SOLUTIONS  Section 2.1

1. (a) Solve
   \(2y'' + 2y = \delta(t)\) with IC \(y(0) = 0, y'(0) = 0\).

   To do this, switch to \(2y'' + 2y = 0\) with IC \(y(0) = 0, y'(0) = 1/2\).

   \(m = \pm i\), \(y_h = A \cos x + B \sin x\). The IC make \(A = 0, B = 1/2\).

   Answer is \(h(t) = \frac{1}{2} \sin t\) for \(t \geq 0\).

   (b) To solve

   \(2y'' - y' - y = \delta(t)\) with \(y(0) = 0, y'(0) = 0\)

   switch to \(2y'' - y' - y = 0\) with \(y(0) = 0, y'(0) = 1/2\).

   \(m = -1/2, 1\); \(y = Ae^{-t/2} + Be^t\). The IC make \(A = -1/3, B = 1/3\).

   \(h(t) = -\frac{1}{3} e^{-t/2} + \frac{1}{3} e^t\) for \(t \geq 0\).

   Question I get asked a lot In (*) and (**), do you always use IC \(y(0) = 0, y'(0) = 0\).
   The problem didn’t say anything about IC.

   Answer If you want to find the impulse response then, yes, you must use
   IC \(y(0) = 0, y'(0) = 0\) and use \(\delta(t)\) as the forcing function. The impulse response
   is defined as the response of an initially-at-rest system to the delta function input.

   You can use the delta function as the input into a system that is not initially
   at rest but then the response is not called the impulse response.

2. (a) Solve \(y'' + 4y = 0\) with IC \(y(0) = 0, y'(0) = 1\)

   \(m = \pm 2i\), \(y = A \cos 2x + B \sin 2x\). The IC make \(A = 0, B = 1/2\)

   \(h(t) = \frac{1}{2} \sin 2t\) for \(t \geq 0\).

   (b) Take the \(h(t)\) from part (a), multiply by 6 and delay.

   Answer is \(y(t) = 6h(t-2) = \begin{cases} 0 & \text{if } t \leq 2 \\ 3 \sin 2(t-2) & \text{if } t \geq 2 \end{cases}\)

3. Response \(y(t)\) to \(\delta(t)\) is \(h(t)\).

   Response \(y(t)\) to \(6\delta(t)\) is \(6h(t)\)

   Response \(y(t)\) to \(\delta(t-2)\) is \(h(t-2)\)

   (a) \(y(3) = h(3) = 1/9\)

   (b) Response \(y(3) = 6h(3) = 6/9\)

   (c) Response \(y(3) = h(3-2) = h(1) = 1\)

4. To get the location of the max value of \(h(t)\), find \(h'(t)\) and set it equal to 0.

   \(-\frac{1}{4} e^{-t} + \frac{1}{4} 3e^{-3t} = 0\)

   \(e^{2t} = 3\)

   \(2t = \ln 3\)

   \(t = \frac{1}{2} \ln 3\).

   This is either a relative max or a rel min or a point of inflection but since we already have the graph, this value of \(t\) must go with a rel max.
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5. (See the superposition rule for IC in Section 1.6.)
To get the solution to
\[ ay'' + by' + cy = \delta(t) \text{ with IC } y(0) = 4, \ y'(0) = 5 \]

take \( h(t) \) and add the solution to
\[ ay'' + by' + cy = \text{ZERO} \text{ with IC } y(0) = 4, \ y'(0) = 5 \]
SOLUTIONS Section 2.3

1. (a)

**Graph of h(u)**  
**Intermediate step**  
**Graph of h(t-u)**

**Question** There is a y-axis (vertical axis) in the left hand diagram. Why isn't there a y-axis in the other two diagrams.

**Answer** Where the y-axis goes in the last two diagrams depends on the size of t. Here is how it works for the right hand diagram.

**case 1** \( t \leq 0 \)

**case 2** \( 0 \leq t \leq 5 \) (so that \( t < 0 \) but \( t-5 \leq 0 \))

**case 3** \( 5 \leq t \leq 9 \) (so that \( t-5 \geq 0 \) but \( t-9 \leq 0 \))

**case 4** \( t \geq 9 \) (so that \( t-9 \geq 0 \))

(b)

**Graph of h(u)**  
**Intermediate step**  
**Graph of h(t-u)**

2. The response is \( h(t)f(t) \). I’ll use the version \( \int_{u=-\infty}^{\infty} h(t-u)f(u) \) du  
Here are the graphs of \( h(t-u) \) and \( f(u) \)
case 1  \( t - 3 \leq 0 \), i.e., \( t \leq 3 \)

\[
h(t) \cdot f(t) = \int_{u=-\infty}^{\infty} 0 \, du = 0
\]

(no overlap yet)

warning In this case, \( h(t) \cdot f(t) \) is not \( \int_{u=-\infty}^{\infty} 4 \cdot 5 \, du \). It's \( \int_{u=-\infty}^{\infty} 0 \, du \)

case 2  \( t - 7 \leq 0 \) and \( t - 3 \geq 0 \), i.e., \( 3 \leq t \leq 7 \)

\[
h(t) \cdot f(t) = \int_{u=0}^{t-3} 5 \cdot 4 \, du = 20t - 60
\]

case 3  \( t - 7 \geq 0 \) and \( t - 3 \leq 6 \), i.e., \( 7 \leq t \leq 9 \)

\[
h(t) \cdot f(t) = \int_{u=t-7}^{t-3} 20 \, du = 80
\]

case 4  \( t - 7 \leq 6 \) and \( t - 3 \geq 6 \), i.e., \( 9 \leq t \leq 13 \)

\[
h(t) \cdot f(t) = \int_{u=t-7}^{6} 20 \, du = -20t + 260
\]

case 5  \( t - 7 \geq 6 \), i.e., \( t \geq 13 \)

No more overlap.

\[
h(t) \cdot f(t) = \int_{u=-\infty}^{\infty} 0 \, du = 0
\]

All in all,

\[
h(t) \cdot f(t) = \begin{cases} 
0 & \text{if } t \leq 3 \\
20t - 60 & \text{if } 3 \leq t \leq 7 \\
80 & \text{if } 7 \leq t \leq 9 \\
-20t + 260 & \text{if } 9 \leq t \leq 13 \\
0 & \text{if } t \geq 13 
\end{cases}
\]
3. The response is \( f(t) \cdot h(t) \).
   I’ll use the version which flips \( f \).
   You should get the same final answer no matter which you flip.

(a) No overlap iff \( t - 13 \geq 5 \), i.e.,
    if \( t \geq 18 \) then the product \( h(u)f(t-u) = 0 \)
    So response dies at time \( t = 18 \)

(b) Overlap never stops.
    Response never dies out

4.

\[ y(t) = \int_{u=0}^{t} 8(4-u) \, du = -4t^2 + 32t \]

**case 1**: \( t \leq 0 \)

\[ y(t) = 0 \]

**case 2**: \( 0 \leq t \leq 4 \)

\[ y(t) = \int_{u=0}^{t} 8(4-u) \, du = -4t^2 + 32t \]

**case 3**: \( t - 6 \leq 0, t \geq 4 \)
    i.e., \( 4 \leq t \leq 6 \)

\[ y(t) = \int_{u=0}^{4} 8(4-u) \, du = 64 \]
case 4 0 \leq t-6 \leq 4, i.e., \[6 \leq t \leq 10\]

\[y(t) = \int_{u=t-6}^{4} 8(4-u) \, du = 4t^2 - 80t + 400\]

\[
\begin{array}{c}
\text{Graph of } f(u) \\
\text{Graph of } h(u) \\
\text{Graph of } h(t-u)
\end{array}
\]

\[
\begin{align*}
\text{warning} & \quad \text{The integrand contains } f(u) = \frac{1}{4}u, \text{ not } \frac{1}{4}t
\end{align*}
\]

case 5 t-6 \geq 4, i.e., \[t \geq 10\]

\[y(t) = 0 \text{ (no more overlap)}\]

5. If the system has impulse response \(f(t)\) then input \(g(t)\) produces output \(c(t)\).

If the system has impulse response \(g(t)\) then input \(f(t)\) produces output \(c(t)\).

6. The response is \(y(t) = h(t)f(t)\). I'll use the version \(\int_{-\infty}^{\infty} h(t-u)f(u) \, du\)

\[
\begin{align*}
\text{Graph of } f(u) \\
\text{Graph of } h(u) \\
\text{Graph of } h(t-u)
\end{align*}
\]

Case 1 \(t \leq 0\)

\[y(t) = 0 \text{ (no overlap yet)}\]

Case 2 \(t \geq 0\) and \(t-3 \leq 0\), i.e.,

\[0 \leq t \leq 3\]

\[y(t) = \int_{u=0}^{t} \frac{1}{4} \, (3+u-t) \, du = \frac{1}{4} \left( \frac{3}{2} u^2 + \frac{1}{3} u^3 - \frac{1}{2} tu^2 \right) \quad \begin{array}{c}
\text{at } u=t \\
\text{at } u=0
\end{array}
\]

\[
\begin{align*}
\text{warning} & \quad \text{The integrand contains } f(u) = \frac{1}{4}u, \text{ not } \frac{1}{4}t
\end{align*}
\]

Case 3 \(t \leq 8\) and \(t-3 \geq 0\), i.e.,

\[3 \leq t \leq 8\]

\[y(t) = \int_{u=t-3}^{t} \frac{1}{4} \, (3+u-t) \, du = \frac{1}{4} \left( \frac{3}{2} u^2 + \frac{1}{3} u^3 - \frac{1}{2} tu^2 \right) \quad \begin{array}{c}
\text{at } u=t \\
\text{at } u=t-3
\end{array}
\]

\[\text{Case 4 } t \geq 8\) and \(t-3 \leq 8\), i.e.,

\[8 \leq t \leq 11\]

\[y(t) = \int_{u=t-3}^{8} \frac{1}{4} \, (3+u-t) \, du = \frac{1}{4} \left( \frac{3}{2} u^2 + \frac{1}{3} u^3 - \frac{1}{2} tu^2 \right) \quad \begin{array}{c}
\text{at } u=t \\
\text{at } u=t-3
\end{array}
\]
7. (a) Response is \( f(t) \star h(t) \) or equivalently \( h(t) \star f(t) \).

(b) By superposition, the solution to
\[
2y'' + 8y' + 6y = f(t)
\]
with IC \( y(0) = 7, y'(0) = 9 \)

is the sum of the solutions to the following two problems:

(1) \( 2y'' + 8y' + 6y = f(t) \) with IC \( y(0) = \text{ZERO}, y'(0) = \text{ZERO} \)

(2) \( 2y'' + 8y' + 6y = \text{ZERO} \) with IC \( y(0) = 7, y'(0) = 9 \)

Solution to (1) is \( h(t) \star f(t) \).

To get solution to (2):
\[
m^2 + 4m + 3 = 0, \ m = -1, -3, \ y_h = Ae^{-t} + Be^{-3t}.
\]

To get the IC, need
\[
A + B = 7, \ -A - 3B = 9, \ A = 15, \ B = -8, \ y = 15e^{-t} - 8e^{-3t}
\]

Final answer is \( h(t) \star f(t) + 15e^{-t} - 8e^{-3t} \)

8. (a) I'll flip \( p \) because it's the simpler function.

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**Graphs**

*Graph of \( q(u) \)

*Graph of \( p(u) \)

*Graph of \( p(t-u) \)

---

**Case 1** \( t \leq 0 \)

\( p \star q(t) = 0 \) (no overlap yet)

**Case 2** \( 0 \leq t \leq 3 \)

\[
p \star q(t) = \int_{u=0}^{t} 2u(u - t + 5) \, du
\]

\[
= \int_{u=0}^{t} 2u^2 \, du + (5-t)\int_{u=0}^{t} 2u \, du
\]

\[
= 5t^2 - \frac{1}{3} t^3
\]

**Case 3** \( t-5 \leq 0 \) and \( t \geq 3 \), i.e.,

\( 3 \leq t \leq 5 \)

\[
p \star q(t) = \int_{u=0}^{3} 2u(u-t+5) \, du + \int_{3}^{t} (-u+9)(u-t+5) \, du
\]
case 4 $0 \leq t - 5 \leq 3$, i.e., $5 \leq t \leq 8$

$$p\ast q(t) = \int_{u=t-5}^{3} 2u(u-t+5)\,du + \int_{3}^{t} (-u+9)(u-t+5)\,du$$

case 5 $t - 5 \geq 3$ and $t \leq 9$, i.e., $8 \leq t \leq 9$

$$p\ast q(t) = \int_{u=t-5}^{t} (-u+9)(u-t+5)\,du$$

case 6 $t - 5 \leq 9$, $t \geq 9$, i.e., $9 \leq t \leq 14$

$$p\ast q(t) = \int_{u=t-5}^{9} (-u+9)(u-t+5)\,du$$

case 7 $t - 5 \geq 9$, i.e., $t \geq 14$

$p\ast q(t) = 0$

(b) $p(t)$ is the response of the system to the input $\delta(t)$ when the system is initially at rest.

(c) $p(t)\ast q(t)$ is the response of the system to input $q(t)$ if the system is initially at rest.

9. Let $v = t - u$, $dv = -du$. Then

$$\int_{u=-\infty}^{\infty} h(t-u)\,f(u)\,du = \int_{v=-\infty}^{\infty} h(v)\,f(t-v)\,-dv$$

$$= \int_{v=-\infty}^{\infty} h(v)\,f(t-v)\,dv$$

(reversing the limits of integration changes the sign of the integral)

$$= \int_{u=-\infty}^{\infty} h(u)\,f(t-u)\,du$$

(change from the dummy variable $v$ to dummy variable $u$)

10. (a) The response is $f(t)\ast h(t)$. I'll flip $f$ because it's the simpler function.

$$e^{-u}$$

Graph of $h(u)$

$$\begin{cases} 6 \\ \text{u-axis} \end{cases}$$

Graph of $f(t-u)$

case 1 $t \leq 0$

$f(t)\ast h(t) = 0$ (no overlap yet)

case 2 $t - 5 \leq 0$, $t > 0$, i.e., $0 \leq t \leq 5$

$$f(t)\ast h(t) = \int_{u=0}^{t} 6e^{-u}\,du = 6 - 6e^{-t}$$
case 3  \( t - 5 \geq 0 \), i.e., \( t \geq 5 \)

\[ f(t) \ast h(t) = \int_{u=t-5}^{t} 6e^{-u} \, du \]
\[ = -6e^{-t} + 6e^{5-t} \]

(b) The response to \( \delta(t) \) is the given impulse response \( h(t) \). That's the whole point of the impulse response.

11. (a) I flipped \( f \).

If \( t \leq 0 \) then \( f(t) \ast h(t) = 0 \)

Suppose \( t \geq 0 \). Then

\[ f(t) \ast g(t) = \int_{u=0}^{t} (t-u) \, e^{-u} \, du \]
\[ = \left[ -te^{-u} - (ue^{-u} - e^{-u}) \right]_{u=0}^{t} \]
\[ = e^{-t} - 1 + t \]

(b) I flipped \( f \).

If \( t \leq 0 \) then \( f(t) \ast h(t) = 0 \).

Suppose \( t \geq 0 \). Then

\[ f(t) \ast g(t) = \int_{u=0}^{t} \sin(t-u) \cos u \, du \]
\[ = \frac{1}{2} \left[ \sin t + \sin(t-2u) \right]_{u=0}^{t} \]
\[ = \frac{1}{2} \left[ ut \sin t + \frac{1}{4} \cos(t-2u) \right]_{u=0}^{t} \]
\[ = \frac{1}{2} t \sin t \]

12. (a)

If \( t \leq 0 \) then \( f \ast f(t) = 0 \)

Suppose \( t \geq 0 \). Then

\[ f(t) \ast f(t) = \int_{u=0}^{t} e^{t-u} \, e^{u} \, du \]
\[ = \int_{0}^{t} e^{t} \, du = te^{t} \]
(b)

Graph of \( f(u) \)  
\[ \overbrace{\text{Graph of } f(t-u)}^{t-b \text{ to } t+b} \]

Case 1: \( t + b < -b \), i.e., \( t < -2b \)
\[ f \star f = 0 \]

No overlap yet

Case 2: \( -b \leq t + b \leq b \), i.e., \( -2b \leq t \leq 0 \)
\[ f(t) \star f(t) = \int_{u=-b}^{t+b} a^2 \, du = a^2(t + 2b) \]

\[ \int_{t-b}^{t+b} a^2 \, du = a^2(t + 2b) \]

Case 3: \( -b \leq t-b \leq b \), i.e., \( 0 \leq t \leq 2b \)
\[ f(t) \star f(t) = \int_{u=t-b}^{b} a^2 \, du = a^2(2b-t) \]

\[ \int_{t-b}^{b} a^2 \, du = a^2(2b-t) \]

Case 4: \( t-b \geq b \), i.e., \( t \geq 2b \)
\[ f(t) \star f(t) = 0 \]

No more overlap

Here's the graph of \( f(t) \star f(t) \)

13. The function in the diagram is the impulse response so call it \( h(t) \).
The response to \( f(t) \) is \( f(t) \star h(t) \).

Case 1: \( t \leq 0 \)
\[ f(t) \star h(t) = 0 \]

No overlap
**case 2**  \[0 \leq t \leq 3\]

\[f(t) \ast h(t) = \int_0^t e^{u-t} (2-u) \, du\]

\[= e^{-t} \int_0^t (2e^u - ue^u) \, du\]

\[= e^{-t} (2e^t - te^t + e^t) \bigg|_{u=0}^{t}\]

\[= 3 - t - 3e^{-t}\]

*Note* You don’t need separate cases for \(0 \leq t \leq 2\) and then \(2 \leq t \leq 3\). No matter which of the two pictures you look at you still get the integral. The fact that the \(h(u)\) graph goes below the axis at \(u=2\) is irrelevant. What is relevant is when a function changes formula, which \(h(u)\) does at \(u=0\) and then again at \(u=3\) but not at \(u=2\).

If you do use the two cases \(0 \leq t \leq 2\) and then \(2 \leq t \leq 3\) you will get the same integral and the same final answer for each case which is the signal that you only needed one case in the first place.

**case 3**  \([t \geq 3]\)

\[f(t) \ast h(t) = \int_0^3 e^{u-t} (2-u) \, du\]

\[= e^{-t} (2e^u - ue^u + e^u) \bigg|_{u=0}^{3}\]

\[= -3e^{-t}\]

Steady state solution is 0 since \(-3e^{-t} \to 0\) as \(t \to \infty\).
HONORS

14. There are two ways to think of the convolution $\delta(t) \ast g(t)$ (see problem 5). So there are two ways to do this problem.

*method 1 (easier)*

$\delta(t) \ast g(t)$ is the response of an initially-at-rest system to input $\delta(t)$ provided the system has impulse response $g(t)$.

But the response of an initially-at-rest system to input $\delta(t)$ is the impulse response. So $\delta(t) \ast g(t) = g(t)$.

(Makes your head spin a little.)

Omit this second explanation.

Here's the same explanation, phrased slightly differently.

Look at this question.

A system has impulse response $g(t)$.

Find the response of the system to $\delta(t)$.

The answer is $\delta(t) \ast g(t)$.

But the answer is also $g(t)$ because the response to $\delta(t)$ is the impulse response.

So the two "answers" $\delta(t) \ast g(t)$ and $g(t)$ must agree. So $\delta(t) \ast g(t) = g(t)$.

Footnote For the operation "convolution", the unit impulse function $\delta(t)$ is the identity element. Just as for the operation "ordinary multiplication", the number 1 is the identity element.

*method 2*

Think of $\delta(t) \ast g(t)$ as the response of an initially-at-rest system to input $g(t)$ provided the system has impulse response $\delta(t)$ (and, as usual, the system satisfies superposition and is time invariant).

Since the system has impulse response $\delta(t)$ this means that the input $\delta(t)$ produces the output $\delta(t)$ (it's a copy-cat system so far).

And the response to $\delta(t-4)$ is $\delta(t-4)$ (see "response to a delayed impulse" in Section 2.1).

And by superposition, if the input is $2\delta(t)$ then the output is $2\delta(t)$.

So any kind of input delta produces the same output delta (more copy-cat).

Now what about the output of this system when $g(t)$ is the input (because that's what the problem says to find).

You can think of $g(t)$ as a sum of various deltas (Fig A) (delayed and not necessarily unit deltas but it doesn't matter).

Each of the various delta inputs produces a copy of itself as an output. By superposition, if you put in the sum of various deltas you get out a sum of the same deltas, which is $g(t)$ again. So when you input $g(t)$, you get out $g(t)$; the system copy-cats every input, not just deltas.

So $\delta(t) \ast g(t)$ [the response of the system to $g(t)$] = $g(t)$.  
15. method 1 The input \( f(t) \) can be thought of as a sum of delta functions. The typical delta function in the sum occurs at time \( t=u \), has width \( du \), area \( f(u)du \) and is named \( f(u) du \, \delta(t-u) \) (Fig A).

So \( f(t) \) is a sum (integral) of \( f(u)du \, \delta(t-u) \)'s.

If the response of the system to \( \delta(t) \) is \( \delta(t) \) then the response to \( \delta(t-u) \) is \( \delta(t-u) \) (see "response to a delayed impulse" in Section 2.1).

If the response to \( \delta(t-u) \) is \( \delta(t-u) \) then the response to \( f(u)du \, \delta(t-u) \) is \( f(u)du \, \delta(t-u) \) by superposition.

And the response to the sum of \( f(u)du \, \delta(t-u) \)'s is that same sum of \( f(u)du \, \delta(t-u) \)'s (more superposition). But the sum is \( f(t) \). So the response to \( f(t) \) is \( f(t) \). QED

method 2

The response to \( f(t) \) is \( h(t) \ast f(t) \) which in this case is \( \delta(t) \ast f(t) \).

Use the previously proved fact (method 1) (or prove it again using transforms) that \( \delta(t) \ast f(t) = f(t) \).

16. (a) Solve \( 2y'' + 8y = \delta(t) \) with IC \( y(0) = 0, y'(0) = 0 \).

To do this, switch to

\[
2y'' + 8y = 0 \text{ with IC } y(0) = 0, y'(0) = 1/2
\]

\[
2m^2 + 8 = 0, m^2 = -4, m = \pm 2i
\]

\[
y = A \cos 2t + B \sin 2t
\]

To get \( y(0) = 0 \) you need \( 0 = A \)

Then \( y'(t) = 2B \cos 2t \)

To get \( y'(0) = 1/2 \) you need \( 1/2 = 2B, B = 1/4 \)

Impulse response is \( h(t) = \frac{1}{4} \sin 2t \)

(b) The convolution \( \frac{1}{4} \sin 2t \ast \frac{t^5 \tan t}{1 + t^2} \) is the solution to
\[2y'' + 8y = \frac{t^5 \tan t}{1 + t^2}\] with IC \(y(0) = 0, y'(0) = 0\).

Use superposition to get the solution to
\[2y'' + 8y = \frac{t^5 \tan t}{1 + t^2}\] with IC \(y(0) = 2, y'(0) = 3\).

Add to the convolution the solution to
\[2y'' + 8y = 0\] with IC \(y(0) = 2, y'(0) = 3\).

\[y_{\text{gen}} = A \cos 2t + B \sin 2t\]

To get \(y(0) = 2\) you need \(A = 2\).

Then \(y' = -4 \sin 2t + 2B \cos 2t\).

To get \(y'(0) = 3\) you need \(2B = 3\), \(B = 3/2\).

Solution here is \(y = 2 \cos 2t + \frac{3}{2} \sin 2t\).

Final answer is \(\frac{1}{4} \sin 2t \cdot \frac{t^5 \tan t}{1 + t^2} + 2 \cos 2t + \frac{3}{2} \sin 2t\).
SOLUTIONS review problems for Chapter 2

1. (a) Solve \( 4y'' + y' = 0 \) with IC \( y(0) = 0 \), \( y'(0) = 1/4 \).

\[
y = A \cos \frac{t}{2} + B \sin \frac{t}{2}.
\]

The IC make \( A = 0 \), \( B = 1/2 \) so \( h(t) = \frac{1}{2} \sin t/2 \) for \( t \geq 0 \).

(b) \( h(t-11) = \begin{cases} 
0 & \text{for } t \leq 11 \\
\frac{1}{2} \sin \frac{1}{2} (t-11) & \text{for } t > 11 
\end{cases} \)

2. The response is \( h(t) \ast f(t) \). I'll use the version \( \int_{-\infty}^{\infty} h(u) f(t-u) \, du \).

\[
y = -2u+6
\]

\[
u-axis
\]

\[
3
\]

\[
y = u
\]

\[
u-axis
\]

\[
t-5
t
\]

\[
t-u
\]

\[
u-axis
\]

Graph of \( h(u) \)    Graph of \( f(u) \)    Graph of \( f(t-u) \)

\( t \leq 0 \)  \h(0,f(0)) = 0

\( 0 \leq t \leq 3 \)

\[
h(t) \ast f(t) = \int_{u=0}^{t} (-2u+6)(t-u) \, du = (-tu^2 + 6tu + \frac{2}{3} u^3 - 3u^2)\bigg|_{u=0}^{u=t}
\]

\[
= 3t^2 - \frac{1}{3} t^3
\]

\( t \geq 3 \) and \( t-5 \leq 0 \), i.e. \( 3 \leq t \leq 5 \)

\[
h(t) \ast f(t) = \int_{u=0}^{3} (-2u+6)(t-u) \, du = (-tu^2 + 6tu + \frac{2}{3} u^3 - 3u^2)\bigg|_{u=0}^{u=3}
\]

\[
= 9t - 9
\]

\( 0 \leq t-5 \leq 3 \), i.e., \( 5 \leq t \leq 8 \)

\[
h(t) \ast f(t) = \int_{u=-5}^{3} (-2u+6)(t-u) \, du = (-tu^2 + 6tu + \frac{2}{3} u^3 - 3u^2)\bigg|_{u=-5}^{u=3}
\]

\( t-5 \geq 3 \), i.e., \( t \geq 8 \)

No more overlap. \( h(t) \ast f(t) = 0 \)