CHAPTER 4 SOME FIRST ORDER DIFFERENTIAL EQUATIONS

SECTION 4.1 LINEAR FIRST ORDER WITH NOT NECESSARILY CONSTANT COEFFICIENTS

solution to \( y' + P(x)y = Q(x) \)

A typical first order linear DE has the form

\[ ay' + by = f(x). \]

If \( a \) and \( b \) are constants then the methods of the preceding sections work for \( y_h \) and \( y_p \) and the gen solution is \( y_h + y_p \).

There is another method which not only works in the case of constant \( a \) and \( b \) but also works if they are not constant.

To solve

\[ ay' + by = f(x) \]

divide by \( a \) to get the form

\[ y' + \frac{b}{a}y = \frac{f(x)}{a} \]

Find \( \int P(x) \, dx \) and let

\[ I = e^{\int P(x) \, dx} \]

Then the solution for \( y \) is given by

\[ Iy = \int IQ \, dx \]

Add an arbitrary constant when you do this integral

To finish up, solve for \( y \) by dividing by \( I \)

Don't bother inserting an arbitrary constant when you find \( \int P(x) \, dx \) (if you do it will only cancel out later anyway — see problem #3(a)). But do put one in when you find \( \int IQ \, dx \). Otherwise you won't get a general solution.

proof

Take the equation

\[ y' + P(x)y = Q(x) \]

and multiply on both sides by an as-of-yet undetermined function \( I(x) \), called an integrating factor:

\[ (*) \quad I(x)y' + I(x)P(x)y = I(x)Q(x). \]

The lefthand side \( I(x)y' + I(x)P(x)y \) would be the derivative of the product \( I(x)y(x) \) if you had

\[ (***) \quad I(x)P(x) = I'(x) \]

So if \( I(x) \) is chosen to satisfy \( (***) \) then the DE in \( (*) \) becomes

\[(Iy)' = IQ \]

and its solution is

\[ Iy = \int IQ \, dx \]
To finish up you need a function $I(x)$ satisfying (**), i.e., you need a function whose derivative equals the original function times $P(x)$. One such function is

$$I(x) = e^{\int P(x) \, dx};$$

This works because its derivative is $e^{\int P(x) \, dx}$ times the derivative of $\int P(x) \, dx$, i.e., its derivative is the original function times $P(x)$.

**Example 1**

To find a general solution to

(2) $xy' - 3y = x^5$  (first order linear, variable coeffs)

rearrange to get

(3) $y' - \frac{3}{x} y = x^4$

Then

(4) $P(x) = -\frac{3}{x}$, $\int P(x) \, dx = -3 \ln x$

$$I = e^{\int P(x) \, dx} = e^{-3 \ln x} = e^{\ln x^{-3}} \quad \text{review} \quad \ln x^a = a \ln x$$

$$= x^{-3} \quad \text{review} \quad e^0 = 1$$

Now use (1):

(5) \[ \frac{y}{x^3} = \int \frac{x^4}{x^3} \, dx = \int x \, dx = \frac{1}{2} x^2 + K \]

**Warning**

1. $Q(x)$ is $x^4$ from line (3), not $x^5$ from line (2).
2. The arbitrary constant must be inserted here. It's wrong to leave it out completely and it's wrong it casually insert it at some later stage. Do it now.

Final solution is

$$y = \frac{1}{2} x^5 + Kx^3$$

**Warning**

Make sure that the coeff of $y'$ is 1 before going into the $P,Q$ routine.

**Example 1 continued**

I'll check that the solution is correct.

Let $y = \frac{1}{2} x^5 + Kx^3$. Find $xy' - 3y$ to see if it comes out to be $x^5$:

$$xy' - 3y = x \left( \frac{5}{2} x^4 + 3Kx^2 \right) - 3 \left( \frac{1}{2} x^5 + Kx^3 \right)$$

$$= \frac{5}{2} x^5 + 3Kx^3 - 3 \left( \frac{1}{2} x^5 + Kx^3 \right)$$

$$= x^5 \quad \text{QED}$$
warning about mathematical style

To do this check in example 1, i.e., to show that \( y = \frac{1}{2}x^5 + Kx^3 \) satisfies \( xy' - 3y = x^5 \), it is neither good style nor good logic to write like this.

Don't write like this:

\[
\begin{align*}
xy' - 3y &= x^5 \\
x\left( \frac{5}{2}x^4 + 3Kx^2 \right) - 3\left( \frac{1}{2}x^5 + Kx^3 \right) &= x^5 \\
\frac{5}{2}x^5 + 3Kx^3 - \frac{3}{2}x^5 - 3Kx^3 &= x^5 \\
x^5 &= x^5 \text{ TRUE!}
\end{align*}
\]

First of all, it is silly keep repeating the \( x^5 \)s on the righthand side; the essence of the argument is in the lefthand sides where \( xy' - 3y \) turned into \( x^5 \).

And any "proof" in mathematics that begins with what you want to prove and ends with something TRUE, like \( x^5 = x^5 \) (or \( B = B \) or \( 0 = 0 \)) is at best badly written and at worst incorrect and drives me crazy.

With this "method" I can prove that \( 3 = 4 \):

\[
\begin{align*}
3 &= 4 \\
4 &= 3 \\
7 &= 7 \text{ (add) TRUE!!} \\
\text{So conclude that } 3 &= 4 \text{ ??????}
\end{align*}
\]

What you should do to check that \( xy' - 3y \) equals \( x^5 \) is work on one of them until it turns into the other or work on each one separately until they turn into the same thing. Don't write \( xy' - 3y = x^5 \) as the first line of your proof. It should be your last line, as in my example 1 continued.

example 2

Find a gen solution to

\[
y' + 2y = e^{3x} \quad \text{(first order linear, constant coeffs)}
\]

\textit{method 1} You can use the methods of Chapter 1 since the DE has constant coeffs.

\[
m + 2 = 0, \quad m = -2 \\
y_h = Ae^{-2x}.
\]

Try \( y_p = Be^{3x} \)

Substitute into the DE to get

\[
3Be^{3x} + 2Be^{3x} = e^{3x}
\]

Equate coeffs of \( e^{3x} \): \( 5B = 1, \quad B = 1/5 \)

So \( y_{gen} = y_h + y_p = Ae^{-2x} + \frac{1}{5}e^{3x} \)
method 2  
You can also use the method of this section:
\[ P(x) = 2, \quad Q(x) = e^{3x}, \quad \int P(x) \, dx = 2x, \quad I = e^{2x}. \]

By (1),
\[
ye^{2x} = \int e^{5x} \, dx = \frac{1}{5} e^{5x} + K
\]
\[
y = Ke^{-2x} + \frac{1}{5} e^{3x}
\]

example 3
The DE
\[ xy'' + y' = 0 \]
is second order when the unknown is the function y but if you consider the unknown to be \( y' \) then it is first order. Rearrange to get
\[
(y')' + \frac{1}{x} (y') = 0.
\]
So
\[
\int P(x) \, dx = \int \frac{1}{x} \, dx = \ln x, \quad I = e^{\ln x} = x
\]
and by (1),
\[
xy' = \int x \cdot 0 \, dx = K,
\]
\[
y' = \frac{K}{x}.
\]
Finally, antidifferentiate to get
\[
y = K \ln x + C
\]

PROBLEMS FOR SECTION 4.1

1. Solve
   
   (a) \((x + 2)^2 y' + 4(x + 2) y = -6\)
   
   (b) \(y' = x - 4xy\)
   
   (c) \(2y' + 2y = e^{2x}\)
   
   (d) \(y' - y \cot x = \csc x\)  
   For reference: \(\int \cot x \, dx = \ln \sin x\)

2. The DE \(xy'' + y' = 4x\) is second order with variable coeffs but if you consider \(y'\) to be the unknown then it is linear first order.
   Solve it for \(y'\) first and then find \(y\).

3. (a) Solve problem #1(b) again and this time insert an arbitrary constant when you find \(\int P(x) \, dx\) (you're entitled). What happens?

   (b) What happens if you solve problem #1(b) and you leave out the arbitrary constant when you find \(\int Q \, dx\)
4. Solve $y' = ky$ two ways (k is a fixed constant).

5. Solve $y' + y = e^{-x}$ with IC $y(-1) = 3$.

6. (a) Why is it not quite legal to say that if $K$ is an arbitrary constant then $e^K$ is just another arbitrary constant and can be renamed $C$.
(b) Is it OK to turn $\ln K$ into a new arbitrary constant called $C$.

7. Solve $xy' + 2y = x^2 + 1$.

8. Let
$$f(x) = \begin{cases} 
  x & \text{for } x \leq 3 \\
  0 & \text{for } x \geq 3
\end{cases}$$

Solve $y' - \frac{1}{x} y = f(x)$ with condition $y(1) = 2$ and make the solution continuous.
9. Let $y(t)$ be the fish population in a lake at time $t$. If the fish are left alone then the population grows at a rate proportional to the size of the population, i.e., $y'(t) = ry(t)$ where $r$ is a positive constant.

If the fish are harvested at the constant rate of $h$ fish per unit of time (where $h$ is a positive constant) then the differential equation becomes

$$y'(t) = ry(t) - h$$

Suppose there are $N$ fish initially.

(a) Find $y(t)$.
(b) Let $N = 40$, $r = 2$ specifically. For what values of $h$ does the lake get fished out.
(c) Continue from part (b) with $N = 40$ and $r = 2$. One of those values of $h$ for which the lake gets fished out is $h = 100$. For this value of $h$, when does the lake get fished out.
SECTION 4.2 SEPARABLE DIFFERENTIAL EQUATIONS

the algebra of arbitrary constants

If A and B are arbitrary constants then so are A + B, 3A, A-B, AB etc. and may be re-named C₁, C₂, C₃ etc.

separating variables

One way to solve

(1) \( y' = \frac{x}{y^2} \)

is to rewrite the equation as

(2) \( y^2(x) \cdot y'(x) = x \)

and antidifferentiate on both sides with respect to x to get

(3) \( \int y^2(x) \cdot y'(x) \, dx = \int x \, dx \)

(4) \( \frac{1}{3} y^3(x) = \frac{1}{2} x^2 + C \)  (An antideriv of \( y^2(x)y'(x) \) is \( \frac{1}{3} y^3(x) \). Differentiate it back, using the chain rule, to see.)

The procedure in (1)-(4) is usually written in the following more convenient style:

| (1A) \( \frac{dy}{dx} = \frac{x}{y^2} \) |
| (2A) \( y^2 \, dy = x \, dx \) |
| (3A) \( \int y^2 \, dy = \int x \, dx \) |
| (4A) \( \frac{1}{3} y^3 = \frac{1}{2} x^2 + C \) |

In (4A) there's an arbitrary constant on one side only because if you put in two constants you get

\( \frac{1}{3} y^3 + A = \frac{1}{2} x^2 + B \)

which reduces to (4A) anyway when you let \( C = B - A \).

So far the solution \( y \) has been found implicitly in (4'A). The explicit solution is

(5) \( y = \sqrt[3]{\frac{3}{2} x^2 + 3C} \)

or equivalently

(6) \( y = \sqrt[3]{\frac{3}{2} x^2 + D} \)

To check the solution in (6) find \( y' \) and \( x/y^2 \) to see if they are equal:

\( y' = \frac{1}{3} (\frac{3}{2} x^2 + D)^{-2/3} \cdot 3x \)

\( \frac{x}{y^2} = \frac{x}{(\frac{3}{2} x^2 + D)^{2/3}} \)

They came out equal so the solution checks out.
In general:

If it's possible to separate variables so that the DE has the form

\[ x\text{-stuff } dx = y\text{-stuff } dy \]

(as in (2')) then the DE is called separable and is solved by antidiffing on both sides and inserting an arbitrary constant on one side.

Only first order DE, that is, equations involving \( y' \) but not \( y'' \), \( y''' \) etc., can be separated.

The separation process usually leads to an implicit solution for \( y \). If it is feasible to solve for \( y \) explicitly, do it, but otherwise settle for an implicit solution.

The solution will contain one arbitrary constant and is called the general solution. If you are given some condition then the constant can be determined to get the specific solution satisfying the DE plus condition.

**warning**

*The variables must be separated* before this method of integrating w.r.t. \( y \) on one side and w.r.t. \( x \) on the other side can be used. The DE

\[ y' = \frac{2x}{x + 3y^2} \]

can be written as

\[ (x + 3y^2) \, dy = 2x \, dx \]

but there is no way to continue and separate the variables. The DE can't be solved by the method in this section.

Here's a **WRONG** way to try to solve it. Write the DE as

\[ \int (x + 3y^2) \, dy = \int 2x \, dx \quad \text{dangerous to have } x\text{'s on the } dy \text{ side} \]

or \( y\text{'s on the } dx \text{ side} \)

\[ xy + y^3 = x^2 \quad \text{WRONG} \]

It's wrong to go from (7) to (8) because in (7), \( y \) is a function of \( x \), \( dy \) is \( y'(x) \, dx \), and the left side of (7) is an abbreviation for \( \int [x + 3y^2(x)] \, y'(x) \, dx \). It does not equal \( xy + y^3 \) because \( xy + y^3 \) differentiates back to \( xy' + y + 3y^2y' \), not to \( x + 3y^2 \).

**warning**

Don't wait until the end of the problem to insert an arbitrary constant. At line (4A) don't write

\[ \frac{1}{3} \, y^3 = \frac{1}{2} \, x^2, \]

then solve for \( y \) to get

\[ y = \sqrt[3]{\frac{3}{2} \, x^2} \]

and then (too late) insert the arbitrary constant to get

\[ y = \sqrt[3]{\frac{3}{2} \, x^2} + C \quad \text{WRONG WRONG} \]

The constant must be inserted *at the antidifferentiation step*, not later.
antiderivative for 1/x
The usual choice is

\[ (9) \quad \int \frac{1}{x} \, dx = \ln x + C \]

but it is also true that

\[ (10) \quad \int \frac{1}{x} \, dx = \ln Kx \]

because

\[ D \ln Kx = \frac{1}{Kx} \cdot K = \frac{1}{x} \]

Here's another way to see (10):

\[ \ln x + C = \ln x + \ln K \quad \text{(rewrite the arbitrary constant)} \]
\[ = \ln Kx \quad \text{(log algebra)} \]

I think the version in (10) is often more useful than (9).

example 1
Use separation to find the general solution to \( w'(t) = 2 - \frac{1}{5} w(t) \).

solution

\[
\frac{dw}{dt} = 2 - \frac{1}{5} w \\
\frac{dw}{2 - \frac{1}{5} w} = dt
\]

Now antidiff and use either (9) or (10).

version I (better)

\[-5 \ln K (2 - \frac{1}{5} w) = t \quad \text{(antidiff and use (10) to insert the constant)} \]
\[ \ln K (2 - \frac{1}{5} w) = -\frac{1}{5} t \quad \text{(Divide by -5)} \]
\[ K (2 - \frac{1}{5} w) = e^{-t/5} \quad \text{(take exp)} \]
\[ 2 - \frac{1}{5} w = A e^{-t/5} \quad \text{(divide by } K \text{ and let } 1/K \text{ be renamed } A) \]
\[ w = 10 - B e^{-t/5} \quad \text{(solve for } w \text{ and let } 5A \text{ be renamed } B) \]

version 2

\[-5 \ln (2 - \frac{1}{5} w) = t + K \quad \text{(antidiff and use (9) to insert the constant)} \]
\[ \ln (2 - \frac{1}{5} w) = -\frac{1}{5} t + C \quad \text{(let } -\frac{1}{5} K \text{ be renamed } C) \]
\[ 2 - \frac{1}{5} w = e^{-t/5} + C \quad \text{(take exp on both sides)} \]
\[ 2 - \frac{1}{5} w = e^{-t/5} e^C \quad \text{(rule of exponents)} \]
\[ 2 - \frac{1}{5} w = B e^{-t/5} \quad \text{(let } e^C \text{ be renamed } B) \]
\[ w = 10 - D e^{-t/5} \quad \text{(solve for } w \text{ and let } 5B \text{ be renamed } D) \]
warning
When you take exp on both sides of (11) it's wrong to get

\[(2 - \frac{1}{5}w) = e^{-t/5} \text{ PLUS } e^C \text{ WRONG WRONG}\]

which turns into

\[(2 - \frac{1}{5}w) = e^{-t/5} + A\]

You should have

\[2 - \frac{1}{5}w = e^{-t/5} + C \text{ RIGHT}\]

which turns into

\[2 - \frac{1}{5}w = e^{-t/5} \text{ TIMES } e^C = Ae^{-t/5}\]

If you use version 1 you won't run the risk of this mistake.

example 2
Solve \(y'(t) = -\frac{1}{3}y(t)\) with IC \(y(0) = 150\)

solution

\[
\frac{dy}{dt} = -\frac{1}{3}y
\]

\[
\frac{dy}{y} = -\frac{1}{3} dt
\]

\[\ln Ky = -\frac{1}{3} t\]

\[Ky = e^{-t/3}\]

\[y = \frac{1}{K} e^{-t/3}\]

(12) \[y = Ce^{-t/3}\]

To determine the specific solution satisfying the IC, set \(y = 150, t = 0\) in (8) to get

\[150 = ce^0, \quad c = 150.\]

The final answer is \(y = 150 e^{-t/3}\)

example 3
Find a general solution to \(y' = 4xy + 3x\)

solution

\[
\frac{dy}{dx} = x(4y + 3)
\]

\[
\frac{dy}{4y + 3} = x \; dx
\]

\[
\frac{1}{4} \ln K(4y + 3) = \frac{1}{2} x^2
\]

\[\ln K(4y + 3) = 2x^2 \quad \text{(by (10))}\]

\[K(4y + 3) = e^{2x^2}\]

\[4y + 3 = Ae^{2x^2}\]

\[y = \frac{Ae^{2x^2} - 3}{4}\]
warning

It is true that if A and B are arbitrary constants then \( A + B = C \) (i.e., \( A+B \) is just another arbitrary constant). And \( \frac{1}{A} = C \). And \( e^A = C \). But it is not true that \( Ae^{2x^2} = C \). An arbitrary constant can't swallow \( x \)-stuff.

orthogonal families

The equation

\[ x^2 + 3y^2 = k, \]

where \( k \) is an arbitrary constant, describes a family of ellipses. I'll find the family of curves orthogonal to the ellipse family.

step 1 Go backwards from the family of ellipses to the differential equation for the family: Differentiate w.r.t. \( x \) on both sides of the equation and remember to treat \( y \) as \( y(x) \).

\[
2x + 6yy' = 0
\]

(13) \( y' = \frac{-x}{3y} \)

For each point \((x,y)\), the differential equation in (13) gives the slope of the curve in the family that passes through that point. For example, the ellipse in the family that passes through point \((7,2)\) has slope \(-7/6\) at that point.

step 2 Find the differential equation for the orthogonal family.

The slopes on the orthogonal family should be the negative reciprocals of the slopes on the original family. So the orthogonal family satisfies the DE

(14) \( y' = \frac{3y}{x} \).

step 3 Solve the DE in (14).

\[
\frac{dy}{y} = 3 \frac{dx}{x} \\
\ln ky = 3 \ln kx \\
\ln ky = \ln x^3 \\
y = Ax^3
\]

This is the equation of the orthogonal family. Fig 1 shows some of the members in each family.

FIG 1
PROBLEMS FOR SECTION 4.2

1. Find a general solution if the equation is separable.
   (a) \( y' = -x \sec y \)  
   (b) \( dx + x^3 y \, dy = 0 \)  
   (c) \( x^2 + y^4 \frac{dy}{dx} = 0 \)
   (d) \( y' = \frac{y}{2x + 3} \)
   (e) \( x^2 \, dy = e^y \, dx \)
   (f) \( y' = \frac{5x + 3}{y} \)
   (g) \( y' = \frac{y}{x + y} \)
   (h) \( y' = \frac{1}{xy + x} \)

2. Take your solutions to #1(e) and (f) and check that they really satisfy the differential equations.

3. Find the particular solution satisfying the given condition.
   (a) \( y' = xy \) with \( y(1) = 3 \)
   (b) \( yy' + 5x = 3 \) with \( y(2) = 4 \)
   (c) \( y' \frac{e^y}{x} = 3 \) with \( y(0) = 2 \)
   (d) \( y' = y^4 \cos x \) with \( y(0) = 2 \)

4. The DE \( w'(t) = 2 - \frac{1}{5} w(t) \) in example 1 is not only separable but also first order linear. Solve it again.

5. If \( y \) is implicitly given by \( xy^2 = y + 7 \), find \( y \) explicitly.

6. You have some radioactive stuff which is decaying at a rate proportional the amount there, where the constant of proportionality is 10. In particular, if \( y(x) \) is the amount of stuff at time \( x \) then \( y' = -10y \).
   (a) If you start with \( G \) grams, at what time will you have only \( G/2 \) grams left.
   (b) If you would like your initial \( G \) grams to decay to \( G/2 \) grams by time \( 3 \), you should start with new radioactive stuff with what constant of proportionality instead of 10.

7. Find the orthogonal family and draw a picture.
   (a) \( xy = K \)
   (b) \( y = Ax^2 \)
      Suggestion: Before you differentiate w.r.t. \( x \) on both sides of the equation of the family, isolate the arbitrary constant so that it will differentiate away.
SECTION 4.3 EXACT DIFFERENTIAL EQUATIONS

the differential of a function

(1-dim version) Suppose $y = f(x)$ and $x$ changes by $dx$ producing a corresponding change in $y$. The differential of $f$ is defined by

$$dy = f'(x)dx.$$ 

It was shown in calculus that the differential approximates the change in $y$.

(2-dim version) Suppose $z = f(x,y)$ and $x$ changes by $dx$, $y$ changes by $dy$ producing a corresponding change in $z$. The differential of $f$ is defined by

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$ 

It was shown in calculus that the differential approximates the corresponding change in $z$.

Mathematicians use the notation $\Delta z$ for the change in $z$ and use $dz$ for the differential in (1) which approximates the change in $z$. Outside of pure mathematics the distinction between (1) and $\Delta z$ is blurred and often both are referred to as $dz$.

example 1

Let $z = x^2y^3$. Then

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 2xy^3 dx + 3x^2y^2 dy$$

meaning that if $x$ and $y$ change by $dx$ and $dy$ respectively there is a corresponding change in $z$ given approximately by $2xy^3 dx + 3x^2y^2 dy$.

example 2

To find $d(3q^2)$ use the one-dimensional differential formula $dy = f'(x) dx$ to get $d(3q^2) = 6q dq$.

sum, product, quotient and chain rules for differentials

Let $u$ and $v$ be functions of one or more variables. Then

(2) $d(u + v) = d(u) + d(v)$

(3) $d(uv) = u d(v) + v d(u)$

(4) $d\left(\frac{u}{v}\right) = \frac{v d(u) - u d(v)}{v^2}$

(5) $d[f(u)] = f'(u) d(u)$

For example, by (5),

$$d(\ln u) = \frac{1}{u} d(u),$$
$$d(\sin u) = \cos u \, d(u).$$

A differential can always be found directly using (1) but sometimes (2)–(5) are more convenient. For example, by (1),

$$d \ln(2x + 3y) = \frac{\partial \ln(2x+3y)}{\partial x} dx + \frac{\partial \ln(2x+3y)}{\partial y} dy$$
$$= \frac{2}{2x+3y} dx + \frac{3}{2x+3y} dy.$$
But also
\[
\begin{align*}
\frac{d}{d(2x + 3y)} \ln(2x + 3y) &= \frac{1}{2x + 3y} d(2x + 3y) \quad \text{(by (5))} \\
&= \frac{1}{2x + 3y} (2 \, dx + 3 \, dy) \quad \text{(by (2))} \\
&= \frac{2 \, dx + 3 \, dy}{2x + 3y}
\end{align*}
\]

For example, By (1),
\[
\begin{align*}
d \sin(x^2 y^3) &= \frac{\partial \sin(x^2 y^3)}{\partial x} \, dx + \frac{\partial \sin(x^2 y^3)}{\partial y} \, dy \\
&= 2xy^3 \cos(x^2 y^3) \, dx + 3x^2 y^2 \cos(x^2 y^3) \, dy
\end{align*}
\]

And also
\[
\begin{align*}
d \sin(x^2 y^3) &= \cos(x^2 y^3) \, d(x^2 y^3) \quad \text{by (5)} \\
&= \cos(x^2 y^3) (x^2 \, 3y^2) \, dy + 2xy^3 \, dx) \quad \text{by (3)}.
\end{align*}
\]

**exact differentials**

Example 1 began with a function \( f(x,y) = x^2 y^3 \) and found \( df = 2xy^3 \, dx + 3x^2 y^2 \, dy \). To identify and solve exact differential equations you have to consider the opposite problem: given the differential expression \( 2xy^3 \, dx + 3x^2 y^2 \, dy \), find a function \( f(x,y) \) with that differential. In general, an expression of the form
\[
(6) \quad p(x,y) \, dx + q(x,y) \, dy
\]
is called a differential form. It is possible (in fact, likely) that (6) simply is not \( df \) for any \( f \).

If there does exist a function \( f(x,y) \) such that
\[
(7) \quad df = p(x,y) \, dx + q(x,y) \, dy
\]
then the differential \( p \, dx + q \, dy \) is called exact.

In other words \( p \, dx + q \, dy \) is exact if there is an \( f(x,y) \) such that
\[
(8) \quad \frac{\partial f}{\partial x} = p(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = q(x,y)
\]

For example, consider the differential
\[
(9) \quad \begin{align*}
p(3x^2 y^2 + 2y^3 + x) \, dx + q(2x^3 y + 6xy^2 + \cos y + 7) \, dy
\end{align*}
\]

The problem is to find \( f(x,y) \), if possible, so that (7) and (8) hold. Begin by antidifferentiating \( p \) with respect to \( x \):
\[
(10) \quad \text{tentative } f = x^3 y^2 + 2xy^3 + \frac{1}{2} x^2
\]
The derivative w.r.t. \( y \) of this tentative answer is

\[(11) \quad 2x^3y + 6xy^2.\]

Compare this with \( q \) in (9). Since (11) is missing the terms \( \cos y + 7 \), fix up (10) by adding \( \sin y + 7y \) to get

\[
\text{better } f = x^3y^2 + 2xy^3 + \frac{1}{2}x^2 + \sin y + 7y.
\]

Now it has the correct partial w.r.t. \( y \). Note that fixing up the answer like this does not change its partial derivative w.r.t \( x \) since the additional terms do not contain the variable \( x \). So the final answer, including the standard arbitrary constant, is

\[
f(x,y) = x^3y^2 + 2xy^3 + \frac{1}{2}x^2 + \sin y + 7y + C
\]

You can check the answer by finding its partials to see that you do get \( p \) and \( q \).

**Example 3**

Let

\[(12) \quad p = 3x^2y^2 + 2y^3 \quad \text{and} \quad q = 2x^3y + 6xy^2 + 8xy^3.\]

Try, but find it impossible, to obtain an \( f \) such that \( df = p \, dx + q \, dy \). In other words, show that \( p \, dx + q \, dy \) is not exact.

**Solution** Antidifferentiate \( p \) to get

\[(13) \quad x^3y^2 + 2xy^3\]

Differentiate this tentative answer w.r.t. \( y \) to get

\[(14) \quad 2x^3y + 6xy^2\]

and compare it with \( q \). The term \( 8xy^3 \) is missing from (14) and can be produced only if you expand (13) to \( x^3y^2 + 2xy^3 + 2xy^4 \). But the extra term \( 2xy^4 \) contains the variable \( x \) so when you differentiate the expanded tentative answer w.r.t. \( x \) you no longer get the desired \( p \). So it is not possible to find a function \( f \) with partials \( p \) and \( q \); the differential \( p \, dx + \, dy \) is not exact.

**A criteria for exactness**

Given \( p \, dx + q \, dy \), one way to decide if there exists an \( f \) such that \( df = p \, dx + q \, dy \) holds is to simply try to find it as in the preceding examples. It is also possible to get a test for determining *in advance* if an \( f \) exists. Then the antidifferentiating process for finding \( f \) need be used only when the criterion guarantees the existence of \( f \). I'll find the criterion and then use it in examples.

If (7) holds then

\[
\frac{\partial f}{\partial x} = p \quad \text{and} \quad \frac{\partial f}{\partial y} = q
\]

so

\[
\frac{\partial q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial p}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}
\]

and so

\[
\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}
\]

In more advanced courses, the converse can be proved: if \( \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} \) then (7) holds.
So here's the criterion:

\[
\begin{align*}
(15) & \text{ If } \frac{\partial q}{\partial x} \neq \frac{\partial p}{\partial y} \text{ then } p \, dx + q \, dy \text{ is not exact} \\
(16) & \text{ If } \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} \text{ then } p \, dx + q \, dy \text{ is exact}
\end{align*}
\]

example 3 repeated

To see if

\[
(3x^2y^2 + 2y^3)\,dx + (2x^3y + 6xy^2 + 8xy^3)\,dy
\]

is exact, find

\[
\frac{\partial q}{\partial x} = 6x^2y + 6y^2 + 8y^3, \quad \frac{\partial p}{\partial y} = 6x^2y + 6y^2.
\]

The two are not identical so, by (15), the differential is not exact.

exact differential equations

Consider the equation

\[
y' = \frac{2x - y^3}{3xy^2}
\]

Then

\[
\frac{dy}{dx} = \frac{2x - y^3}{3xy^2},
\]

\[
3xy^2 \, dy = (2x - y^3) \, dx.
\]

The equation is not separable so try a second approach. Write the equation as

\[
(17) \quad (y^3 - 2x) \, dx + 3xy^2 \, dy = 0
\]

Since

\[
\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} \quad \text{(both are } 3y^2)\]

the left side of (17) is an exact differential df. To find f, antidifferentiate p w.r.t. x to get the terms

\[
xy^3 - x^2.
\]

The derivative w.r.t. y of this tentative f is 3xy^2, precisely q, so the tentative f is final: the differential equation can be written as

\[
d(xy^3 - x^2) = 0
\]

Since the differential is 0, if x changes by dx and y changes by dy, the function \(xy^3 - x^2\) itself does not change. Therefore it is a constant function.

In general, \(f(x,y)\) is constant if and only if \(df = 0\), analogous to the 1-dim rule that \(f'(x)\) is constant if and only if \(f'(x) = 0\). So

\[
xy^3 - x^2 = k
\]
where $K$ is an arbitrary constant, and this describes an *implicit* solution to the original diff equ. The *explicit* solution is found by solving for $y$ to get

$$y = \sqrt[3]{x^2 + K}$$

Here's the overall idea.

(18) Consider the differential equation

$$p \, dx + q \, dy = 0 \quad \text{where} \quad \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$  

The left side of the DE is an exact differential, the equation is called exact, and there is a function $f(x,y)$ such that the equation can be written as $df = 0$. The solution $y(x)$ to the differential equation is given implicitly by

$$f(x,y) = K.$$  

An explicit solution is found by solving the implicit solution for $y$, if possible.

(19) More generally, if a differential equation can be written as

$$df = dg$$

(rather than as $df = 0$) then its solution is given implicitly by

$$f(x,y) = g(x,y) + K$$

example 4

Find the particular solution to

$$y' = \frac{x^2 - y}{x}$$

satisfying the condition $y(3) = 1$.

solution

The equation is

$$\frac{dy}{dx} = \frac{x^2 - y}{x}$$

$$\left(\frac{x^2 - y}{p}\right) \, dx - x \, \frac{dy}{q} = 0$$

Since $\frac{\partial q}{\partial x}$ and $\frac{\partial p}{\partial y}$ both equal $-1$, the equation is exact. In particular, it can be written as

$$d\left(\frac{1}{3}x^3 - xy\right) = 0$$

so the solution is given implicitly by

$$\frac{1}{3}x^3 - xy = K$$

and explicitly by
To find K, substitute x = 3, y = 1 in either (20) or (21). Using (20) which is more convenient, you have \(9 - 3 = K\), \(K = 6\), so the solution is

\[
y = \frac{1}{3} x^2 - \frac{6}{x}.
\]

**warning**

After you find \(\frac{1}{3} x^3 - xy\) in example 4, you must know what to do with it.

Here are some non-solutions:

\[
\frac{1}{3} x^3 - xy
\]

not the solution

\[
\frac{1}{3} x^3 - xy = 0
\]

not the solution

\[
f(x,y) = \frac{1}{3} x^3 - xy
\]

not the solution

\[
y = \frac{1}{3} x^3 - xy
\]

not the solution

The solution is the function \(y(x)\) defined implicitly by the equation

\[
\frac{1}{3} x^3 - xy = K
\]

**warning**

In example 4 here are some wrong ways to identify \(p\) and \(q\):

\[
\frac{(x^2 - y)}{p} dx = \frac{x}{q} dy
\]

wrong

\[
\frac{(x^2 - y)}{p} dx - \frac{x}{q} dy
\]

wrong

The correct identification is

\[
\frac{(x^2 - y)}{p} dx - \frac{x}{q} dy
\]

right (q gets a minus sign)

**warning**

Here's another wrong way to do example 4:

\[
x^2 \ dy = (x^2 - y) \ dx \quad \text{(so far, so good)}
\]

\[
\int x^2 \ dy = \int (x^2 - y) \ dx
\]

Hmmmmm!

\[
x^2y = \frac{1}{3} x^3 - xy + K \quad \text{WRONG \ WRONG}
\]

This was an attempt to separate variables (Section 4.1) but they aren't separated (there is an \(x\) on the dy side and a \(y\) on the dx side).
a brief table of exact differentials  (this is on the reference page)

(22) \[ \frac{y \, dx - x \, dy}{y^2} = d\left(\frac{x}{y}\right) \]

(23) \[ \frac{x \, dy - y \, dx}{x^2} = d\left(\frac{y}{x}\right) \]

(24) \[ \frac{-2x \, dx - 2y \, dy}{(x^2 + y^2)^2} = d\left(\frac{1}{x^2 + y^2}\right) \]

(25) \[ \frac{x \, dx + y \, dy}{\pm \sqrt{x^2 + y^2}} = d\left(\pm \sqrt{x^2 + y^2}\right) \]

(26) \[ \frac{2x \, dx + 2y \, dy}{x^2 + y^2} = d \ln(x^2 + y^2) \]

(27) \[ \frac{-y \, dx + x \, dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right) \]

integrating factors

Look at the equation

\[ y \, dx - x \, dy = y^3 \, dy. \]

The right side is an exact differential, namely \(d\left(\frac{1}{4}y^4\right)\), but the left side is not exact since \(p(x,y) = y, \ q(x,y) = -x, \ \frac{\partial q}{\partial x} \neq \frac{\partial p}{\partial y}\). But compare the left side with (22) to see that it can be made exact if you multiply by \(1/y^2\). So multiply on both sides to get

\[ \frac{y \, dx - x \, dy}{y^2} = y \, dy \]

The left side is now the exact differential in (22) and fortunately the right side is still exact. The equation can be written as

\[ d\left(\frac{x}{y}\right) = d\left(\frac{1}{2}y^2\right) \]

By (19), the implicit solution is

\[ \frac{x}{y} = \frac{1}{2}y^2 + K \]

It is not convenient to solve for \(y\) and get the explicit solution so I'll settle for the implicit version.

A factor, \(1/y^2\) in this case, which changes a differential equation from non-exact to exact is called an integrating factor. A table of exact differentials like (22)-(27) can serve as goals.
PROBLEMS FOR SECTION 4.3

1. Check formulas (22)-(27) by finding the differential indicated on the righthand side to see if you get the lefthand side.

2. Suppose a point has polar coords r,θ and rectangular coords x,y. If r changes by dr and θ changes by dθ, find dx and dy.

3. Decide if the expression is an exact differential df and if so, find f.
   (a) 2xy dx + y dy
   (b) (x^3 + 3x^2y) dx + (x^3 + y^3) dy
   (c) \( \frac{y}{x^2} \) dx + (5 - \( \frac{1}{x} \)) dy

4. Find q so that xy^3 dx + q dy is exact

5. Solve the DE if it is exact. Find the explicit solution whenever possible.
   (a) (6x^2 + y^2) dx + (2xy + 3y^2) dy = 0
   (b) (3x^2 + y) dx + x dy = 0
   (c) y' = \( \frac{x - y \cos x}{y + \sin x} \)
   (d) y' = e^{xy}
   (e) (2r \cos θ - 1) dr = r^2 \sin θ dθ
   (f) (x+y) dx + (x^2 + y^2) dy = 0
   (g) \cos x \cos y dx - \sin x \sin y dy = x^3 dx
   (h) (ye^{-x} - \sin x) dx = (e^{-x} + 2y) dy

6. Check that your answer to #5(h) really does satisfy the DE.

7. Solve.
   (a) 2xy dx + (x^2 + y) dy = 0 with y(1) = 4
   (b) 2 \sin(2x + 3y) dx + 3 \sin(2x + 3y) dy = 0 with y(0) = π/2
   (c) \( \frac{1}{x+y} \) dx + \( \frac{1}{x+y} \) dy = dx with y=1 when x = 0

8. The equation (x^2 + 2) dx + 3y dy = 0 is both exact and separable. Solve it twice.

9. Find an integrating factor and then solve.
   (a) (x^2 + y^2) dx = x dy - y dx
   (b) y dx - x dy = y^2 dx
   (c) \( \sqrt{x^2 + y^2} \) dy = x dx + y dy
   (d) y' = \( \frac{x}{x^2 + y^2 - y} \)
   (e) x dy - y dx = 2x^3 dx + 2x^2y dy
SECTION 4.4 DIRECTION FIELDS

the direction field of a first order DE

Look at the differential equation

\[ y' = y - x \]  

The direction field of the DE consists of a lot of little line segments such that the segment at point \((x,y)\) has slope \(y-x\). For example, at point \((4,1)\) draw a small line segment with slope 3.

Here's a Mathematica program which sketches the direction field.

```mathematica
directionField[f_, {x_, x1_, x2_}, {y_, y1_, y2_}, d_, s_] := 
  Table[Graphics[Line[{{x, y}, {x + s1*f/Sqrt[1 + f^2], y + s1}}]],
  {x, x1, x2}, {y, y1, y2}, {d, d}]

MyField = directionField[y - x, {x, -3, 3}, {y, -3, 3}, .4, .4];
Show[MyField, Axes -> True, Ticks -> None];
```

Here's the connection between the direction field and the solutions to the DE. The general solution to the DE is a family of curves in the plane such that the slope at the point \((x,y)\) on any curve in the family is \(y-x\).

The DE is first order linear so here's one way to solve it:

\[ y' - y = -x \]

\[ P(x) = -1, \quad Q(x) = -x \]

\[ I = e^\int P(x)\,dx = e^x \]

\[ e^x y = \int -xe^{-x}\,dx = xe^{-x} + e^{-x} + K \quad \text{(tables)} \]

The general solution is

\[ y = x + 1 + Ke^x \]

Here's the graph of some of the solutions (namely the ones where \(K = -2, -1.5, -1, -0.5, 0, .5, 1, 1.5, 2\)) superimposed on top of the direction field.
If you solve (1) with the IC $y(2) = 1$ you will get the particular solution that passes through the point $(2,1)$. Substitute the condition into (2) to get

$$1 = 2 + 1 + ke^2$$

$$k = -2e^2$$

So the solution satisfying the IC is

$$y = x + 1 - 2e^{-2} e^x$$

Here's a picture of the solution along with the direction field.
In general, you can get a rough sketch of the solution corresponding to IC \( y(x_0) = y_0 \) by sketching the "path" in the direction field that goes through point \((x_0, y_0)\).

**PROBLEMS FOR SECTION 4.4**

1. Mathematica can draw nice direction fields but you can sketch them by hand too. Try doing it for these equations and then solve the equation and see if the solutions go with the direction field.
   - (a) \( y' = x/y \)
   - (b) \( y' = y/x \)

2. Here's the direction field of a first order DE. Sketch the solution satisfying the IC \( y(0) = 1 \).

3. Look at the DE \( y' = xy \)
   The diagram shows its direction field.

   (a) What are all those little segments supposed to be?
   (b) Solve the differential equation.
   (c) Plot some of the solutions.
   (d) Find the curve in the family of solutions through point \((4,3)\).
REVIEW PROBLEMS FOR CHAPTER 4

1. Solve in as many ways as possible, for practice (using this chapter and/or earlier chapters)

(a) \((x^2 + 2) \, dx + 3y \, dy = 0\)

(b) \(y' = -y\)

(c) \(y' = \frac{2x - y}{x}\) with \(y(1) = 2\)

(d) \(y'' = y\)

(e) \(y'' = 3y' + 12\)

(f) \(y' = e^{x+y}\)

(g) \(xy'' - y' = 1\) with conditions \(y(1) = 2, y'(1) = 3\)

2. Back in the review problems for Chapters 1, 2 there was a problem about the velocity \(v(t)\) of a falling object with mass \(m\),

\[mv' = mg - cv\]

where \(g, c, m\) are constants, It was solved by treating the equation as linear first order with constant coefficients. Try it again with the methods of this chapter.

3. Sketch the direction field of the DE \(y' = x + y\) and then solve the equation and see if the solutions go with your direction field.

For reference: \(\int xe^{ax} \, dx = \frac{1}{a} xe^{ax} - \frac{1}{a^2} e^{ax}\).

4. Find the family orthogonal to \(y = Ax^3\) and draw a picture.

5. Show that any separable DE can be rearranged to be exact (you have to be general here) but not vice versa (you must find a specific counterexample here).