SOLUTIONS  Section 3.1

1. \( y_2 = y_1 + y_0 = 1 + 0 = 1, \quad y_3 = y_2 + y_1 = 1 + 1 = 2, \)
   \( y_4 = y_3 + y_2 = 2 + 1 = 3, \quad y_5 = y_4 + y_3 = 3 + 2 = 5 \)

2. \( S_{n+1} = S_n + n+1 \) or equivalently \( S_n = S_{n-1} + n \) with IC \( S_1 = 1 \)

3. Need 3 terms, say \( y_1, y_2, y_3 \) to get started. Then \( y_4 = \frac{2y_2 - 3y_1}{6} \),
   \( y_5 = \frac{2y_3 - 3y_2}{6} \) etc. The rr is 3-rd order and can be rewritten as
   \( 6y_{n+3} - 2y_{n+1} + 3y_n = 0 \)

4. Set \( n = 1 \). Then \( y_3 = 1^3 + 2y_1 = 1 + 2 \cdot 2 = 5 \)
   Set \( n = 2 \). Then \( y_4 = 2^3 + 2y_2 = 8 + 2 \cdot 3 = 14 \)
   Set \( n = 3 \). Then \( y_5 = 3^3 + 2y_3 = 27 + 2 \cdot 5 = 37 \)

5. LHS = \((n + 2) \cdot 2^{n+2} \) - \(4(n+1) \cdot 2^{n+1} + 4n \cdot 2^n = (n+2) \cdot 2^n \cdot 2^2 - 4(n+1) \cdot 2^n \cdot 2 + 4n \cdot 2^n \)
   = \((4n+8) \cdot 2^n - (8n+8) \cdot 2^n + 4n \cdot 2^n = 0 \) QED

6. (a) \( 3y_{n+4} + 5y_{n+1} - 2y_n = \sin \pi n + \sin \pi n = 2 \sin \pi n \)
   (b) \( 3y_{n+4} + 5y_{n+1} - 2y_n = 3 \sin \pi n \)
   (c) \( 3y_{n+4} + 5y_{n+1} - 2y_n = \sin \pi n - \sin \pi n = 0 \)

7. All are solutions to \( ay_{n+2} + by_{n+1} + cy_n = 0 \)

8. The rr is always supposed to say
   term = preceding term + (the number of the term)^2
   (a) \( S_n = S_{n-1} + n^2 \)
   (b) \( S_{n+6} = S_{n+5} + (n+6)^2 \)
SOLUTIONS Section 3.2

1. (a) \( \lambda^2 - 3\lambda - 10 = 0 \), \( \lambda = -2,5 \), \( y_n = A(-2)^n + B5^n \)

(b) \( \lambda^2 + 3\lambda - 4 = 0 \), \( \lambda = 1, -4 \), \( y_n = A + B(-4)^n \)

(c) \( 2\lambda^2 + 2\lambda - 1 = 0 \), \( \lambda = -\frac{1 \pm \sqrt{3}}{2} \)

\[ y_n = A \left[ \frac{-1 + \sqrt{3}}{2} \right]^n + B \left[ \frac{-1 - \sqrt{3}}{2} \right]^n \]

(d) same as (b)

2. \( \lambda^2 + 2\lambda - 15 = 0 \), \( \lambda = -5, 3 \), gen \( y_n = A(-5)^n + B3^n \)

Need \( A + B = 0 \), \(-5A + 3B = 1\). So \( A = -\frac{1}{8}, B = \frac{1}{8} \). Answer is \( -\frac{1}{8}(-5)^n + \frac{1}{8}3^n \)

3. (a) \( y_{n+2} = y_{n+1} + 6y_n \) so \( y_2 = y_1 + 6y_0 = 0 + 6 \cdot 1 = 6 \)

\( y_3 = y_2 + 6y_1 = 6 + 6 \cdot 0 = 6 \)

(b) \( \lambda^2 - \lambda - 6 = 0 \), \( \lambda = 3, -2 \). Gen sol is \( y_n = A3^n + B(-2)^n \)

Need \( 1 = A + B \), \( 0 = 3A - 2B \) so \( A = \frac{2}{5}, B = \frac{3}{5} \). Sol is \( y_n = \frac{2}{5}3^n + \frac{3}{5}(-2)^n \)

(c) \( y_3 = \frac{2}{5}3^3 + \frac{3}{5}(-2)^3 = 6 \)

4. (a) \( \lambda^2 + 2\lambda + 2 = 0 \), \( \lambda = -1 \pm i \). The number \(-1 + i\) has mag \( \sqrt{2} \) and angle \( \frac{3\pi}{4} \) so

\[ y_n = (\sqrt{2})^n \left( A \cos \frac{3\pi}{4} + B \sin \frac{3\pi}{4} \right) \]

(b) \( \lambda^2 + \lambda + 1 = 0 \), \( \lambda = -\frac{1 \pm i\sqrt{3}}{2} \). Mag of \( \frac{-1 + i\sqrt{3}}{2} \) is 1, angle is \( \frac{2\pi}{3} \) so

\[ y_n = A \cos \frac{2n\pi}{3} + B \sin \frac{2n\pi}{3} \]

5. \( \lambda = -2 \pm 2i \). The number \(-2 + 2i\) has mag \( \sqrt{8} \) and angle \( \frac{3\pi}{4} \) so

\[ y_n = (\sqrt{8})^n \left( A \cos \frac{3n\pi}{4} + B \sin \frac{3n\pi}{4} \right) \]

To get \( y_0 = 0 \) need \( A = 0 \). To get \( y_1 = 2 \) need \( 2 = \sqrt{8} B \frac{1}{2} \sqrt{2}, B = 1 \).

So \( y_n = (\sqrt{8})^n \sin \frac{3n\pi}{4} \) and \( y_{102} = (\sqrt{8})^{102} \sin \frac{153\pi}{2} = (\sqrt{8})^{102} = 8^{51} \)
6. (a) \( \sqrt{3} + i \) has mag 2 and angle \( \frac{5\pi}{6} \).

\[ y_n = A(-3)^n + B 4^n + Cn 4^n + 2^n \left( D \cos \frac{5\pi n}{6} + E \sin \frac{5\pi n}{6} \right) \]

(b) \( 2i \) has mag 2 and angle \( \pi/2 \).

\[ y_n = A + B 2^n + C(-2)^n + D 3^n + 2^n \left( E \cos \frac{n\pi}{2} + F \sin \frac{n\pi}{2} \right) + n2^n \left( G \cos \frac{n\pi}{2} + H \sin \frac{n\pi}{2} \right) \]

7. \( \lambda = \frac{1 \pm \sqrt{5}}{2} \), gen sol is \( y_n = A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n \).

To get the IC we need \( A + B = 0 \), \( \frac{1}{2} A(1 + \sqrt{5}) + \frac{1}{2} B(1 - \sqrt{5}) = 1 \), \( A = \frac{1}{\sqrt{5}} \), \( B = -\frac{1}{\sqrt{5}} \).

Answer is \( y_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \).

8. (a) \( y_1 = 2 \), \( y_2 = 5 \) (or you can begin with \( y_0 = 2 \), \( y_1 = 5 \). Makes no difference)

Then \( y_3 = \frac{2 + 5}{2} = \frac{7}{2} \), \( y_4 = \frac{1}{2}(5 + \frac{7}{2}) = \frac{17}{4} \), \( y_5 = \frac{1}{2}(\frac{17}{4} + \frac{17}{4}) = \frac{31}{8} \)

(b) \( y_{n+2} = \frac{y_{n+1} + y_n}{2} \), \( 2y_{n+2} - y_{n+1} - y_n = 0 \), \( \lambda = -\frac{1}{2}, 1 \); general sol is \( y_n = A \left( -\frac{1}{2} \right)^n + B \)

Plug in IC \( y_1 = 2 \), \( y_2 = 5 \) to get \( 2 = -\frac{1}{2} A + B \), \( 5 = \frac{1}{4} A + B \).

So \( B = 4 \), \( A = 4 \) and answer is \( y_n = 4 \left( -\frac{1}{2} \right)^n + 4 \).

(c) \( y_5 = 4 \left( -\frac{1}{2} \right)^5 + 4 = \frac{31}{8} \)

9. (a) \( y_n = A(-3)^n + B 4^n + Cn 4^n \)
(b) \( y_n = A 5^n + Bn 5^n + Cn^2 5^n + Dn^3 5^n + E 2^n \)
(c) \( y_n = A + Bn + Cn^2 + D 6^n + E(-7)^n \)

10. \( \lambda = -3, -3 \), \( y_n = A(-3)^n + Bn(-3)^n \)
11. \(\lambda = 1,1,2, (\lambda-1)^2 (\lambda-2) = 0, \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0.\)

Recurrence is \(y_{n+3} - 4y_{n+2} + 5y_{n+1} - 2y_n = 0.\)

12. \((\lambda - 1)^3 = 0, \lambda = 1,1,1.\) Gen sol is \(y_n = A + Bn + Cn^2.\)

Need \(A + B + C = 0, A + 2B + 4C = 1, A + 3B + 9C = 0.\) So \(A = -3, B = 4, C = -1.\)

Sol is \(y_n = -3 + 4n - n^2.\)

13. (a) by inspection If \(y_1 = 4\) and \(y_{n+1} = y_n\) then \(y_2 = 4, y_3 = 4\) and in general, \(y_n = 4\) for every \(n.\)

overkill \(\lambda = 1.\) General sol is \(y_n = A.\) Plug in the IC \(y_1 = 4\) to get \(A = 4.\) Sol is \(y_n = 4.\)

(b) by inspection If \(y_0 = 0, y_1 = 0\) and thereafter \(y_{n+2} = \frac{-b}{a} y_{n+1} - \frac{c}{a} y_n\) then every term is 0, i.e., \(y_n = 0\) for all \(n.\)

overkill Say the roots of the auxiliary equ are \(\lambda_1\) and \(\lambda_2.\) Then the general sol is \(A \lambda_1^n + B \lambda_2^n\) (assuming \(\lambda_1 \neq \lambda_2\) which is another story). Plug in the IC and we get \(0 = A + B, 0 = A \lambda_1 + B \lambda_2.\)

The only solution for \(A\) and \(B\) is \(A = 0, B = 0\) which makes \(y_n = 0.\)

14. (a) Sequence is 5,7,9,11,13,... Pattern is \(y_n = 2n + 3.\)

(b) \(y_{n+1} = \frac{y_{n+2} + y_n}{2}, y_{n+2} - 2y_{n+1} + y_n = 0, \lambda = 1,1, y_n = A + Bn.\)

To get the IC \(y_1 = 5, y_2 = 7\) need \(A + B = 5, A + 2B = 7.\)

So \(A = 3, B = 2.\) Answer is \(y_n = 3 + 2n.\)
SOLUTIONS Section 3.3

1. (a) \( y_1 = 2, \ y_2 = 2y_1 + 6 \cdot 2 = 4 + 12 = 16, \ y_3 = 2y_2 + 6 \cdot 3 = 50, \)

\( y_4 = 2y_3 + 6 \cdot 4 = 124 \)

(b) \( \lambda = 2, \ h_n = A \ 2^n. \ Try \ p_n = Bn + C. \ Need \)

\[ Bn + C = 6 \]

Equate \( n \) coeffs \( -B = 6 \)

Equate constant terms \( 2B - C = 0 \)

\( B = -6, \ C = -12, \ y_n = A \ 2^n - 6n - 12 \)

The IC makes \( 2 = 2A - 6 - 12, \ A = 10. \ Answer \ y_n = 10 \cdot 2^n - 6n - 12 \)

(c) \( y_4 = 10 \cdot 16 - 6 \cdot 4 = 124 \)

2. (a) \( \lambda = 2, -1, \ h_n = A \ 2^n + B(-1)^n. \ Try \ p_n = C. \)

Then \( p_{n+1} = C, \ p_{n+2} = C. \ Substitute \) into \( r \) to get \[ C - C - 2C = 1, \ C = -\frac{1}{2} \]

General sol is \( y_n = A \ 2^n + B(-1)^n - \frac{1}{2} \)

Plugging in the IC makes \( 1 = 2A - B - \frac{1}{2}, \ 3 = 4A + B - \frac{1}{2} \)

So \( A = \frac{5}{6}, \ B = \frac{1}{6}. \ Answer \ y_n = \frac{5}{6} \cdot 2^n + \frac{1}{6}(-1)^n - \frac{1}{2} \)

(b) \( \lambda = -5, 3, \ h_n = A \ 3^n + B(-5)^n. \ Try \ p_n = Cn + D \)

Need \( C(n+2) + D + 2 \left[ C(n+1) + D \right] - 15(Cn + D) = 6n + 10 \)

Equate \( n \) coeffs \( -12C = 6, \ C = -\frac{1}{2} \)

Equate constant terms \( 4C - 12D = 10, \ D = -1 \)

General sol is \( y_n = A \ 3^n + B(-5)^n - \frac{1}{2} n - 1 \)

The IC make \( A = \frac{11}{8}, \ B = \frac{5}{8}. \ Answer \ y_n = \frac{11}{8} \cdot 3^n + \frac{5}{8}(-5)^n - \frac{1}{2} n - 1 \)

3. \( \lambda = \frac{3 + \sqrt{5}}{2}, \ h_n = A \left(\frac{3 + \sqrt{5}}{2}\right)^n + B \left(\frac{3 - \sqrt{5}}{2}\right)^n \)

Try \( p_n = C \ 4^n. \ Need \)

\[ C \ 4^{n+2} - 3C \ 4^{n+1} + C \ 4^n = 10 \cdot 4^n \]

\[ 16C \ 4^n - 12C \ 4^n + C \ 4^n = 10 \cdot 4^n \]

\[ 5C = 10, \ C = 2 \]

Gen sol is \( y_n = A \left(\frac{3 + \sqrt{5}}{2}\right)^n + B \left(\frac{3 - \sqrt{5}}{2}\right)^n + 2 \cdot 4^n \)
4. \( \lambda = 3, -2, \ h_n = D \ 3^n + E \ (-2)^n \). Try \( p_n = A n^2 + B n + C \). Need

\[ A(n+2)^2 + B(n+2) + C - \left( A(n+1)^2 + B(n+1) + C \right) - 6 \left( A n^2 + B n + C \right) = 18n^2 + 2 \]

Match \( n^2 \) coeffs: \(-6A = 18, \ A = -3\)
Match \( n \) coeffs: \(2A - 6B = 0, \ B = -1\)
Match constant terms: \(3A + B - 6C = 2, \ C = -2\)

Gen sol is \( y_n = D \ 3^n + E \ (-2)^n - 3n^2 - n - 2 \)

The IC make \( D = \frac{8}{5}, \ E = -\frac{3}{5} \), Answer is \( y_n = \frac{8}{5} \cdot 3^n - \frac{3}{5} \cdot (-2)^n - 3n^2 - n - 2 \)

5.(a) Method 1 Switch to \( y_{n+2} - 2y_n = 5 e^{n\pi i} \)

Need the homog sol to see if I need to step up. \( \lambda = \pm \sqrt{2}, \ h_n = A(\sqrt{2})^n + B(-\sqrt{2})^n \).

No interference. No stepping up, try \( p_n = D e^{n\pi i} \). Need

\[ D e^{(n+2)\pi i} - 2De^{n\pi i} = 5e^{n\pi i} \]

\[ De^{\pi i} e^{n\pi i} - 2De^{n\pi i} = 5e^{n\pi i} \]

\[ De^{n\pi i} - 2De^{n\pi i} = 5e^{n\pi i} \because e^{2\pi i} = 1 \]

So \( D = 2D = 5 \). Switched \( p_n = -5e^{n\pi i} = -5(\cos n\pi + i \sin n\pi) \).

Original \( p_n = \text{real part} = -5 \cos n\pi \)

Method 2 Try \( p_n = A \sin n\pi + B \cos n\pi \). Need

\[ A \sin \pi(n+2) + B \cos \pi(n+2) - 2(A \sin n\pi + B \cos n\pi) = 5 \cos n\pi \]

\[ A \sin (n\pi + 2\pi) + B(\cos(n\pi + 2\pi) - 2(A \sin n\pi + B \cos n\pi) = 5 \cos n\pi \]

\[ A \sin n\pi + B \cos n\pi - 2(A \sin n\pi + B \cos n\pi) = 5 \cos n\pi \]

(Use identities \( \sin(x + 2\pi) = \sin x, \ \cos(x + 2\pi) = \cos x \))

\[-A \sin n\pi - B \cos n\pi = 5 \cos n\pi \]

\(-A = 0, -B = 5, B = -5, \ p_n = -5 \cos n\pi \)

(b) Try \( p_n = D(-1)^n \) (no need to step up since \( h_n = A(\sqrt{2})^n + B(-\sqrt{2})^n \))

Need \( D(-1)^{n+2} - 2D(-1)^n = 5(-1)^n \)

\[ D(-1)^2(-1)^n - 2D(-1)^n = 5(-1)^n \]

Equate coeff of \( (-1)^n \): \( D - 2D = 5, \ D = -5 \)

\( p_n = -5(-1)^n \) which agrees with the solution \(-5 \cos n\pi \) from part (a).

(c) Method 1 Switch to \( \frac{n\pi i}{2} \)

and try \( \frac{n\pi i}{2} \)

Need

\[ \frac{(n+1)\pi i}{2} - \frac{n\pi i}{2} = 10e^{\pi i/2} \]
\[ \frac{\pi i}{2} - \frac{n\pi i}{2} = 10e^{\pi i/2} \]

(\( \because e^{\pi i} = 1 \)) (mag 1, angle \( \pi/2 \))
\((-2 + i)D = 10, \ D = -4-2i\)

Switched \(p_n = (-4-2i) e^{\frac{n\pi i}{2}}\) = \((-4-2i)(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2})\)

Take imag part to get original \(p_n = -4 \sin \frac{n\pi}{2} - 2 \cos \frac{n\pi}{2}\)

**Method 2** Try \(p_n = A \sin \frac{n\pi}{2} + B \cos \frac{n\pi}{2}\). Need

\[
A \sin \left(\frac{(n+1)\pi}{2}\right) + B \cos \left(\frac{(n+1)\pi}{2}\right) - 2 \left[ A \sin \frac{n\pi}{2} + B \sin \frac{n\pi}{2} \right] = 10 \sin \frac{n\pi}{2}
\]

\[
A \left[ \sin \frac{n\pi}{2} \cos \frac{\pi}{2} + \cos \frac{n\pi}{2} \sin \frac{\pi}{2} \right] + B \left[ \cos \frac{n\pi}{2} \cos \frac{\pi}{2} - \sin \frac{n\pi}{2} \sin \frac{\pi}{2} \right] - 2 \left[ A \sin \frac{n\pi}{2} + B \cos \frac{n\pi}{2} \right] = 10 \sin \frac{n\pi}{2}
\]

\((-2A - B) \sin \frac{n\pi}{2} + (A-2B) \cos \frac{n\pi}{2} = 10 \sin \frac{n\pi}{2}\)

Match coeffs: \(-2A-B = 10, \ A - 2B = 0\).

So \(A = -4, \ B = -2, \ p_n = -4 \sin \frac{n\pi}{2} - 2 \cos \frac{n\pi}{2}\)

6. (a) Try \(p_n = An^4 + Bn^3 + Cn^2 + Dn + E\)

(b) Ordinarily you would try \(p_n = An^4 + Bn^3 + Cn^2 + Dn + E\) but since \(E, n, n^2\) are all homog sols, try \(p_n = n^4(An^4 + Bn^3 + Cn^2 + Dn + E) = An^8 + Bn^7 + Cn^6 + Dn^5 + En^4\)

(c) Try \(p_n = An^2\) (step up because \(2^n\) is a homog sol)

(d) Try \(p_n = A2^n\)

(e) Try \(p_n = An^2 3^n\) (step up because \(3^n\) and \(n3^n\) are both homog sols)

(f) One method is to try \(p_n = n(A \cos \frac{n\pi}{2} + B \sin \frac{n\pi}{2})\)

(Step up because \(i\) has mag 1 and angle \(\pi/2\) so \(\cos \frac{n\pi}{2}\) and \(\sin \frac{n\pi}{2}\) are homog sols)

Another method is to switch to forcing function \(5e^{n\pi i/2}\), try \(p_n = Ae^{n\pi i/2}\) (step up here too) and eventually take real part

(g) Either try \(p_n = A \cos \frac{n\pi}{2} + B \sin \frac{n\pi}{2}\) or switch to the forcing function \(5e^{n\pi i/2}\) and try \(p_n = Ae^{n\pi i/2}\) and eventually take real part.

Why not step up here the way you had to in part (f)? Because \(2i\) has mag 2 and angle \(\pi/2\) so the homog sols are \(2^n \cos \frac{n\pi}{2}\) and \(2^n \sin \frac{n\pi}{2}\)

When the forcing function is \(5 \cos \frac{n\pi}{2}\) you should step up \(p_n\) only if plain \(\cos \frac{n\pi}{2}\) is a homog solution as it was in part (f), not if \(2^n \cos \frac{n\pi}{2}\) is a homog sol.
7. \( \lambda = \frac{1}{2}, \ h_n = A \left( \frac{1}{2} \right)^n \). Try \( p_n = Bn \left( \frac{1}{2} \right)^n \) (step up). Need

\[\begin{align*}
2B(n+1) \left( \frac{1}{2} \right)^{n+1} - Bn \left( \frac{1}{2} \right)^n &= \left( \frac{1}{2} \right)^n \\
2B(n+1) \frac{1}{2} \left( \frac{1}{2} \right)^n - Bn \left( \frac{1}{2} \right)^n &= \left( \frac{1}{2} \right)^n \\
B \left( \frac{1}{2} \right)^n &= \left( \frac{1}{2} \right)^n
\end{align*}\]

So \( B = 1 \). General sol is \( y_n = A \left( \frac{1}{2} \right)^n + n \left( \frac{1}{2} \right)^n \)

The IC make \( A = 3 \). Answer is \( y_n = 3 \left( \frac{1}{2} \right)^n + n \left( \frac{1}{2} \right)^n \)

8. \( \lambda = 1,1, \ h_n = A + Bn \). Ordinarily you would try \( p_n = C \) but since \( C \) and \( n \) are both homog sols, step up twice and try \( p_n = Cn^2 \). Need

\[\begin{align*}
C(n+2)^2 - 2C(n+1)^2 + Cn^2 &= 1
\end{align*}\]

The \( n^2 \) terms and \( n \) terms cancel out.

Equate constant terms: \( 2C = 1, \ C = \frac{1}{2} \)

Gen sol is \( y_n = A + Bn + \frac{1}{2} n^2 \)

The IC make \( A = 1, \ \frac{1}{2} = A + B + \frac{1}{2} \). So \( B = -1 \). Answer is \( y_n = 1 - n + \frac{1}{2} n^2 \)

9. \( S_{n+1} = S_n + (n+1)^2, \ S_{n+1} - S_n = (n+1)^2 \) with IC \( S_1 = 1 \).

\( \lambda = 1, \ n = D \). Try \( p_n = n(An^2 + Bn + C) = An^3 + Bn^2 + Cn \). (Step up because \( C \) is a homog sol.) Need

\[\begin{align*}
A(n+1)^3 + B(n+1)^2 + C(n+1) - (An^3 + Bn^2 + Cn) &= n^2 + 2n + 1
\end{align*}\]

The \( n^3 \) coeffs drop out.

Equate \( n^2 \) coeffs \( 3A = 1, \ A = \frac{1}{3} \)

Equate \( n \) coeffs \( 3A + 2B = 2, \ B = \frac{1}{2} \)

Equate constant terms \( A + B + C = 1, \ C = \frac{1}{6} \)

Gen sol is \( S_n = D + \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \)

The IC \( S_1 = 1 \) makes \( D = 0 \). Answer is \( S_n = \frac{n(n+1)(2n+1)}{6} \), usually written as \( S_n = \frac{n(n+1)}{2} \cdot \frac{(2n+1)}{3} \).

10. (a) Need \( C2^{n+2} - 3C \ 2^{n+1} + 2C \ 2^n = 6 \cdot 2^n, \ 4C \ 2^n - 6C \ 2^{n+2} + 2C \ 2^n = 6 \cdot 2^n \),

The \( C \)'s cancel out leaving \( 0 = 6 \cdot 2^n \) which can't be satisfied. There is no value of \( C \) which makes \( C \ 2^n \) work.

(b) Need \( 2A(n+2) + 3A(n+1) + 4An = 18n, \ 9An + 7A = 18n \).

Equate \( n \) coeffs \( 9A = 18, \ A = 2 \)

Equate constant terms \( 7A = 0, \ A = 0 \)

Impossible. So there is no solution of the form \( An \).
11. \( \lambda = 2 \), \( h_n = A(-2)^n \). Try \( p_n = B + C 4^n \). Need

\[
B + C 4^{n+1} + 2(B + C 4^n) = 3 + 4^n
\]

Match coeffs \( 3B = 3, \ 6C = 1 \)

So \( B = 1, \ C = \frac{1}{6} \). General sol is \( y_n = A(-2)^n + 1 + \frac{1}{6} \cdot 4^n \)

IC make \( 2 = A + 1 + \frac{1}{6} \). So \( A = \frac{5}{6} \). Answer is \( y_n = \frac{5}{6} (-2)^n + 1 + \frac{1}{6} \cdot 4^n \)

12. \( \lambda^4 - 16 = 0 \) \( \lambda^2 = \pm 4, \ \lambda = \pm 2, \pm 2i \). The number \( 2i \) has mag 2 and angle \( \pi/2 \) so

\[
h_n = A 2^n + B(-2)^n + 2^n (C \cos \frac{n\pi}{2} + D \sin \frac{n\pi}{2})
\]

Try \( p_n = A^n + B + C 3^n \). Need

\[
A(n+4) + B + C 3^{n+4} - 16(A + B + C 3^n) = n + 3^n
\]

\[
A(n+4) + B + 81 C 3^n - 16(A + B + C 3^n) = n + 3^n
\]

Equate n coeffs \( -15A = 1, \ A = -\frac{1}{15} \)

Equate constant terms \( 4A - 15B = 0, \ B = -\frac{4}{225} \)

Equate \( 3^n \) terms \( 65C = 1, \ C = \frac{1}{65} \)

Gen sol is

\[
y_n = A 2^n + B(-2)^n + 2^n (C \cos \frac{n\pi}{2} + D \sin \frac{n\pi}{2}) - \frac{1}{15} n - \frac{4}{225} + \frac{1}{65} 3^n
\]

13. \( \lambda = \frac{1+i \sqrt{3}}{2} \). Mag of \( \frac{1+i \sqrt{3}}{2} \) is 1, angle is \( \pi/3 \). So

\[
h_n = A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3}
\]

Try \( p_n = C 2^n \) Need

\[
c 2^{n+2} - c 2^{n+1} + c 2^n = 2^n
\]

\[
4c 2^n - 2c 2^n + c 2^n = 2^n
\]

\[
4c - 2c + c = 1, \ c = \frac{1}{3}
\]

Gen sol is \( y = A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3} + \frac{1}{3} 2^n \)

The IC make \( 1 = A + \frac{1}{3}, \ 3 = A \cos \frac{\pi}{3} + B \sin \frac{\pi}{3} \), \( A = \frac{2}{3}, \ B = \frac{4}{\sqrt{3}} \)

Answer is \( y_n = \frac{2}{3} \cos \frac{n\pi}{3} + \frac{4}{\sqrt{3}} \sin \frac{n\pi}{3} + \frac{1}{3} 2^n \)
14. $\lambda = 2,1$, $h_n = A2^n + B$. Try $p_n = (Cn + D)3^n$. Then

$$p_{n+1} = [c(n+1) + d] 3^{n+1} + (Cn + C + D)3^3 n$$
$$p_{n+2} = [c(n+2) + d] 3^{n+2} = (Cn + 2C + D)3^2 \cdot 3^n$$

We need

$$(9Cn + 18C + 9D)3^n - 3(3Cn + 3C + 3D)3^n + 2(Cn + D)3^n = 8n3^n$$

$$2Cn 3^n + (9C + 2)3^n = 8n3^n$$

$$2C = 8, \quad 9C + 2D = 0$$

$$C = 4, \quad D = -18$$

General sol is $y_n = A2^n + B + (4n - 18)3^n$

The IC make $-16 = A + B$, $-40 = 2A + B - 42$

So $A = 0, B = 2$. Answer is $y_n = 2 + (4n - 18)3^n$
SOLUTIONS review problems for Chapter 3

1. \(\lambda^2 - 9 = 0, \ \lambda = \pm 3\), \(h_n = A3^n + B(-3)^n\)

Try \(p_n = cn^2 + dn + e\)
\[c(n + 2)^2 + d(n + 2) + e = 9(cn^2 + dn + e) = 56n^2\]

Equate \(n^2\) coeffs: \(-8c = 56\), \(c = -7\)
Equate \(n\) coeffs: \(4c - 8d = 0\), \(d = -7/2\)
Equate constant terms: \(4c + 2d - 8e = 0\), \(e = -35/8\)

\(y_n = h_n + p_n = A3^n + B(-3)^n - 7n^2 - \frac{7}{2}n - \frac{35}{8}\)

2. \(\lambda = 1 \pm i\sqrt{3}\). The number \(1 + i\sqrt{3}\) has mag 2, angle \(\pi/3\) so

\(y_n = 2^n(A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3})\)

3. \(\lambda = -2, \ h_n = A(-2)^n\). Try \(p_n = B7^n\).

Need \(2B7^{n+1} + 4B7^n = 6\cdot 7^n\)

\(14B7^n + 4B7^n = 6\cdot 7^n\)

So \(18B = 6\), \(B = \frac{1}{3}\), \(y_n = A(-2)^n + \frac{1}{3}7^n\)

To get the IC you need \(5 = -2A + \frac{7}{3}\), \(A = -\frac{4}{3}\).

Answer is \(y_n = -\frac{4}{3}(-2)^n + \frac{1}{3}7^n\)

4. \(\lambda = \frac{-5 \pm \sqrt{29}}{2}\), \(h_n = A\left[\frac{-5 + \sqrt{29}}{2}\right]^n + B\left[\frac{-5 - \sqrt{29}}{2}\right]^n\)

Try \(p_n = D\). Need \(D + 5D = 6\), \(D = \frac{6}{5}\).

Answer is \(y_n = A\left[\frac{-5 + \sqrt{29}}{2}\right]^n + B\left[\frac{-5 - \sqrt{29}}{2}\right]^n + \frac{6}{5}\)

5. \(S_{n+1} = S_n + n+1\), \(S_{n+1} - S_n = n+1\) with IC \(S_1 = 1\).

\(\lambda = 1, \ h_n = A\). Try \(p_n = n(Bn + C)\) (step up) = \(Bn^2 + Cn\)

Need \(B(n+1)^2 + C(n+1) - (Bn^2 + Cn) = n+1\)

The \(n^2\) terms drop out on each side

Equate \(n\) coeffs: \(2B + C - C = 1\), \(B = \frac{1}{2}\)

Equate constant terms: \(B + C = 1\), \(C = \frac{1}{2}\)

\(S_n = S_n = A + \frac{1}{2}n^2 + \frac{1}{2}n\)

To get \(S_1 = 1\) you need \(1 = A + \frac{1}{2} + \frac{1}{2}\), \(A = 0\).

Answer is \(S_n = \frac{1}{2}n^2 + \frac{1}{2}n\) usually written as \(S_n = \frac{n(n+1)}{2}\)
6. \( \lambda = \pm 3 \), \( h_n = A 3^n + B(-3)^n \). Try \( p_n = Dn 3^n \) (step up).

Need \( D(n+2) 3^{n+2} - 9Dn 3^n = 5 \cdot 3^n \)

\[ 9D(n+2) 3^n - 9Dn 3^n = 5 \cdot 3^n \]

The \( n3^n \) terms drop out.

Match the \( 3^n \) coeffs: \( 18D = 5 \), \( D = \frac{5}{18} \). Answer is \( y_n = A 3^n + B(-3)^n + \frac{5}{18} n3^n \)

7. The \( rr \) can be written as \( y_{n+1} - 2y_n = 0 \) and it is only first order. It would come with only one IC and its general sol (namely \( B 2^n \)) should only have one constant. So nothing is wrong.

8. (a) To get all the rings moved you have to pass through the following stages

![Diagram of the Towers of Hanoi puzzle stages](image)

Start here

Move the top \( n-1 \) rings from peg 1 to peg 3 using peg 2 as storage. Takes \( y_{n-1} \) moves to do it in the best way

Move the largest ring to peg 2

Takes one move

Move the \( n-1 \) rings from peg 3 to peg 2 using peg 1 as storage.

Takes \( y_{n-1} \) moves

So \( y_n = 2y_{n-1} + 1 \).

And since it only takes one move in a 1-ring game the IC is \( y_1 = 1 \).

(b) \( \lambda = 2 \), \( h_n = A 2^n \). Try \( p_n = B \).

Need \( B - 2B = 1 \), \( B = -1 \)

Gen sol is \( y_n = A 2^n - 1 \). To get the IC you need \( 1 = 2A - 1 \), \( A = 1 \).

Sol is \( y_n = 2^n - 1 \).

For example to move a 10-ring tower it takes a minimum of \( 2^{10} - 1 \) moves.