Pt + Q + \sum_{n=1}^{\infty} \left[ C_n \cos \frac{n\pi a t}{L} + D_n \sin \frac{n\pi a t}{L} \right] \cos \frac{n\pi x}{L}
\]

Now you have to determine the constants to satisfy the IC.

To get \( y(x,0) = f(x) \) for \( x \) in \([0,L]\) you need

\[ f(x) = Q + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L} \text{ for } x \text{ in } [0,L] \]

which you can get with

\[
(2) \quad Q = \frac{1}{L} \int_{0}^{L} f(x) \, dx, \quad C_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx
\]

Now you have to satisfy the second IC. First, get

\[
\frac{\partial y}{\partial t} = P + \sum_{n=1}^{\infty} \left[ -\frac{n\pi a}{L} C_n \sin \frac{n\pi a t}{L} + \frac{n\pi a}{L} D_n \cos \frac{n\pi a t}{L} \right] \cos \frac{n\pi x}{L}
\]

Then to get \( \frac{\partial y}{\partial t}(x,0) = g(x) \) for \( x \) in \([0,L]\) you need

\[ g(x) = P + \sum_{n=1}^{\infty} \frac{n\pi a}{L} D_n \cos \frac{n\pi x}{L} \text{ for } x \text{ in } [0,L] \]

To do this you need
(3) \[ P = \frac{1}{L} \int_0^L g(x) \, dx \]

and

\[ \frac{n\pi a}{L} D_n = \frac{2}{L} \int_0^L g(x) \cos \frac{n\pi x}{L} \, dx \]

(4) \[ D_n = \frac{2}{n\pi a} \int_0^L g(x) \cos \frac{n\pi x}{L} \, dx \]

The solution is (1), with the constants in the solution given in (2)-(4).

Footnote: The solution in (1) is unrealistic in the sense that \( y \to \infty \) as \( t \to \infty \) (unless the coefficient \( P \) is 0). That's because the wave equation doesn't include a term representing gravity and when the ends are on rollers, it's possible for the idealized rope to move unboundedly high.

4. This is like problem 3 but with \( L = 2 \). I won't repeat the whole separation. Begin with the case where the \( X \) solutions are sines and cosines:

\[ X = A \cos \lambda x + B \sin \lambda x, \quad T = C \cos \lambda t + D \sin \lambda t \]

The BC \( X'(0) = 0, X'(2) = 0 \) make \( B = 0, \lambda = \frac{n\pi}{2} \) so \( X = A \cos \frac{n\pi x}{2} \)

The second IC separates to \( T'(0) = 0 \) which makes \( D = 0 \) so \( T = C \cos \frac{n\pi at}{2} \)

From this case you have \( y = E \cos \frac{n\pi at}{2} \cos \frac{n\pi x}{2} \)

In the \( \lambda = 0 \) case,

\[ X = Ax + B, \quad T = Ct + D \]

The BC \( X'(0) = 0, X'(2) = 0 \) make \( A = 0 \)

The IC \( T'(0) = 0 \) makes \( C = 0 \).

From this case you get \( y = BD = 0 \)

By superposition, \( y = Q + \sum_{n=1}^{\infty} E_n \cos \frac{n\pi x}{2} \cos \frac{n\pi at}{2} \)

To get the final (nohomog) IC you need

\[ x = Q + \sum_{n=1}^{\infty} E_n \cos \frac{n\pi x}{2} \quad \text{for } x \text{ in } [0,2] \]

\[ Q = \text{average value of } x \text{ in } [0,2] = 1 \]
\[ E_n = \frac{2}{2} \int_{0}^{2} x \cos \frac{n\pi x}{2} \, dx = \begin{cases} 
0 & \text{if } n \text{ is even} \\
-\frac{8}{n^2\pi^2} & \text{if } n \text{ is odd} 
\end{cases} \quad \text{(ref page (3))} \]

Answer is
\[ y = 1 - \frac{8}{\pi^2} \cos \frac{\pi a}{2} \cos \frac{\pi x}{2} - \frac{8}{9\pi^2} \cos \frac{3\pi a}{2} \cos \frac{3\pi x}{2} - \frac{8}{25\pi^2} \cos \frac{5\pi a}{2} \cos \frac{5\pi x}{2} - \ldots \]

5. (a) Initially, the wire lies flat on the x-axis but has velocity 3, i.e., is in the process of moving up at 3 meters per second. The ends are looped around poles, free to move up or down. There is no gravity or air resistance. So as time goes on the wire simply continues moving up at the rate of 3 meters per second. So
\[ y(x,t) = 3t \]

(b) I won't repeat the separation. One of the good cases is where
\[ X = A \cos \lambda x + B \sin \lambda x, \quad T = C \cos \lambda at + D \sin \lambda at \]
\[ X' = -\lambda A \sin \lambda x + \lambda B \cos \lambda x \]
\[ X'(0) = 0 \text{ makes } B = 0 \]
\[ X'(L) = 0 \text{ makes } -\lambda A \sin L\lambda = 0, \quad L\lambda = n\pi, \quad \lambda = \frac{n\pi}{L} \]

The first IC is homog and it separates to \[ T(0) = 0 \] which makes \[ C = 0 \]

In the \[ \lambda = 0 \] case,
\[ X = Ax + B, \quad T = Ct + D \]
\[ X'(0) = 0 \text{ and } X'(L) = 0 \text{ make } A = 0 \]
\[ T(0) = 0 \text{ makes } D = 0 \]

Put it all together to get
\[ (*) \quad y = kt + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi a}{L} \cos \frac{n\pi x}{L} \]

Still need \[ \frac{\partial Y(x,0)}{\partial t} = 3 \text{ for } x \text{ in } [0,L]. \] We have
\[ \frac{\partial Y}{\partial t} = k + \sum_{n=1}^{\infty} \frac{n\pi a}{L} A_n \cos \frac{n\pi a}{L} \cos \frac{n\pi x}{L} \]

so you need \[ 3 = k + \sum_{n=1}^{\infty} \frac{n\pi a}{L} A_n \cos \frac{n\pi x}{L} \text{ for } x \text{ in } [0,L] \]

You don't need fancy Fourier coeff formulas for this. By inspection, pick \[ k = 3, \quad A_n = 0. \] Plug this into \((*)\) to get final answer \[ y = 3t. \]
SOLUTIONS  Section 6.4  

1. (a) The equation was separated in Part I of example 1 so I won't repeat it. Use the case where \( X = A \cos \lambda x + B \sin \lambda x \), \( Y = C \cosh \lambda y + D \sinh \lambda y \). 

Then \( X' = -\lambda A \sin \lambda x + \lambda B \cos \lambda x \) 

\( X'(0) = 0 \) so \( B = 0 \). 

\( X'(6) = 0 \) so \(-\lambda A \sin 6\lambda = 0\), \( 6\lambda = n\pi \), \( \lambda = \frac{n\pi}{6} \). 

\( Y(0) = 0 \) so \( C = 0 \). 

Use the \( \lambda = 0 \) separation case where \( X = Ex + F \), \( Y = Gy + H \) 

\( X'(0) = 0 \) and \( X'(6) = 0 \) make \( E = 0 \) 

\( Y(0) = 0 \) makes \( H = 0 \). 

This case produces solution \( v = FGy = Ky \). 

By superposition, 

\[ v = Ky + \sum_{n=1}^{\infty} D_n \sinh \frac{n\pi y}{6} \cos \frac{n\pi x}{6} \] 

The top BC is \( v(x,18) = f(x) \) for \( x \) in \([0,6]\) where \( f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 3 \\ 3 & \text{if } 3 \leq x \leq 6 \end{cases} \) 

To get it you need 

\( f(x) = 18K + \sum_{n=1}^{\infty} D_n \sinh 3n\pi \cos \frac{n\pi x}{6} \) for \( x \) in \([0,6]\) 

which you can get with 

\( 18K = \text{av value of } f = \frac{3}{2} \) 

\( K = \frac{1}{12} \) 

\[ D_n \sinh 3n\pi = \frac{2}{6} \int_{0}^{6} f(x) \cos \frac{n\pi x}{6} \, dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-6}{n\pi} & \text{if } n = 1,5,9,... \quad \text{(by (4b))} \\ \frac{6}{n\pi} & \text{if } n = 3,7,11,... \end{cases} \] 

so 

\[ D_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{\sinh 3n\pi} \cdot \frac{6}{n\pi} & \text{if } n = 1,5,9,... \\ \frac{1}{\sinh 3n\pi} \cdot \frac{6}{n\pi} & \text{if } n = 3,7,11,... \end{cases} \] 

Solution is 

\[ v = \frac{1}{12} y + \frac{6}{\pi} \left[ -\frac{1}{\sinh 3\pi} \sinh \frac{3\pi y}{6} \cos \frac{3\pi x}{6} + \frac{1}{3 \sinh 9\pi} \sinh \frac{3\pi y}{6} \cos \frac{3\pi x}{6} - \frac{1}{5 \sinh 15\pi} \sinh \frac{5\pi y}{6} \cos \frac{5\pi x}{6} + \ldots \right] \] 

(b) I won't repeat the separation. 

Use the case where \( X = A \cos \lambda x + B \sin \lambda x \), \( Y = C \cosh \lambda y + D \sinh \lambda y \).
Then \( x' = -\lambda A \sin \lambda x + B \lambda \cos \lambda x, \quad y' = C \lambda \sinh \lambda y + D \lambda \cosh \lambda y \)

\[ x'(0) = 0 \text{ so } B = 0 \]

\[ x'(a) = 0 \text{ so } -\lambda A \sin a = 0, \quad \lambda = \frac{n\pi}{a} \]

\[ y'(0) = 0 \text{ so } D = 0 \]

In the case where \( \lambda = 0 \) you have \( X = E x + F, \; Y = G y + H \)

\[ x'(0) = 0 \text{ and } x'(a) = 0 \text{ make } E = 0 \]

\[ y'(0) = 0 \text{ makes } G = 0 \]

From this case you get \( v = Fh = K \)

By superposition

\[
v = K + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}
\]

The last BC is \( v(x,b) = f(x) \) so you need

\[ f(x) = K + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi b}{a} \cos \frac{n\pi x}{a} \text{ for } x \text{ in } [0,a] \]

which you can get with

\[
K = \frac{1}{a} \int_0^a f(x) \, dx, \quad A_n \cosh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a} \, dx
\]

\[
A_n = \frac{2}{a \cosh \frac{n\pi b}{a}} \int_0^a f(x) \cos \frac{n\pi x}{a} \, dx
\]

(c) I won't repeat the separation.

Use the case where \( X = A \cos \lambda x + B \sin \lambda x, \; Y = C \cosh \lambda y + D \sinh \lambda y \)

Then \( x' = -\lambda A \sin \lambda x + B \lambda \cos \lambda x \)

\[ x'(0) = 0 \text{ so } B = 0 \]

\[ x'(4) = 0 \text{ so } -\lambda A \sin 4\lambda = 0, \quad \lambda = \frac{n\pi}{4} \]

\[ y(0) = 0 \text{ so } C = 0 \]

Use the case where \( X = E x + F, \; Y = G y + H \)

\[ x'(0) = 0 \text{ and } x'(4) = 0 \text{ make } E = 0 \]

\[ y(0) = 0 \text{ makes } H = 0 \]

From this case you have \( v = FG = Ky \)

By superposition

\[
v = Ky + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{4} \cos \frac{n\pi x}{4}
\]
Now plug in the final BC \( \frac{\partial v}{\partial y} (x,5) = 2x \) for \( x \) in \([0,4]\). We have

\[
\frac{\partial v}{\partial y} = K + \sum_{n=1}^{\infty} A_n \frac{n\pi}{4} \cosh \frac{n\pi y}{4} \cos \frac{n\pi x}{4}
\]

so you need

\[
2x = K + \sum_{n=1}^{\infty} A_n \frac{n\pi}{4} \cosh \frac{n\pi 5}{4} \cos \frac{n\pi x}{4} \quad \text{for} \ x \in [0,4],
\]

\[
K = \text{av value of } 2x \text{ in } [0,4] = 4
\]

\[
\frac{n\pi}{4} A_n \cosh \frac{5n\pi}{4} = \frac{2}{4} \int_0^4 2x \cos \frac{n\pi x}{4} \, dx = \begin{cases} -\frac{32}{n^2 \pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad (\text{tables (3)})
\]

So \( A_{\text{even}} = 0 \), \( A_{\text{odd}} = -\frac{128}{n^3 \pi^3} \cosh \frac{5n\pi}{4} \)

and the final answer is

\[
v = 4y - \frac{128}{\pi^3 \cosh \frac{5\pi}{4}} \sinh \frac{\pi y}{4} \cos \frac{\pi x}{4} - \frac{128}{27\pi^3 \cosh \frac{15\pi}{4}} \sinh \frac{3\pi y}{4} \cos \frac{3\pi x}{4} - \ldots
\]

2. (a) I won’t repeat the separation. Use the case where \( X = A \cos \lambda x + B \sin \lambda x, \ Y = Ce^{-\lambda y} + De^{-\lambda y} \).

\( X(0) = 0 \) so \( A = 0 \)

\( X(5) = 0 \) so \( B \sin 5\lambda = 0 \) \( 5\lambda = n\pi, \lambda = \frac{n\pi}{5} \)

Make \( C = 0 \) to keep \( Y \) finite.

By superposition, \( v = \sum_{n=1}^{\infty} D_n e^{-\frac{n\pi y}{5}} \sin \frac{n\pi x}{5} \)

To get the last BC, \( v(x,0) = 2 \), you need

\[
2 = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{5} \quad \text{for} \ x \in [0,5],
\]

\[
D_n = \frac{2}{5} \int_0^5 2 \sin \frac{n\pi x}{5} \, dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\]

Solution is

\[
v = \frac{8}{\pi} e^{-\frac{\pi y}{5}} \sin \frac{\pi x}{5} + \frac{8}{3\pi} e^{-\frac{3\pi y}{5}} \sin \frac{3\pi x}{5} + \frac{8}{5\pi} e^{-\frac{5\pi y}{5}} \sin \frac{5\pi x}{5} + \ldots
\]
(b) I won't repeat the separation. Use the case where
\[ X = A \cos \lambda x + B \sin \lambda x, \quad Y = C e^{\lambda y} + D e^{-\lambda y}. \]

\[ X(0) = 0 \quad \text{so} \quad A = 0 \]
\[ X(5) = 0 \quad \text{so} \quad B \sin 5\lambda = 0, \quad 5\lambda = n\pi, \quad \lambda = \frac{n\pi}{5} \]

Make \( C = 0 \) to keep \( Y \) finite.

By superposition,

\[ v = \sum_{n=1}^{\infty} D_n e^{-\frac{n\pi y}{5}} \sin \frac{n\pi x}{5} \]

The last BC is \( \frac{\partial v}{\partial y}(x,0) = 3 \) for \( 0 \leq x \leq 5 \). First find

\[ \frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} D_n \frac{-n\pi}{5} e^{-\frac{n\pi y}{5}} \sin \frac{n\pi x}{5} \]

Then set \( y = 0 \), \( \frac{\partial v}{\partial y} = 3 \): you need

\[ 3 = \sum_{n=1}^{\infty} D_n \frac{-n\pi}{5} \sin \frac{n\pi x}{5} \quad \text{for} \quad 0 \leq x \leq 5 \]

which you can get with

\[ \frac{-n\pi}{5} D_n = \frac{12}{n\pi} \int_{0}^{5} 3 \sin \frac{n\pi x}{5} \, dx = \begin{cases} 0 & \text{if} \ n \text{ is even} \\ \frac{12}{n\pi} & \text{if} \ n \text{ is odd} \end{cases} \]

So \( D_n = \frac{-60}{n^2\pi^2} \) for odd \( n \). Plug this back into (*) to get the solution

\[ v = -\frac{60}{\pi^2} \left( e^{\frac{\pi y}{5}} \sin \frac{\pi x}{5} + \frac{1}{9} e^{\frac{3\pi y}{5}} \sin \frac{3\pi x}{5} + \frac{1}{25} e^{\frac{5\pi y}{5}} \sin \frac{5\pi x}{5} + \ldots \right) \]

3. Plugging it into the cosh sinh version makes \( C = 0 \) and leaves

\[ Y = D \sinh \lambda y \quad \text{for any} \ D \]

Plugging it into the exp version makes \( E + F = 0, \ F = -E \),

\[ Y = E e^{\lambda y} - E e^{-\lambda y} = E(e^{\lambda y} - e^{-\lambda y}) \]

But \( E(e^{\lambda y} - e^{-\lambda y}) = E \cdot 2 \sinh \lambda y \) and \( 2E \) is just another arbitrary constant, say \( Q \), so

\[ Y = Q \sinh \lambda y \quad \text{for any} \ Q, \]

same as before

4. method 1 Try \( u(x,y) = X(x)Y(y) \). Then

\[ X''Y + XY'' = XY \]

\[ X''Y = X(Y - Y'') \]

\[ \frac{X''}{X} = \frac{Y - Y''}{Y} = \lambda \]

You want good \( X \) solutions in anticipation of the condition involving \( f(x) \).
case 1 \( \lambda \) is negative. Call it \(-\lambda^2\)

\[
X = A \cos \lambda x + B \sin \lambda x
\]

\[
y'' - (1 + \lambda^2)y = 0, \quad y = C e^{\sqrt{1 + \lambda^2}} - D e^{-\sqrt{1 + \lambda^2}}
\]

**case 2** \( \lambda = 0 \)

\[
X = Ax + B, \quad Y = C e^{-y} + D e^{y}
\]

**method 2** (The other way is algebraically easier). You can factor differently and end up with

\[
\frac{X - X''}{X} = \frac{Y''}{Y} = \text{con}
\]

\[
xX'' + (\text{con} - 1)xX = 0, \quad y'' - \text{con} y' = 0.
\]

For the \( X \) problem here are the useful cases.

**case 1** \( \text{con} - 1 > 0 \) so let \( \text{con} - 1 = \lambda^2 \)

Then

\[
x'' + \lambda^2 x = 0, \quad X = A \cos \lambda x + B \sin \lambda x
\]

\[
y'' - (1 + \lambda^2)y = 0 \quad \text{and since } 1 + \lambda^2 \text{ is always positive, you have}
\]

\[
m = \pm \sqrt{1 + \lambda^2}, \quad y = C e^{\sqrt{1 + \lambda^2}} - D e^{-\sqrt{1 + \lambda^2}}
\]

**case 2** \( \text{con} - 1 = 0, \ \text{con} = 1 \)

Then

\[
x'' = 0, \quad X = Ax + B
\]

\[
y'' - y = 0, \quad Y = C e^{-y} + D e^{y}
\]

5. Try \( u(x,y) = X(x)Y(y) \) Then

\[
xX'Y = y^2 xY'' + xy', \quad xX'Y = x(y^2 xY'' + y'), \quad \frac{xX''}{X} = \frac{y^2 Y'' + y'}{Y} = \lambda,
\]

\[
xX'' = \lambda X \text{ with BC } X(0) = 0, \quad y^2 Y'' + y' = \lambda Y \text{ with BC } Y(3) = 0
\]

(The BC \( u(x,5) = x^2 \) doesn't separate.)

6. \( \frac{x'' + X}{X} = \text{con}, \quad \frac{y'}{Y} = \text{con}, \)

\[
x'' + (1 - \text{con})X = 0, \quad T' - \text{con} T = 0
\]

The \( T \) solution is always \( C e^{\text{con} t} \) but for the \( X \) part you have \( m = \pm \sqrt{\text{con} - 1} \) and the solution now depends on the sign of \( 1 - \text{con} \) (not on the sign of \( \text{con} \))
case 1 \( \text{con -1} = 0, \) i.e., \( \text{con} = 1 \)
\[
x'' = 0, \quad X = Ax + B, \quad T = Ce^t
\]

\textbf{case 2} \( \text{con - 1} \) is negative. Call it \( -\lambda^2 \) so that \( \text{con} = 1 - \lambda^2 \)
\[
x'' + \lambda^2 x = 0, \quad X = A \cos \lambda x + B \sin \lambda x, \quad T = Ce^{(1-\lambda^2)t}
\]

\textbf{case 3} \( \text{con - 1} \) is positive. Call it \( \lambda^2 \) so that \( \text{con} = 1 + \lambda^2 \)
\[
x'' - \lambda^2 x = 0, \quad X = Ae^{\lambda x} + Be^{-\lambda x}, \quad T = Ce^{(1+\lambda^2)t}
\]

7. You want good \( X \) solutions (\( T \) is never the important variable).
(a) \( x'' = \text{con} x, \quad T'' + (1 - \text{con})T = 0 \)
\textbf{case 1} \( \text{con < 0, say con} = -\lambda^2 \)
\[
x'' = -\lambda^2 x, \quad m = \pm \lambda i, \quad X = A \cos \lambda x + B \sin \lambda x
\]
\[
t'' + (1 + \lambda^2)T = 0, \quad m = \pm i\sqrt{1 + \lambda^2}, \quad T = C \cos t\sqrt{1 + \lambda^2} + D \sin t\sqrt{1 + \lambda^2}
\]

\textbf{case 2} \( \text{con = 0} \)
\[
x'' = 0, \quad X = Ax + B, \quad T'' + T = 0, \quad m = \pm i, \quad T = C \cos t + D \sin t
\]

(b) \( x'' - x = \text{con} x, \quad x'' - (1 + \text{con})x = 0 \)
\[
T'' = \text{con} T
\]
\textbf{case 1} \( 1 + \text{con < 0, say} 1 + \text{con} = -\lambda^2 \) so that \( \text{con} = -1 - \lambda^2 \)
\[
x + \lambda^2 x = 0, \quad X = A \cos \lambda x + B \sin \lambda x
\]
\[
t'' = -(1 + \lambda^2)T = 0, \quad m = \pm i\sqrt{1 + \lambda^2}, \quad T = C \cos t\sqrt{1 + \lambda^2} + D \sin t\sqrt{1 + \lambda^2}
\]

\textbf{case 2} \( 1 + \text{con} = 0 \) so that \( \text{con} = -1 \)
\[
x'' = 0, \quad X = Ax + B
\]
\[
t'' = -T, \quad T = C \cos t + D \sin t
\]
(c) \( x'' - \frac{1}{\text{con}} x = 0, \quad T'' + (1 - \frac{1}{\text{con}})T = 0 \)

To get good \( X \) solutions (namely sines and cosines) here are the cases you need.
case 1 \( \frac{1}{\text{con}} < 0 \), say \( \frac{1}{\text{con}} = -\lambda^2 \) so that \( \text{con} = -\frac{1}{\lambda^2} \)

\[ x'' + \lambda^2 x + 0, \quad x = A \cos \lambda x + B \sin \lambda x \]

\[ t'' + (1 + \lambda^2) t = 0, \quad m = \pm i \sqrt{1 + \lambda^2}, \quad t = c \cos t \sqrt{1 + \lambda^2} + d \sin t \sqrt{1 + \lambda^2} \]

case 2 \( \frac{1}{\text{con}} = 0 \), \( \text{con} = \infty \)

\[ x'' = 0, \quad x = Ax + B \]

\[ t'' = -t, \quad t = C \cos t + D \sin t \]

Notice that you get the same solutions no matter which way you separate. But the titles of the cases depend on how you separate. That's why there can't be a rule like "always use the case where \( \text{con} = -\lambda^2 \)."

The rule is "pick cases that give you good solutions for the important variable". In this problem it means choose cases so that you end up with \( x'' + \lambda^2 x = 0, x = A \cos \lambda x + B \sin \lambda x \).

8. Try \( u(x,y) = X(x)Y(y) \) Then \( XY = XY - XY', \quad \frac{X'}{X} = \frac{Y'}{Y} = \lambda \),

\[ Y' + (\lambda - 1) Y = 0, \quad X' - \lambda X = 0. \]

(No need for cases because both DE's are first order.)

\[ Y = Be^{(1-\lambda)y}, \quad X = Ae^{\lambda x}, \quad u = Ae^{\lambda x} Be^{(1-\lambda)y} = ce^{\lambda x} + (1-\lambda)y \]

9. Try \( u(x,y) = X(x)Y(y) \) Then

\[ x''y = x'y' \]

\[ \frac{x''}{x'} = \frac{y'}{y} = \lambda \]

\[ x'' = \lambda x', \quad y' = \lambda y \]

Case 1 \( \lambda \neq 0 \)

For the \( X \) equ, \( m^2 - \lambda m = 0, m = 0, \lambda, \quad X = A + Be^{\lambda x} \)

For the \( Y \) equ, \( m = \lambda, \quad Y = Ce^{\lambda y} \)

Case 2 \( \lambda = 0 \)

\[ x'' = 0, \quad x = Ax + B \]

\[ y' = 0, \quad y = C \]

10. Try \( u(x,y) = X(x)Y(y) \). Then \( x'y' + xy = 4 \) and you just can't get any further. You can't get \( X \)'s on one side and \( Y \)'s on the other side.
11. Here's why the first one separates.

If \( u(x,y) = X(x)Y(y) \) and \( u(3,y) = 0 \) for \( 0 \leq y \leq b \) then

\[
X(3)Y(y) = 0 \text{ for } 0 \leq y \leq b.
\]

So

\[
X(3) = 0 \text{ or } Y(y) = 0 \text{ for } 0 \leq y \leq b.
\]

If \( Y(y) = 0 \) for \( 0 \leq y \leq b \) then \( u(x,y) = 0 \) for \( 0 \leq y \leq b \) which is not a useful solution.

So use \( X(3) = 0 \).

Here's why the second one doesn't separate. If \( u(0,y) = 3 \) for \( 0 \leq y \leq b \) then

\[
X(0)Y(y) = 3 \text{ for } 0 \leq y \leq b
\]

But if the product of two factors is 3 then you can't conclude that one of the factors has to be 3. In fact you can't conclude anything about the individual factors. So this BC doesn't separate.

12. \( X'(5)T(t) = -3X(5)T(t) \) for all \( t \)

\[
T(t) [X'(5) + 3X(5)] = 0 \text{ for all } t
\]

Either \( T(t) = 0 \) for all \( t \) [which produces only the trivial solution for \( u \) so ignore it] or \( X'(5) + 3X(5) = 0 \). So the BC separates to \( X'(5) = -3X(5) \).
Honors

12. Try a solution of the form $\Psi(x,y,z,t) = \phi(x,y,z) T(t)$ Then

$$-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \phi}{\partial x^2} T + \frac{\partial^2 \phi}{\partial y^2} T + \frac{\partial^2 \phi}{\partial z^2} T \right) + V\phi T = i\hbar \phi T'$$

The left side has no $t$'s in it and the right side has no $x,y,z$'s in it so neither side has any variables in it so each side is a constant which I'll call $E$. So

$$-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + V\phi = \frac{i\hbar \phi T'}{T} = E$$

$$T' = \frac{E}{i\hbar} T \quad \text{(sol is } T = Ae^{-i(E/\hbar)t})$$

And the $\phi$ equation (the time independent Schrödinger equation) is

$$-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + V\phi = E\phi$$
SOLUTIONS Section 6.5

1. The equation was separated in Part I of example 2 so I won't repeat it. Try the case where $R = Ar^\lambda + Br^{-\lambda}$, $\Theta = C \cos \lambda \theta + D \sin \lambda \theta$

$\Theta(0) = 0$ so $C = 0$

$\Theta(\pi/4) = 0$ so $D \sin \frac{\pi \lambda}{4} = 0$, $\frac{\pi \lambda}{4} = n\pi$, $\lambda = 4n$ for $n = 1, 2, 3, \ldots$

To keep $R(0)$ finite set $B = 0$

By superposition, $v = \sum_{n=1}^{\infty} D_n r^{4n} \sin 4\theta$

To get $v(5, \theta) = \frac{\theta}{\pi}$ for $\theta$ in $[0, \pi/4]$ you need

$$\frac{\theta}{\pi} = \sum_{n=1}^{\infty} D_n 5^{4n} \sin 4n\theta \text{ for } \theta \text{ in } [0, \pi/4]$$

$\sin 4n\theta$ is of the form $\sin \frac{n\pi \theta}{L}$ where $L = \pi/4$ so you need

$$D_n 5^{4n} = \frac{2}{\pi/4} \int_0^{\pi/4} \frac{\theta}{\pi} \sin 4n\theta d\theta = \left\{ \begin{array}{ll}
-\frac{1}{2n\pi} & \text{if } n \text{ is even} \\
\frac{1}{2n\pi} & \text{if } n \text{ is odd}
\end{array} \right.$$  

(use (2) in the tables with $L = \pi/4$ and an extra factor of $1/\pi$).

So $D_{\text{even}} n = -\frac{1}{5^{4n} 2n\pi}$, $D_{\text{odd}} n = \frac{1}{5^{4n} 2n\pi}$ and the solution is

$$v = \frac{1}{2\pi} \left[ \left( \frac{r^4}{5} \right) \sin 4\theta - \frac{1}{2} \left( \frac{r^8}{5} \right) \sin 8\theta + \frac{1}{3} \left( \frac{r^{12}}{5} \right) \sin 12\theta - \ldots \right]$$

2. (a) I won't repeat the separation.

Use the case where $\Theta = A \cos \lambda \theta + B \sin \lambda \theta$, $R = Cr^\lambda + Dr^{-\lambda}$

$\Theta(0) = 0$ so $A = 0$

$\Theta(\pi) = 0$ so $B \sin \pi \lambda = 0$, $\pi \lambda = n\pi$, $\lambda = n$ for $n = 1, 2, 3, \ldots$

To keep $R(0)$ finite choose $D = 0$.

By superposition, $v = \sum_{n=1}^{\infty} B_n r^n \sin n\theta$

Then $\frac{\partial v}{\partial r} = \sum_{n=1}^{\infty} n B_n nr^{n-1} \sin n\theta$.

To get the last BC $\frac{\partial v}{\partial r}(2, \theta) = f(\theta)$ you need

$$f(\theta) = \sum_{n=1}^{\infty} n B_n 2^{n-1} \sin n\theta \text{ for } \theta \text{ in } [0, \pi]$$

So you need $nB_n 2^{n-1} = \frac{2}{\pi} \int_0^\pi f(\theta) \sin n\theta d\theta$, $B_n = \frac{1}{n 2^{n-1} \pi} \frac{2}{\pi} \int_0^\pi f(\theta) \sin n\theta d\theta$
(b) \( B_n = \frac{1}{n^{2n-1}} \frac{2}{\pi} \int_0^\pi \sin n\theta \, d\theta = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n^2 \pi^{2n-1}} & \text{if } n \text{ is odd} \end{cases} \) (Tables (2))

Solution is

\[
v = \frac{8}{\pi} \left[ \frac{r}{2} \sin \theta + \frac{1}{9} \left(\frac{r}{2}\right)^3 \sin 3\theta + \frac{1}{25} \left(\frac{r}{2}\right)^5 \sin 5\theta + \ldots \right]
\]

3. (a) I won't repeat the separation.

Use the case where \( \Theta = A \cos \lambda \theta + B \sin \lambda \theta, R = Cr^\lambda + Dr^{-\lambda} \)

Then \( \Theta' = -\lambda A \sin \lambda \theta + \lambda B \cos \lambda \theta \)

\( \Theta'(0) = 0 \) so \( B = 0 \)

\( \Theta'(\pi/4) = 0 \) so \( -\lambda A \sin \frac{\lambda \pi}{4} = 0 \)

\( \lambda \pi \quad 4 \quad n \pi, \lambda = 4n \) for \( n = 1,2,3,\ldots \)

To keep \( R(\infty) \) finite make \( C = 0 \)

Use the \( \lambda = 0 \) case where \( \Theta = E\theta + F, R = G ln r + H \)

\( \Theta'(0) = 0 \) and \( \Theta'(\pi/4) = 0 \) make \( E = 0 \)

To keep \( R(\infty) \) finite make \( G = 0 \) From this case you get \( v = FH = K \)

By superposition, \[
\boxed{v = K + \sum_{n=1}^{\infty} C_n r^{-4n} \cos 4n\theta}
\]

To get \( v(6,\theta) = f(\theta) \) for \( \theta \) in \([0,\pi/4]\) you need

\[
f(\theta) = K + \sum_{n=1}^{\infty} C_n 6^{-4n} \cos 4n\theta \quad \text{for} \quad \theta \in [0,\pi/4],
\]

\[
K = \frac{1}{\pi/4} \int_0^{\pi/4} f(\theta) \, d\theta, \quad C_n 6^{-4n} = \frac{2}{\pi/4} \int_0^{\pi/4} f(\theta) \cos 4n\theta \, d\theta,
\]

\[
C_n = 6^{4n} \frac{2}{\pi/4} \int_0^{\pi/4} f(\theta) \cos 4n\theta \, d\theta
\]

(b) The only difference here is that you must keep \( R(0) \) finite by making \( D = 0 \) in the first case and \( G = 0 \) again in the second case. The net effect is to have \( r^{4n} \) instead of \( r^{-4n} \) in the solution and \( 6^{4n} \) instead of \( 6^{-4n} \) in the \( C_n \) coeff formula. Solution is

\[
\boxed{v = K + \sum_{n=1}^{\infty} C_n r^{4n} \cos 4n\theta}
\]

where \( K = \frac{1}{\pi/4} \int_0^{\pi/4} f(\theta) \, d\theta \) and \( C_n = 6^{-4n} \frac{2}{\pi/4} \int_0^{\pi/4} f(\theta) \cos 4n\theta \, d\theta \)
4. (a) Use the major case where $\Theta = C \cos \lambda \theta + D \sin \lambda \theta$, $R = A r^\lambda + B r^{-\lambda}$ and the minor case where $R = E r^n + F$, $\Theta = G \theta + H$.

For $v$ inside, continue as in example 3. Need $\lambda = n$ and $G = 0$ to keep $\Theta$ periodic. Need $B = 0$ and $E = 0$ to keep $R$ finite. By superposition

$$v = K + \sum_{n=1}^{\infty} r^n (C_n \cos n \theta + D_n \sin n \theta)$$

The BC is $v(5, \theta) = \begin{cases} 1 & \text{if } 0 \leq \theta \leq \pi \\ -1 & \text{if } \pi \leq \theta \leq 2\pi \end{cases}$

To get it you need

$$(*) \quad v(5, \theta) = K + \sum_{n=1}^{\infty} 5^n (C_n \cos n \theta + D_n \sin n \theta) \text{ for } 0 \leq \theta \leq 2\pi$$

$$K = \frac{1}{2\pi} \int_0^{2\pi} v(5, \theta) \, d\theta = \text{average value of } v(5, \theta) = 0$$

$$5^n C_n = \frac{2}{2\pi} \int_0^{2\pi} v(5, \theta) \cos n \theta \, d\theta = \frac{1}{\pi} \left[ \int_0^\pi \cos n \theta \, d\theta + \int_\pi^{2\pi} -\cos n \theta \, d\theta \right]$$

$$= \frac{1}{\pi} \frac{1}{n} \sin n \theta \bigg|_0^\pi - \frac{1}{\pi} \frac{1}{n} \sin n \theta \bigg|_\pi^{2\pi} = 0$$

(Can't use (4b) in the tables because here $L = 2\pi$ but the cosine is not $\cos \frac{n\pi \theta}{2\pi}$.) Similarly

$$5^n D_n = \frac{2}{2\pi} \int_0^{2\pi} f(\theta) \sin n \theta \, d\theta = \frac{1}{\pi} \left[ \int_0^\pi \sin n \theta \, d\theta + \int_\pi^{2\pi} -\sin n \theta \, d\theta \right]$$

and eventually you get

$$D_{\text{even}} n = 0, \quad D_{\text{odd}} n = \frac{4}{5^n} \frac{n\pi}{n\pi}$$

so

$$v_{\text{inside}} = \frac{4}{\pi} \left[ \frac{r}{5} \sin \theta + \frac{1}{3} \left( \frac{r}{5} \right)^3 \sin 3\theta + \frac{1}{5} \left( \frac{r}{5} \right)^5 \sin 5\theta + \ldots \right]$$

Finding $v$ outside is like finding $v$ inside but to keep $R$ finite set $A = 0$ in the main case instead of $B$. The net effect is to have $r^{-n}$ instead of $r^n$ in the solution and $5^{-n}$ instead of $5^n$ in the $D_{\text{odd}} n$ coeff formula. All in all,

$$v(r, \theta) = \begin{cases} \frac{4}{\pi} \left( \frac{r}{5} \sin \theta + \frac{1}{3} \left( \frac{r}{5} \right)^3 \sin 3\theta + \frac{1}{5} \left( \frac{r}{5} \right)^5 \sin 5\theta + \ldots \right) \text{ for } r \leq 5 \\ \frac{4}{\pi} \left( \frac{5}{r} \sin \theta + \frac{1}{3} \left( \frac{5}{r} \right)^3 \sin 3\theta + \frac{1}{5} \left( \frac{5}{r} \right)^5 \sin 5\theta + \ldots \right) \text{ for } r \geq 5 \end{cases}$$

(b) Continue as in part (a) until line $(*)$ which becomes

$$4 \sin 3\theta = K + \sum_{n=1}^{\infty} 5^n (C_n \cos n \theta + D_n \sin n \theta) \text{ for } 0 \leq \theta \leq 2\pi$$

By inspection you can get this with $K = 0$, $5^n C_n = 0$, $C_n = 0$,

$$5^3 D_3 = 4, \quad D_3 = \frac{4}{5^3}, \quad \text{other } D_n \text{'s} = 0.$$

Solution is $v_{\text{inside}} = \frac{4}{5^3} r^3 \sin 3\theta = 4 \left( \frac{r}{5} \right)^3 \sin 3\theta$

Similarly $v_{\text{outside}} = 4 \left( \frac{5}{r} \right)^3 \sin 3\theta$. 
5. (a) This is an Euler's equation with $a = 2$, $b = -12$. Let $x = e^t$.
Get $y'' + y' - 12y = 0$, $m = -4, 3$.
y(t) = $Ae^{-4t} + Be^{3t}$, $y(x) = Ax^{-4} + Bx^3$

(b) Euler with $a = -3$, $b = 4$. Let $x = e^t$.
Get $y'' - 4y' + 4y = 0$, $m = 2, 2$.
y(t) = $Ae^{2t} + Bte^{2t}$, $y(x) = Ax^2 + Bx^2 \ln x$

(c) Euler with $a = 5$, $b = 5$. Let $x = e^t$.
Get $y'' + 4y' + 5y = 0$, $m = -2 \pm i$.
y(t) = $e^{-2t} (A \cos t + B \sin t)$, $y(x) = x^{-2} (A \cos \ln x + B \sin \ln x)$

(d) Let $x = e^t$. Get $y'' - 4y' + 4y = t$, $y_h = Ae^{2t} + Bte^{2t}$
Try $y_p = At + B$, Need

$$-4A + 4At + 4B = t,$$

$$4A = 1, -4A + 4B = 0$$

$A = \frac{1}{4}, B = \frac{1}{4}$

$y(t) = Ae^{2t} + Bte^{2t} + \frac{1}{4} t + \frac{1}{4}$
$y(x) = Ax^2 + Bx^2 \ln x + \frac{1}{4} \ln x + \frac{1}{4}$

(e) Let $x = e^t$ to get

$$(*) \ y''(t) + 2y'(t) - 3y(t) = 10e^{2t}$$

We have

$y_h = Ae^{-3t} + Be^t$
as before. Try

$y_p = Ce^{2t}$

Substitute into $(*)$ to get

$$4Ce^{2t} + 2Ce^{2t} - 3Ce^{2t} = 10e^{2t}$$

$5C = 10, \ C = 2$

$y_{gen}(t) = Ae^{-3t} + Be^t + 2e^{2t}$

Now go back to $x$'s. One way to do it is to think of $e^{-3t}$ and $e^{2t}$ as $(e^t)^{-3}$ and $(e^t)^2$. Substitute $x$ for $e^t$ to get the final answer:

$$y_{gen}(x) = \frac{A}{x^3} + Bx + 2x^2$$
SOLUTIONS Section 6.6

1. (a) even periodic extension

2. sine series converges to the odd periodic extension

(b) even periodic extension

3. cosine series converges to the even periodic extension

full series converges to the periodic extension
3. (a) The picture is the even periodic extension of the \([0,4]\) piece (even function, \(T = 8\), use cos series with \(L = 4\)). Series is \(A_0 + \sum A_n \cos \frac{n \pi x}{4}\) where

\[
A_0 = \frac{1}{4} \int_0^4 (8 - 2x) \, dx = \text{average value of } f(x) \text{ on } [0,4] \text{ which is } 4 \text{ by inspection}
\]

\[
A_n = \frac{2}{4} \int_0^4 (8 - 2x) \cos \frac{n \pi x}{4} \, dx
\]

(Can also in inefficiently find cos series using \([0,8]\) piece or full series using \([0,8]\) piece. All three versions will turn out to be the same series.)

(b) The picture is the odd periodic extension of the \([0,4]\) piece (odd function, \(T = 8\), use sines with \(L = 4\)). Series is \(\sum B_n \sin \frac{n \pi x}{4}\) where

\[
B_n = \frac{2}{4} \int_0^4 f(x) \sin \frac{n \pi x}{4} \, dx = \frac{2}{4} \left[ \int_0^2 -x \sin \frac{n \pi x}{4} \, dx + \int_2^4 (x-4) \sin \frac{n \pi x}{4} \, dx \right]
\]

(can also, inefficiently, find full series for \([0,8]\) piece)

(c) The picture is the periodic extension of the \([0,5]\) piece (not—odd, not—even, \(T = 5\), use full series with \(L = 5\)). Series is

\[
A_0 + \sum \left[ A_n \cos \frac{n \pi x}{5/2} + B_n \sin \frac{n \pi x}{5/2} \right]
\]

where \(A_0 = \frac{1}{5} \int_0^5 f(x) \, dx = \frac{1}{5} \left[ \int_0^2 2x \, dx + \int_2^4 (8 - 2x) \, dx \right]
\]

(average value of \(f(x)\) on \([0,5]\) is 8/5, by inspection)

\[
A_n = \frac{2}{5} \left[ \int_0^2 2x \cos \frac{2n \pi x}{5} \, dx + \int_2^4 (8 - 2x) \cos \frac{2n \pi x}{5} \, dx \right]
\]

\(B_n = \text{ditto but with sines}\)

(d) The picture is the even periodic extension of the \([0, \frac{3}{2} \, 2\, 5/2]\) piece; i.e., I'll find the cos series for

\[
f(x) = \begin{cases} 
4 - 2x & \text{if } 0 \leq x \leq 2 \\
0 & \text{if } 2 \leq x \leq 2.5 
\end{cases}
\]

(even function, \(T = 5\), use cos series with \(L = 5/2\)).

\[
\text{Warning: It's wrong to use } [0,3] \text{ because the even periodic extension of the } [0,3] \text{ piece is}
\]

\[
\text{The flat pieces have length 2 not 1 so this is not the desired picture.}
\]

\(\text{Check your choice of interval carefully.}\)

Series is \(A_0 + \sum A_n \cos \frac{2n \pi x}{5}\) where

\[
A_0 = \frac{1}{5/2} \int_0^{5/2} f(x) \, dx = \frac{2}{5} \int_0^2 (4 - 2x) \, dx
\]

(average value of \(f(x)\) on \([2, 5/2]\) is 8/5, by inspection)

\[
A_n = \frac{2}{5/2} \int_0^{3/2} f(x) \, dx = \frac{4}{5} \int_0^2 (4 - 2x) \cos \frac{2n \pi x}{5} \, dx
\]
(e) The picture is the periodic extension of the [0,2] piece (not—odd, not—even, \( T = 2 \), use full series with \( L = 2 \)). Series is

\[
A_0 + \sum (A_n \cos n\pi x + B_n \sin n\pi x)
\]

where

\[
A_0 = \frac{1}{2} \int_0^2 (2-x) \, dx \quad \text{(average } f \text{ value on } [0,2] \text{ is 1, by inspection)}
\]

\[
A_n = \frac{2}{2} \int_0^2 (2-x) \cos n\pi x \, dx, \quad B_n = \frac{2}{2} \int_0^2 (2-x) \sin n\pi x \, dx
\]

(f) Picture is the odd periodic extension of the [0,1] piece (odd function, \( T = 2 \), use sines with \( L = 1 \)). Series is \( \sum B_n \sin n\pi x \) where \( B_n = 2 \int_0^1 (1-x) \sin n\pi x \, dx \)

(Can inefficiently find the sine series for the [0,2] piece or find the full series for the [0,2] piece)

4. Function is the periodic extension of the [0,2] piece (not—odd, not—even, \( T = 2 \), use full series with \( L = 2 \)). Series is \( A_0 + \sum (A_n \cos n\pi x + B_n \sin n\pi x) \) where

\[
A_0 = \text{average value of } f(x) \text{ on } [0,2] = \frac{3}{4} \\
A_n = \frac{2}{2} \int_0^2 f(x) \cos n\pi x \, dx = \int_0^{1/2} 3 \cos n\pi x \, dx = \frac{3}{n\pi} \sin \frac{n\pi}{2}
\]

\[
= \begin{cases} 
0 & \text{if } n \text{ is even} \\
\frac{3}{n\pi} & \text{if } n = 1,5,9,\ldots \\
-\frac{3}{n\pi} & \text{if } n = 3,7,11,\ldots
\end{cases}
\]

\[
B_n = \int_0^{1/2} 3 \sin n\pi x \, dx = -\frac{3}{n\pi} (\cos \frac{n\pi}{2} - 1)
\]

\[
= \begin{cases} 
\frac{3}{n\pi} & \text{if } n \text{ is odd} \\
0 & \text{if } n = 4,8,12,\ldots \\
\frac{6}{n\pi} & \text{if } n = 2,6,10,\ldots
\end{cases}
\]

\[
f(x) = \frac{3}{4} + \frac{3}{\pi} (\cos \pi x - \frac{1}{3} \cos 3\pi x + \frac{1}{5} \cos 5\pi x - \frac{1}{7} \cos 7\pi x + \ldots) \\
+ \frac{3}{\pi} (\sin \pi x + \frac{2}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x + \frac{2}{6} \sin 6\pi x + \frac{1}{7} \sin 7\pi x + \ldots)
\]

The first harmonic is \( \frac{3}{\pi} (\cos \pi x + \sin \pi x) \) so the fundamental frequency is 1 (cycle per second) with amp \( \sqrt{\frac{9}{\pi^2} + \frac{9}{\pi^2}} = \frac{3}{\pi} \sqrt{2} \)

The first overtone is \( \frac{3}{\pi} \sin 2\pi x \) (there is no \( \cos 2\pi x \) term) so first overtone frequency is 2 with amp \( 3/\pi \)

5. Function is the even periodic extension of the [0,6] piece (even function, \( T = 12 \), use cosines with \( L = 6 \)). Series is of the form \( A_0 + \sum A_n \cos \frac{n\pi x}{6} \). You want the first nonzero cosine term. Assuming \( A_1 \neq 0 \), the fundamental frequency is 1/6 and its amplitude is |A_1| where
6. The function is the even periodic extension of the \( \pi/2 \) piece (even function, \( T = \pi \), use cos series with \( L = \pi/2 \)). Series is \( A_0 + \sum A_n \cos 2nx \) where

\[
A_0 = \frac{1}{\pi/2} \int_{0}^{\pi/2} f(x) \cos 2nx \, dx = \frac{2}{\pi} \left[ \int_{0}^{\pi/2} 2x \cos \frac{\pi x}{6} \, dx + \int_{\pi/2}^{\pi} 4 \cos \frac{\pi x}{6} \, dx \right]
\]

<table>
<thead>
<tr>
<th>( \sin x )</th>
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<tbody>
<tr>
<td>( \pi )</td>
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Note that \( \cos \left( \frac{1 - 2n}{2} \pi \right) \) and \( \cos \left( \frac{1 + 2n}{2} \pi \right) \) are 0. So

\[
A_n = \frac{4}{\pi} \left[ \frac{1}{2(1-2n)} + \frac{1}{2(1+2n)} \right] = \frac{-4}{\pi(2n-1)(2n+1)}
\]

Series is \( \frac{2}{\pi} - \frac{4}{\pi} \left[ \frac{1}{1.3} \cos 2x + \frac{1}{3.5} \cos 4x + \frac{1}{5.7} \cos 6x + \ldots \right] \)

7. The function is the even periodic extension of the \([0,\pi]\) piece (even function, \( T = 2\pi \), use cosine series with \( L = \pi \)).

\[
\pi \quad 2\pi
\]

Series is \( A_0 + \sum A_n \cos nx \) where

\[
A_0 = \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{0}^{\pi/2} \cos x \, dx
\]

\[
A_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_{0}^{\pi/2} \cos x \cos nx \, dx
\]

8. The function is the odd periodic extension of the \([0,4]\) piece (odd function, \( T = 8 \), use sine series with \( L = 4 \)):

\[
\sum B_n \sin \frac{n\pi x}{4} \quad \text{where} \quad B_n = \frac{2}{4} \int_{0}^{2} 5 \sin \frac{n\pi x}{4} \, dx = \begin{cases} 
10/n\pi & \text{if } n \text{ is odd} \\
20/n\pi & \text{if } n = 2,6,10,\ldots
\end{cases}
\]

Series is \( \frac{10}{\pi} \sin \frac{\pi x}{4} + \frac{20}{2\pi} \sin \frac{2\pi x}{4} + \frac{10}{3\pi} \sin \frac{3\pi x}{4} + \frac{10}{5\pi} \sin \frac{5\pi x}{4} + \ldots \)

9. (a)(i) The series converges to the odd periodic extension of the \([0,3]\) piece (ii) At a jump, the series converges to the middle value
(iii) The \([24,30]\) piece is the same as the \([0,6]\) piece. At \(x = 29\) the series has the same value that it did at \(x = 5\) and at \(x = -1\). To get the value at \(x = -1\), find the value at \(x = 1\) and change signs. Answer is \(-(3 + 6 \cdot 1 - 1^2) = -8\).

(b) (i) The series converges to the even periodic extension of the \([0,3]\) piece

(ii) There are no jumps.

(iii) At \(x = 29\) the series has the same value that it did at \(x = -1\) which (since the picture is even) is the same as its value at \(x = 1\). Answer is 8.

(c) (i) The series converges to the periodic extension of the \([0,3]\) piece

(ii) See the diagram.

(iii) The \([27,29]\) piece is the same as the \([0,3]\) piece. At \(x = 29\), the series has the same value as at \(x = 2\). Answer is 11.

10. (a) I'll use the forcing function \(e^{ikx}\) and try \(y_p = Ae^{ikx}\). Substituting the trial \(y_p\) into the DE gives

\[
Ai^2k^2 e^{ikx} + 4Ae^{ikx} = e^{ikx}, \quad A(4 - k^2) = 1, \quad A = \frac{1}{4-k^2},
\]

\[
y_p = \frac{1}{4-k^2} (\cos kx + i \sin kx).
\]

Take imag part to get the particular sol to the original equation:

\[
y_{pk} = \frac{1}{4-k^2} \sin kx.
\]

(b) By (2), the DE is

\[
y'' + 4y = \frac{16}{\pi^2} \left[ \sin \frac{\pi x}{4} - \frac{1}{9} \sin \frac{3\pi x}{4} + \frac{1}{25} \sin \frac{5\pi x}{4} - \ldots \right]
\]

By part (a) and superposition, the particular solution is

\[
y_p = y_{p1} + y_{p3} + y_{p5} + \ldots
\]

\[
= \frac{16}{\pi^2} \left[ \frac{1}{4-(\frac{\pi}{4})^2} \sin \frac{\pi x}{4} - \frac{1}{9} \frac{1}{4-(\frac{3\pi}{4})^2} \sin \frac{3\pi x}{4} + \frac{1}{25} \frac{1}{4-(\frac{5\pi}{4})^2} \sin \frac{5\pi x}{4} - \ldots \right]
\]
SOLUTIONS Section 6.7

1. \( A_3 = \int_0^1 2x^5 \cdot x^4 \, dx = \frac{1}{5} \cdot \frac{9}{1} = \frac{9}{5} \)

2. \( \int_0^1 \cos \frac{\pi x}{L} \, dx = \frac{L}{n \pi} \sin \frac{\pi x}{L} \bigg|_0^L = \frac{L}{n \pi} (\sin \pi - \sin 0) = 0 \)

3. (a) Part I Separate Try \( u = X(x) \cdot T(t) \). Then \( XT' = kX''T \), \( \frac{X''}{X} = \frac{T''}{kT} = \text{constant.} \)

The BC separate to \( X'(0) = 0, X(L) = 0 \)

   case 1 \( \text{con} = -\lambda^2 \)

   \( X'' = -\lambda^2 X, T' = -\lambda^2 T, X = A \cos \lambda x + B \sin \lambda x, T = C e^{-\lambda^2 kt} \)

   case 2 \( \text{con} = 0 \)

   \( X'' = 0, T' = 0, X = P x + Q, T = D \)

Part II Plug in the separated BC

   case 1

   \( X' = -\lambda A \sin \lambda x + B \lambda \cos \lambda x \)

   \( X'(0) = 0 \) makes \( B = 0 \)

   \( X(L) = 0 \) makes \( A \cos \lambda L = 0, \lambda L = \frac{n \pi}{2} \) for odd \( n \)

   \( \lambda = \frac{n \pi}{2L} \) for \( n = 1, 3, 5, ... \)

   \( X = \cos \frac{n \pi x}{2L}, T = C e^{-k(n \pi/2L)t} \) for odd \( n \)

   case 2

   \( X' = P \)

   \( X'(0) = 0 \) makes \( P = 0 \)

   \( X(L) = 0 \) makes \( Q = 0 \)

   Nothing useful here.

Part III

A general solution is

\[
(* \quad u = \sum_{\text{odd } n} A_n e^{-k(n \pi/2L)^2 t} \cos \frac{n \pi x}{2L} \quad \text{for } 0 \leq x \leq L, t \geq 0
\]

To satisfy the IC you need

\[
 f(x) = \sum_{\text{odd } n} A_n \cos \frac{n \pi x}{2L} \quad \text{for } 0 \leq x \leq L
\]

Note that the "ingredients" of the series (the \( \phi \)'s) are not the functions

\( 1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \cos \frac{3\pi x}{L}, ... \)

The \( \phi \)'s here are

\[
(**) \quad \cos \frac{\pi x}{2L}, \cos \frac{3\pi x}{2L}, \cos \frac{5\pi x}{2L}, ... \]

This is a new complete orthogonal family on the interval \([0, L]\). They came from solving

\( X'' = \text{con} \cdot X \) with BC \( X'(0) = 0, X(L) = 0 \)

This is a Sturm Liouville problem with \( p(x) = 1, q(x) = 0 \), so the functions in (**) are orthogonal on the interval \([0, L]\).
The solution is (*) with the constants given by the formula in (5):

\[ A_{\text{odd } n} = \frac{\int_0^L f(x) \cos \frac{n\pi x}{2L} \, dx}{\int_0^L \cos^2 \frac{n\pi x}{2L} \, dx} \]

(b) numerator of the \( A_{\text{odd } n} \) formula

\[ \int_0^L 7 \cos \frac{n\pi x}{2L} \, dx = \frac{14L}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 14L/n\pi & \text{if } n = 1, 5, 9, \ldots \\ -14L/n\pi & \text{if } n = 3, 7, 11, \ldots \end{cases} \]

\[ \text{Answer is } u = \frac{28}{\pi} e^{-k(\pi/2L)^2 t} \left( \cos \frac{\pi x}{2L} - \frac{1}{3} e^{-k(3\pi/2L)^2 t} \cos \frac{3\pi x}{2L} + \frac{1}{5} e^{-k(5\pi/2L)^2 t} \cos \frac{5\pi x}{2L} - \ldots \right) \]

for \( 0 \leq x \leq L, t \geq 0 \)

4. Can't be done in general. The series is an "incomplete" cosine series: it doesn't have a constant term. The building blocks of the series in (*) are

\[ \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \cos \frac{3\pi x}{L}, \ldots \]

but the complete set of orthogonal functions that are used to make a series that will converge to any \( f(x) \) for \( 0 \leq x \leq L \) is

\[ 1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \cos \frac{3\pi x}{L}, \ldots \]

Can't make the incomplete series do what you want it to do for \( 0 \leq x \leq L \). It is not correct to say the series will converge to \( f(x) \) if \( A_n = \frac{2L}{\int_{-\infty}^{\infty} f(x) \cos \frac{n\pi x}{L} \, dx} \).

The only functions you can make the series in (*) converge to for \( 0 \leq x \leq L \) are functions whose average value on \([0, L]\) is 0. The cosine series for that kind of function has \( A_0 = 0 \) anyway so their series are supposed to be missing a constant term.

5. The functions are of the form \( \sin \frac{n\pi x}{L} \) where \( L = 5\pi \). So the interval is \([0, 5\pi]\).
SOLUTIONS Section 6.8

1. (a) \( R(T' + T) = kT(R'' + \frac{1}{r} R') \)

\[
\frac{T' + T}{kT} = \frac{R'' + \frac{1}{r} R'}{R} = \text{con}
\]

\( T' + (1-\text{con} \, k)T = 0, \quad T = C e^{(\text{con} \, k - 1)t} \)

\( rR'' + R' = r \, \text{con} \, R \)

*case 1* \( \text{con} = -\lambda^2 \)

\( rR'' + R' + r\lambda^2 R = 0 \)

\( R = A J_0(\lambda r) + B Y_0(\lambda r), \quad T = C e^{-(1+k\lambda^2)t} \)

*case 2* \( \text{con} = 0 \)

\( rR'' + R' = 0, \quad R = A J_0 (r) + B (\text{ref page}), \quad T = C e^{-t} \)

(b) \( T' - \text{con} \, T = 0, \quad T = C e^{\text{con} \, t} \)

\( rR'' + R' = \frac{1 + \text{con}}{k} rR \)

Sturm Liouville form with \( p(r) = r, q(r) = 0, \quad w(r) = r \) and \( \frac{1 + \text{con}}{k} \) playing the role of the constant.

*case 1* \( \frac{1 + \text{con}}{k} = -\lambda^2, \quad \text{con} = -(1+k\lambda^2) \)

\( rR'' + R' + \lambda^2 rR = 0, \quad R = A J_0(\lambda r) + B Y_0(\lambda r) \)

\( T = C e^{-(1+k\lambda^2)t} \)

*case 2* \( \frac{1 + \text{con}}{k} = 0, \quad \text{con} = -1 \)

\( rR'' + R' = 0, \quad R = A \ln r + B \)

\( T = C e^{-t} \)

Ultimately these are the same solutions as part (a) but they were easier to get with the factoring in (a).

2. The BC \( u(L,t) = 0 \) separates to \( R(L) = 0 \)

Plug it in.

*case 1* \( R = A J_0(\lambda r) + B Y_0(\lambda r) \)

Set \( B = 0 \) to keep \( R \) finite

\( R(L) = 0 \) makes \( A J_0(\lambda L) = 0, \quad \lambda = \frac{a_n}{L} \) where the \( a_n \)'s are the zeros of \( J_0 \).

*case 2* \( R = A \ln r + B \)

Set \( A = 0 \) to keep \( R \) finite.

To get \( R(B) = 0 \) you need \( B = 0 \). So the only solution in this case is \( R = 0 \).
Part III  Use the IC.

By superposition,
\[ u = \sum_{n=1}^{\infty} A_n \left( 1 + k \left( \frac{a_n}{L} \right)^2 \right) e^{-\left(1 + k \left( \frac{a_n}{L} \right)^2 \right) t} J_0 \left( \frac{a_n r}{L} \right) \] for \( 0 \leq r \leq L, \ t \geq 0 \)

To get the IC, set \( t = 0, \ u = f(r) \). You need
\[ f(r) = \sum_{n=1}^{\infty} A_n J_0 \left( \frac{a_n r}{L} \right) \] for \( r \) in \([0,L]\)

which you can get with
\[
A_n = \frac{\int_0^{L} f(r) J_0 \left( \frac{a_n r}{L} \right) r \ dr}{\int_0^{L} J_2 \left( \frac{a_n r}{L} \right) r \ dr}
\]

The solution is (*) and (***) where the \( a_n \)'s are the zeros of \( J_0 \) (Fig 6).

3. Part I  Separate variables.

Try \( v(r,z) = R(r)Z(z) \). Then
\[
R''Z + \frac{1}{r} R' Z + R' Z'' = 0
\]
\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = - \frac{Z''}{Z} = \text{con}
\]
\[
xR'' + R' = \text{con} \ x \ R
\]
\[
z'' + \text{con} \ Z = 0
\]
The BC \( v(5,z) = 0 \) separates to \( R(5) = 0 \)
\[ \text{case 1} \ \ \ \ \text{con} = -\lambda^2 \]
\[ R = A J_0 (\lambda r) + B Y_0 (\lambda r), \ Z = Ce^{\lambda z} + De^{-\lambda z} \]

\[ \text{case 2} \ \ \ \ \text{con} = 0 \]
\[ R = A \ln R + B, \ Z = C \cos \lambda z + D \sin \lambda z \]

Part II  Satisfy the separable BC

\[ \text{case 1} \]
Set \( B = 0 \) to keep \( R \) finite at \( r=0 \).
\[ R(5) = 0 \ \ \text{so} \ \ \ A J_0 (5\lambda) = 0, \ \ \lambda = \frac{a_n}{5} \] where the \( a_n \)'s are the zeros of \( J_0 \).

Set \( C = 0 \) to keep \( Z \) finite as \( z \to \infty \).

\[ \text{case 2} \]
Nothing useful. Just get \( R = 0 \).
Part III Satisfy the nonhomog BC

By superposition

\[ v = \sum_{n=1}^{\infty} A_n \left( J_0 \left( \frac{a_n r}{5} \right) e^{-a_n z/5} \right) \]

Then

\[ \frac{\partial v}{\partial z} = \sum_{n=1}^{\infty} \frac{a_n}{5} A_n \left( J_0 \left( \frac{a_n r}{5} \right) e^{-a_n z/5} \right) \]

and to get the nonhomog BC you need

\[ f(r) = \sum_{n=1}^{\infty} \frac{a_n}{5} A_n \left( J_0 \left( \frac{a_n r}{5} \right) e^{-a_n z/5} \right) \text{ for } r \in [0,5], \]

which you can get with

\[ \frac{a_n}{5} A_n = \frac{\int_{0}^{5} f(r) J_0 \left( \frac{a_n r}{5} \right) r \, dr}{\int_{0}^{5} J_0^2 \left( \frac{a_n r}{5} \right) r \, dr} \]

\[ (***) \quad A_n = -\frac{5}{a_n} \frac{\int_{0}^{5} f(r) J_0 \left( \frac{a_n r}{5} \right) r \, dr}{\int_{0}^{5} J_0^2 \left( \frac{a_n r}{5} \right) r \, dr} \]

The solution is (*) and (***) where the \( a_n \)'s are the zeros of \( J_0 \) (Fig 6).

4. The graph of \( J_0(x) \) crosses the x-axis at \( a_1, a_2, a_3, \ldots \)

The graph of \( J_0 \left( \frac{a_1 x}{100} \right) \) crosses at \( \frac{100}{a_1} a_1 (= 100), \frac{100}{a_1} a_2, \frac{100}{a_1} a_3, \ldots \)

Think that if \( J_0 \) were periodic (which it isn't at the beginning but almost is eventually) then to get \( J_0 \left( \frac{a_1 x}{100} \right) \), multiply the old period by \( \frac{100}{a_1} \) (just the way the period of \( \sin \frac{1}{2} x \) is twice the period of \( \sin x \)).

Footnote \( a_1 \) is approximately 2.4 so \( \frac{100}{a_1} \) is approx 42.
SOLUTIONS  review problems for Chapter 6

1. Let \( u(x,t) = X(x)T(t) \). Then

\[
X'T' + XT = kX''T
\]

\[
\frac{X''}{X} = \frac{T' + T}{kT} = \text{const}
\]

\( X'' = \text{const} X, \quad T' + (1 - k \text{const})T = 0 \)

Try the case where \( \text{const} \) is negative, renamed \(-\lambda^2\). Then

\[
X = A \cos \lambda x + B \sin \lambda x, \quad T = Ce^{-(1+k\lambda^2)t}
\]

\[
X' = A\lambda \sin \lambda x + B\lambda \cos \lambda x
\]

\( X'(0) = 0 \) so \( B = 0 \)

\( X'(4) = 0 \) so \(-4\lambda \sin 4\lambda = 0, \quad 4\lambda = n\pi, \quad \lambda = \frac{n\pi}{4} \) for \( n = 1, 2, 3, \ldots \)

Try the case where \( \text{const} = 0 \). Then \( X'' = 0, \quad T' + T = 0, \quad X = Dx + E, \quad T = Fe^{-t} \)

\( X'(0) = 0 \) and \( X'(4) = 0 \) make \( D = 0 \).

Solution in this case is \( u = Be^{-t} = Ge^{-t} \)

By superposition \( u = Ge^{-t} + \sum_{n=1}^{\infty} A_n e^{-(1+k(n\pi/4)^2)t} \cos \frac{n\pi x}{4} \)

To get the IC we need

\[
f(x) = G + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{4} \quad \text{for } x \text{ in } [0,4]
\]

\( C = \text{average value of } f(x) \text{ on } [0,4] = 5 \)

\[
A_n = \frac{2}{4} \int_0^4 f(x) \cos \frac{n\pi x}{4} \, dx = \frac{2}{4} \left[ \int_0^2 3 \cos \frac{n\pi x}{4} \, dx + \int_2^4 7 \cos \frac{n\pi x}{4} \, dx \right]
\]

\[
= \begin{cases} 
-\frac{8}{n\pi} & \text{if } n = 1, 5, 9, \ldots \\
\frac{8}{n\pi} & \text{if } n = 3, 7, 11, \ldots \\
0 & \text{if } n \text{ is even}
\end{cases}
\]

Solution is

\[
u = 5e^{-t} - \frac{8}{\pi} \left[ e^{-(1+k\frac{\pi}{4})^2t} \cos \frac{\pi x}{4} - \frac{1}{3} e^{-(1+k\frac{3\pi}{4})^2t} \cos \frac{3\pi x}{4}
+ \frac{1}{5} e^{-(1+k\frac{5\pi}{4})^2t} \cos \frac{5\pi x}{4} - \ldots \right]
\]
2. (a) **Part I** Separate variables.

Try a solution of the form \( v(r,\theta) = R(r) \Theta(\theta) \)

Then

\[
R'' + \frac{1}{r} R' + \frac{1}{r^2} R \Theta'' = 0
\]

\[
\Theta \left[ R'' + \frac{1}{r} R' \right] = -\frac{1}{r^2} r \Theta''
\]

\[
-\frac{r^2}{R} \left[ R'' + \frac{1}{r} R' \right] = \frac{\Theta''}{\Theta} = \text{constant}
\]

\[
\Theta'' - \text{constant} \Theta = 0,
\]

\[
r^2 R'' + r R' + \text{constant} R = 0
\]

**case 1** \( \text{con} = -\lambda^2 \)

\( \Theta = C \cos \lambda \theta + D \sin \lambda \theta, \ R = A \lambda + B \lambda^{-1} \).

**case 2** Constant = 0

\( \Theta'' = 0, \ \Theta = A \theta + B \)

\( r^2 R'' + r R' = 0, \ R(r) = A \lambda + B \lambda^{-1} \) (reference page)

The BC separate to \( \Theta'(0) = 0, \ \Theta'(\pi) = 0 \)

**Part II** Plug in the separated BC.

**case 1**

\( \Theta'(0) = 0 \) so \( D = 0 \)

\( \Theta'(\pi) = 0 \) so \( -C \sin \pi = 0, \ \lambda = n \) for \( n = 1,2,3,... \)

Need \( B = 0 \) to keep \( R \) finite

**case 2**

\( \theta = A \theta + B, \ R = C \theta \ln r + D \)

\( \Theta'(0) = 0 \) and \( \Theta'(\pi) = 0 \) make \( A = 0 \).

Need \( C = 0 \) to keep \( R \) finite From this case we have solution \( v = BD = F_0 \)

By superposition, \( v = F_0 + \sum_{n=1}^{\infty} F_n r^n \cos n\theta \)

**Part III** Get the last (nonhomog) BC.

We need

\[
f(\theta) = F_0 + \sum_{n=1}^{\infty} F_n 5^n \cos n\theta \text{ for } \theta \text{ in } [0,\pi],
\]

\[
F_0 = \frac{1}{\pi} \int_0^\pi f(\theta) \ d\theta
\]

\[
F_n 5^n = \frac{2}{\pi} \int_0^\pi f(\theta) \cos n\theta \ d\theta,
\]

\[
F_n = \frac{1}{5^n} \frac{2}{\pi} \int_0^\pi f(\theta) \cos n\theta \ d\theta
\]

(b) Now we need \( 3 + 6 \cos 2\theta = F_0 + \sum_{n=1}^{\infty} F_n 5^n \cos n\theta \text{ for } \theta \text{ in } [0,\pi], \)

By inspection we can get it with \( F_0 = 3, \ F_2 5^2 = 6, , F_2 = \frac{6}{25}, \) other \( F's = 0 \)

Solution is \( v = 3 + \frac{6}{25} \ r^2 \cos 2\theta \)
3. Try $y = X(x)T(t)$ Then $XT'' = gX'X'T + gX'T$, $\frac{T''}{gT} = \frac{xX'' + X'}{X} = \text{constant}$, $T'' - \text{const} gT = 0$, $xX'' + X' - \text{const} X = 0$

4. Try $y = X(x)T(t)$ Then $XT'' + XT' = X'T$, $\frac{T'' + T'}{T} = \frac{X'}{X} = \text{const}$. 

   case 1  $\text{const}$ is negative, call it $-\lambda^2$. Then
   
   $X = A \cos \lambda x + B \sin \lambda x$
   
   $T'' + T' = -\lambda^2 T$, $T'' + T' + \lambda^2 = 0$, $m = -1 \pm \sqrt{1 - 4\lambda^2}$ and we need subcases because the $T$ solution depends on whether $1 - 4\lambda^2$ is pos, neg or zero.

   case 1(a) $1 - 4\lambda^2$ positive, i.e., $0 \leq \lambda^2 < \frac{1}{4}$
   
   $T = Ce^{-1 + \sqrt{1 - 4\lambda^2}} t + De^{-1 - \sqrt{1 - 4\lambda^2}} t$

   case 1(b) $1 - 4\lambda^2$ negative, i.e., $\lambda^2 > \frac{1}{4}$
   
   $T = e^{-t/2} \left[ c \cos \frac{\sqrt{4\lambda^2 - 1}}{2} t + d \sin \frac{\sqrt{4\lambda^2 - 1}}{2} t \right]$

   case 1(c) $1 - 4\lambda^2 = 0$, i.e., $\lambda^2 = 1/4$
   
   $T = C e^{-t/2} + D t e^{-t/2}$

   case 2  $\text{const} = 0$
   
   $X = Ax + B$, $T'' + T' = 0$, $T = C + De^{-t}$

5. (a) With these axes, the picture is the odd periodic extension of the [0,5] piece (odd function, $T = 10$, use sines with $L = 5$)

   Series is $\sum b_n \sin \frac{n\pi x}{5}$ where $b_n = \frac{2}{5} \int_0^5 7 \sin \frac{n\pi x}{5} \, dx = \frac{28}{n\pi}$ if n is odd

   Series is $\frac{28}{\pi} \sin \frac{\pi x}{5} + \frac{28}{3\pi} \sin \frac{3\pi x}{5} + \frac{28}{5\pi} \sin \frac{5\pi x}{5} - \ldots$

   Fund frequency is $\frac{1}{5}$ with amp $\frac{28}{\pi}$

   First overtone freq is $\frac{3}{5}$ with amp $\frac{28}{3\pi}$

   (b) With these axes the function is the even periodic extension of the [0,5] piece (even function, $T = 10$, use cosines with $L = 5$).
Series is \( A_0 + \sum A_n \cos \frac{n\pi x}{5} \) where \( A_0 = \) average \( f(x) \) on \([0,5]\) = 7,
\[
A_n = \frac{2}{5} \int_0^5 f(x) \cos \frac{n\pi x}{5} \, dx = \frac{2}{5} \int_0^{5/2} 14 \cos \frac{n\pi x}{5} \, dx = \frac{28}{n\pi} \sin \frac{n\pi x}{5} \bigg|_0^{5/2}
\]
Series is \( 7 + \frac{28}{\pi} \cos \frac{\pi x}{5} - \frac{28}{3\pi} \cos \frac{3\pi x}{5} + \frac{28}{5\pi} \cos \frac{5\pi x}{5} - \ldots \)

Same frequencies and amplitudes as in method 1.

(c) Here's the long way to get the series.
The function is the periodic extension of the \([0,10]\) piece (not—odd, not—even, \( T = 10 \), use full series with \( L = 10 \)). Series is
\[
A_0 + \sum_{n=1}^\infty \left( A_n \cos \frac{n\pi x}{5} + B_n \sin \frac{n\pi x}{5} \right) \text{ where}
\]
\[
A_0 = \text{average value of the } [0,10] \text{ piece } = 7
\]
I got this by inspection. Half the time, \( f(x) = 14 \) and half the time \( f(x) = 0 \).
You can also use
\[
\frac{1}{10} \int_0^{10} f(x) \, dx = \frac{1}{10} \left[ \int_0^5 14 \, dx + \int_5^{10} 0 \, dx \right]
\]
\[
A_n = \frac{2}{10} \int_0^{10} f(x) \cos \frac{n\pi x}{5} \, dx = \frac{2}{10} \left[ \int_0^5 14 \cos \frac{n\pi x}{5} \, dx + \int_5^{10} 0 \cos \frac{n\pi x}{5} \, dx \right]
\]
\[
= \frac{2}{10} \cdot 14 \cdot \frac{5}{n\pi} \sin \frac{n\pi x}{5} \bigg|_0^5 = 0
\]
\[
B_n = \text{ditto but with sines } = \frac{2}{10} \cdot 14 \cdot \frac{5}{n\pi} \cos \frac{n\pi x}{5} \bigg|_0^5 = -\frac{14}{n\pi} \left( \cos n\pi - 1 \right)
\]
\[
= \frac{28}{n\pi} \text{ if } n \text{ is odd}
\]
Series is \( 7 + \frac{28}{\pi} \sin \frac{\pi x}{5} + \frac{28}{3\pi} \sin \frac{3\pi x}{5} + \frac{28}{5\pi} \sin \frac{5\pi x}{5} - \ldots \)
Like the series in part (a) but with the extra term 5 (which makes it a full series, not a sine series). Same frequencies and amplitudes as in part (a)

The fast way to get the series is to notice that

function in Fig (c) = 7 + function in Fig (a)

(i.e., translate Fig (a) up 7 to get Fig (c))

So

series for (c) = 7 + series for (a)
6. (a) You need

\[ u(x,2) = 8A_0 + \sum_{n=1}^{\infty} A_n e^{-2n} \cos \frac{n\pi x}{4} \quad \text{for } 0 \leq x \leq 4 \]

which you can get with

\[
8A_0 = \frac{1}{4} \int_0^4 u(x,2) \, dx
\]

\[
A_n e^{-2n} = \frac{2}{4} \int_0^4 u(x,2) \cos \frac{n\pi x}{4} \, dx
\]

\[
u(x,2) = \begin{cases} 
0 & \text{if } 0 \leq x \leq 2 \\
5x - 10 & \text{if } 2 \leq x \leq 3 \\
5 & \text{if } 3 \leq x \leq 4
\end{cases}
\]

so the solution is

\[
u(x,y) = A_0 y^3 + \sum_{n=1}^{\infty} A_n e^{-ny} \cos \frac{n\pi x}{4} \quad \text{for } 0 \leq x \leq 4, \ y \geq 2
\]

where

\[
A_0 = \frac{1}{8} \left[ \int_2^3 (5x - 10) \, dx + \int_3^4 5 \, dx \right]
\]

\[
A_n = e^{2n} \frac{2}{4} \left[ \int_2^3 (5x - 10) \cos \frac{n\pi x}{4} \, dx + \int_3^4 5 \cos \frac{n\pi x}{4} \, dx \right]
\]

(b) Now you would need

\[ u(x,2) = 8A_0 + \sum_{n=1}^{\infty} A_n e^{-2n} \cos \frac{n\pi x}{3} \quad \text{for } 0 \leq x \leq 4 \]

But the functions \(1, \cos \frac{\pi x}{3}, \cos \frac{2\pi x}{3}, \cos \frac{3\pi x}{3}, \ldots\) are not a complete set on the interval \([0,4]\). You can't make a series out of them that will do what you want it to do for \(0 \leq x \leq 4\) (you can only control what happens for \(0 \leq x \leq 3\)). So you can't get the condition satisfied. This shouldn't happen in the course of solving a PDE which comes from a real life problem. You should realize how lucky you are.