CHAPTER 2 THE IMPULSE RESPONSE

SECTION 2.1 THE UNIT IMPULSE AND THE IMPULSE RESPONSE

This chapter is about systems in which inputs f(t) and outputs y(t) are related by a DE of the form

$$ay'' + by' + cy = f(t)$$

where a,b,c, are constants.
So the systems satisfy superposition and are time-invariant meaning that the ingredients such as mass, resistance etc. do not change with time.

the delta function

Fig 1 shows the function $\delta(t)$, called the *unit impulse* at time 0. It is thought of as an "infinite" force applied for a "split second" at time t = 0, producing an impulse (area under the curve) of 1 (Fig 1).

The function $\delta(t-t_0)$ is a unit impulse occurring at time t_0 (Fig 2).
The function $7\delta(t)$ is an impulse of "size" (i.e., area) 7 at time t = 0 (Fig 3).

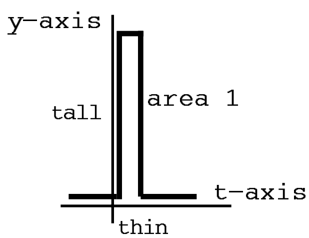


FIG 1 $\delta(t)$

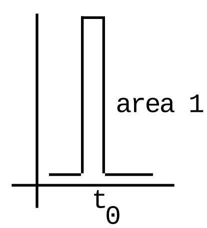


FIG 2 $\delta(t-t_0)$

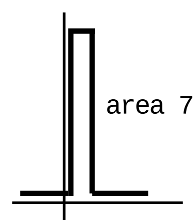


FIG 3 $7\delta(t)$

A delta function is sometimes drawn as a vertical arrow f with height equal to the "area enclosed". Fig 4 shows the arrow representation of $7\delta(t - t_0)$. Fig 5 shows the

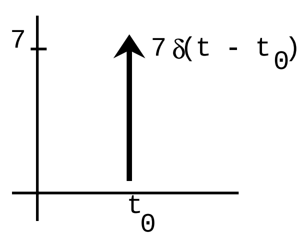


FIG 4

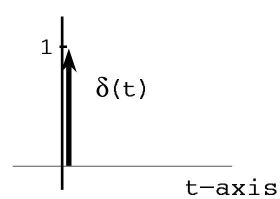


FIG 5

the impulse response h(t)

The *impulse response* of a system is its response to the input $\delta(t)$ when the system is initially at rest. The impulse response is usually denoted h(t). Sometimes it's called Green's function.
In other words, if the input to an initially-at-rest system is $\delta(t)$ then the output is named h(t).

finding the impulse response

Suppose inputs f(t) and outputs y(t) are related by the DE

$$ay'' + by' + cy = f(t)$$

By definition, the system's impulse response h(t) is the solution to

$$ay'' + by' + cy = \delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0.$$

To find the impulse response, solve the new problem

$$ay'' + by' + cy = \boxed{0} \text{ with IC } y(0) = 0,\ y'(0) = \boxed{1/a}$$

Here's a physical interpretation of the rule. You have given the block at the end of a resting spring a quick hard kick at time 0. Now that the kick is over, it is still time 0 and no more kick is coming in, the block is not yet displaced but the quick kick gave it initial velocity 1/a.

quickie pseudo proof

Consider the differential equation

(1) $ay'' + by' + cy = \delta(t)$ with IC $y(0) = 0,\ y'(0) = 0$

Think of y(t) as the displacement of a block on a spring at time t. You can do this if a,b,c aren't negative. The boxed rule still holds if one of the coeffs is negative but this proof would be no good in that case.

Physicists say that

(2) impulse = change in momentum

where

(3) momentum = mass × velocity

The mass of the block is the coefficient a (this is stated, but without explanation unfortunately, on page 2 of Section 1.2).

The velocity of the block when the impulse hits is 0 because the IC in (1) is $y'(0) = 0$.

The impulse imparted to the block is 1 because the forcing function in (1) is the unit impulse function $\delta(t)$.

So (2) becomes

1 = change in a × velocity

But a is a fixed constant so

1 = a × change in velocity

and

change in velocity = 1/a

Look at what happens right after the spike part of the input $\delta(t)$ has acted on the block. The block's velocity goes up by 1/a, the block hasn't changed position yet, it's still time 0 (practically) and for the rest of time the input in (1) is the 0 part of the delta function. So to get the response,i.e., the solution to (1), solve

$$ay'' + by' + cy = 0 \text{ with IC } y(0) = 0,\ y'(0) = 0 + 1/a = 1/a$$

example 1

Given a system where the input f(t) and response y(t) are related by

$$2y'' + 8y' + 6y = f(t)$$

Find the system's impulse response.

solution The problem says to solve

$$2y'' + 8y' + 6y = \delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0$$

To do it, switch to the problem

$$2y'' + 8y' + 6y = 0 \text{ with IC } y(0) = 0,\ y'(0) = 1/2.$$

Solve $2m^2 + 8m + 6 = 0$, $m = -1, -3$. So $y = Ae^{-t} + Be^{-3t}$.

To satisfy IC y(0) = 0 you need

$$A + B = 0$$

We have $y' = -Ae^{-t} - 3Be^{-3t}$ so to satisfy IC y'(0) = 1/2 you need

$$-A - 3B = 1/2$$

So A = 1/4, B = − 1/4 and

(1) $$h(t) = \frac{1}{4} e^{-t} - \frac{1}{4} e^{-3t} \text{ for } t \geq 0$$

Note that h(t) is always 0 for $t \leq 0$ since there is no response until the impulse hits at time t = 0

In other words, for this system, if the input is the delta function in Fig 5 then the response is the function h(t) in Fig 6.

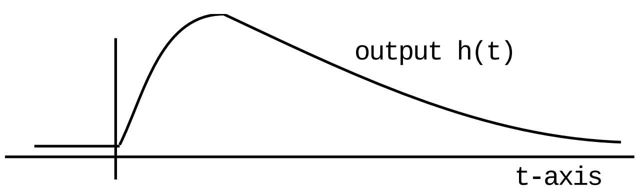


FIG 6

warning If you leave out "for t ≥ 0" in (1) then you are suggesting that the impulse response is $\frac{1}{4} e^{-t} - \frac{1}{4} e^{-3t}$ for *all* t (Fig 7) which is *wrong*.

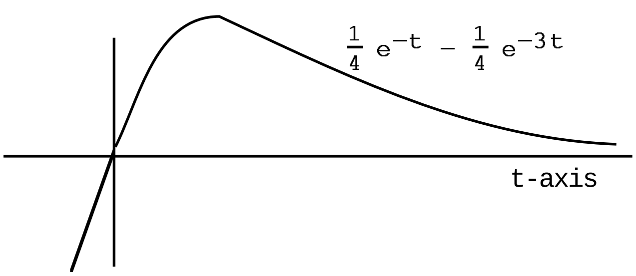


FIG 6A

response to a non-unit impulse

If the response to δ(t) is h(t) then, by superposition, the response to the impulse 7δ(t) is 7h(t).

example 1 continued

The solution to

$$2y'' + 8y' + 6y = 5\delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0$$

is

$$y = 5h(t) = 5\left(\frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}\right) \quad \text{for } t \geq 0$$

response to a delayed impulse

Suppose a system is initially at rest.
And, as usual, suppose its response at time t to input $\delta(t)$ is $h(t)$. Let $t_0 \geq 0$.
Then the system's response at time t to $\delta(t-t_0)$, a unit impulse at time $t = t_0$, is $h(t-t_0)$.

In other words, if the impulse is delayed then the response is delayed.

footnote

This is not as obvious as it seems. It hold only because the system is initially at rest and is time invariant so *nothing happens between time t=0 and t=t_o* (no block moves, no current flows, no ice melts, no birds sing); the system remains suspended in time and therefore responds to the delayed impulse in the same way in which it would have responded to the original impulse.

example 1 continued

The solution to

$$2y'' + 8y' + 6y = \delta(t-4) \quad \text{with zero IC}$$

is

$$y = h(t-4) = \begin{cases} 0 & \text{for } t \leq 4 \\ \frac{1}{4}e^{-(t-4)} - \frac{1}{4}e^{-3(t-4)} & \text{for } t \geq 4 \end{cases}$$

warning

The solution is *not* simply $y = \frac{1}{4}e^{-(t-4)} - \frac{1}{4}e^{-3(t-4)}$.
The sol has this formula *only* for $t \geq 4$.
The solution is 0 until time t=4 since the impulse hasn't hit yet.

Fig 7 shows the input $\delta(t-4)$ and Fig 8 shows the response $h(t-4)$.

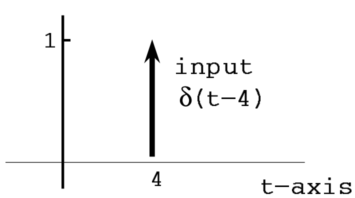


FIG 7

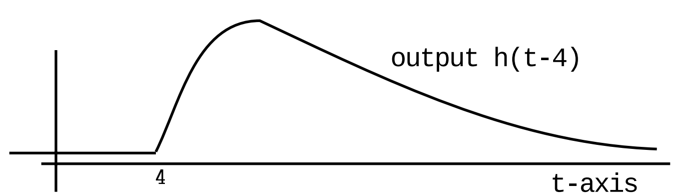


FIG 8

PROBLEMS FOR SECTION 2.1

1. Find the impulse response of a system whose input f(t) and output y(t) are related by

(a) $2y'' + 2y = f(t)$ (b) $2y'' - y' - y = f(t)$

2. (a) Solve $y'' + 4y = \delta(t)$ with IC $y(0) = 0$, $y'(0) = 0$.

(b) Solve $y'' + 4y = 6\delta(t-2)$ with IC $y(0) = 0$, $y'(0) = 0$.

3. Let $h(t) = 1/t^2$, $t \geq 0$, be the impulse response of a system. If the system is initially at rest, find the response of the system at time 3 to

(a) a unit impulse at time 0
(b) an impulse of size 6 at time 0
(c) a unit impulse at time 2

4. Go back to example 1. The impulse response is in (1) and its graph is in Fig 6. Where does the peak occur; i.e., when does the response stop growing and start dying out.

HONORS

5. Suppose that for a certain physical system, inputs $f(t)$ and outputs $y(t)$ are related by

$$ay'' + by' + cy = f(t).$$

Your roommate found the impulse response of the system, $h(t)$, i.e., the solution to

$$ay'' + by' + cy = \delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0$$

You were asked to find the response of the system at time t to the input $\delta(t)$ when the system is *not* initially at rest; namely, with $y(0) = 4$, $y'(0) = 5$.

You are going to take advantage of your roommate's answer, plus superposition, to come up with your answer. In particular, fill in the following blank:

The response at time t to the input $\delta(t)$ when the IC are $y(0) = 4$, $y'(0) = 5$

is $h(t)$ [that I stole from my roommate] + ______________________

Don't actually try to compute what goes in the blank (because you don't have a, b, c). Just explain briefly what you would do to get the blank and why.

SECTION 2.2 GETTING READY TO CONVOLVE

integrating a "multi-piece" function

If $g(x) = \begin{cases} x^3 & \text{for } 1 \le x \le 2 \\ 4 & \text{for } 2 \le x \le 5 \\ 0 & \text{otherwise} \quad \text{(Fig 1)} \end{cases}$

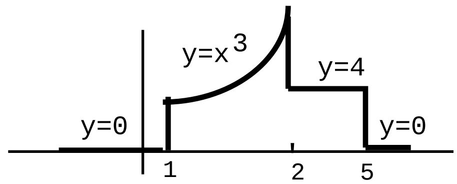


FIG 1

then

$$\int_{-\infty}^{\infty} g(x)\,dx = \underbrace{\int_{-\infty}^{1} 0\,dx}_{0} + \int_1^2 x^3\,dx + \int_2^5 4\,dx + \underbrace{\int_5^{\infty} 0\,dx}_{0} = \int_1^2 x^3\,dx + \int_2^5 4\,dx$$

In general, to integrate a multi-piece function, integrate the pieces and add, ignoring the intervals where the function is 0

vertical lines in a graph

The graph of g(x) in Fig 1 is ambiguous because of the vertical segments at x=1,2,5. In other words, you can't find g(1), g(2), g(5) from the diagram. For our purposes, it doesn't matter.

integrating a product of multi-piece functions

If $f(x) = \begin{cases} e^x & \text{for } 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$

and

$g(x) = \begin{cases} x^3 & \text{if } 1 \le x \le 4 \\ 7 & \text{if } 4 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$

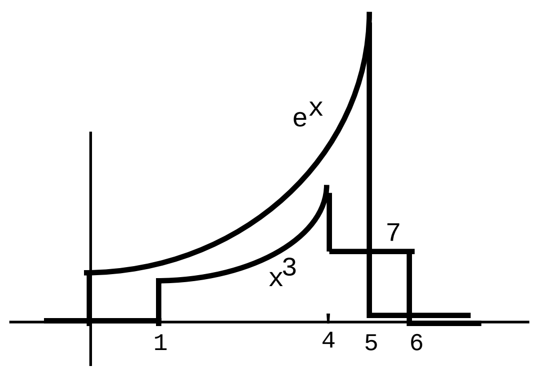


FIG 2

then (look at at the picture)

$f(x)g(x) = \begin{cases} x^3 e^x & \text{for } 1 \le x \le 4 \\ 7e^x & \text{for } 4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$

and

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_1^4 x^3 e^x\,dx + \int_4^5 7e^x\,dx$$

the graph of $y = g(t-x)$ in an x,y coord system

Consider say $y = x^3$. The graph of $y = (2 - x)^3$ can be found by first translating left 2 and then reflecting in the y-axis. Fig 3 shows the several steps.

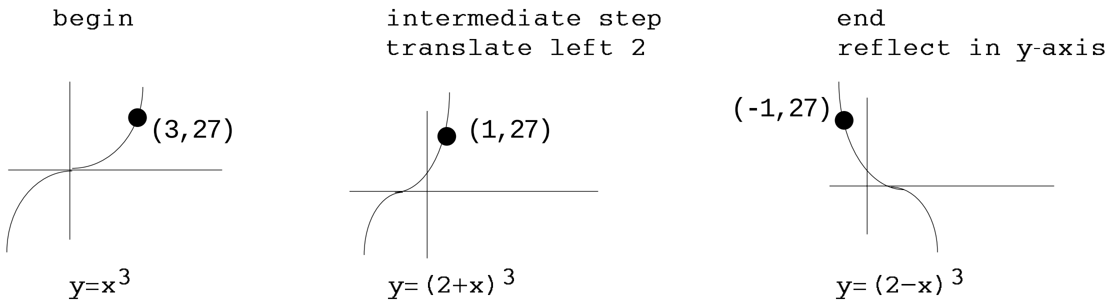


FIG 3 Going from $y = x^3$ to $y = (2-x)^3$

In general, to sketch the graph of $g(t-x)$ in an x,y coord system, translate the graph of $y = g(x)$ *left* by t and then reflect in the y-axis.

Alternatively, you can reflect in the y-axis *first* and then translate *right* by t.

example 1 Let

$$g(x) = \begin{cases} xe^x & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise (Fig 4)} \end{cases}$$

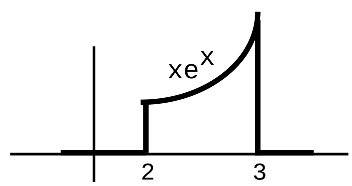


FIG 4 graph of $g(x)$

Fig 5 shows the graph of $g(7-x)$.

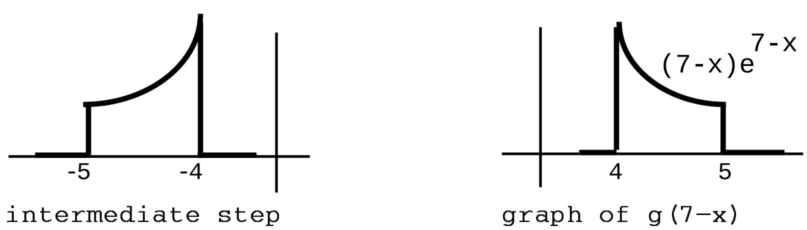


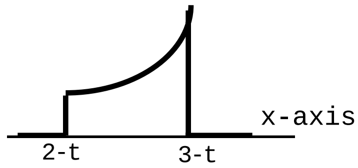
FIG 5

The graph shows that the nonzero piece lies between $x = 4$ and $x = 5$. The formula for the nonzero piece in the new graph is found by replacing x by $7-x$ so all in all

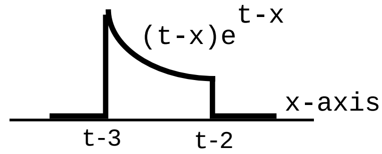
$$g(7-x) = \begin{cases} (7-x)e^{7-x} & \text{for } 4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

More generally, Fig 6 shows the graph of $g(t-x)$ in an x,y coordinate system (the y-axis is not drawn because where it is depends on the size of t):

$$g(t-x) = \begin{cases} (t-x)e^{t-x} & \text{for } t-3 \le x \le t-2 \\ 0 & \text{otherwise} \end{cases}$$



intermediate step



graph of g(t−x)

FIG 6

drawing the graph of f(x) + g(x)

To get the graph of say x + sin x, you can draw the graph of x and the graph of sin x (Fig 7) and add (painstakingly) their heights (Fig 7).

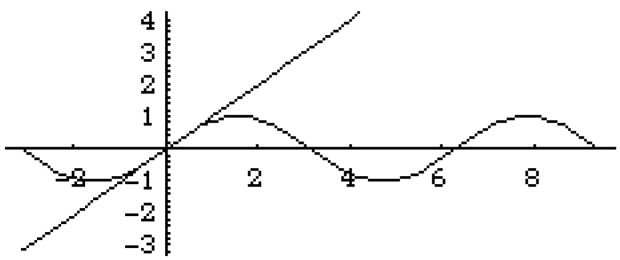


FIG 7

FIG 8

example 2

Figs 9, 10, 11 show the graphs of f(x), g(x) and k(x).

The graph of f(x) + g(x) + k(x) is in Fig 12.

The curve in Fig 12 was obtained by adding *heights* from Figs 9, 10, 11. Don't refer to it as "adding areas".

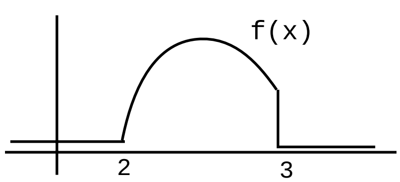


FIG 9

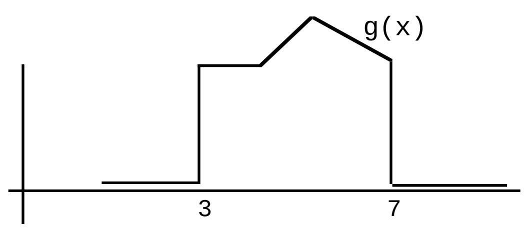


FIG 10

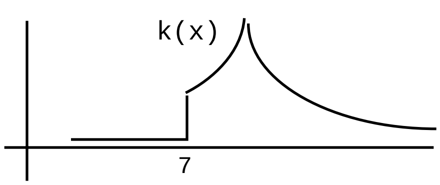


FIG 11

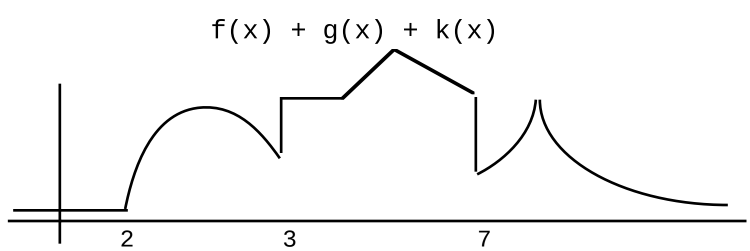


FIG 12

SECTION 2.3 CONVOLUTION

This chapter is about systems in which inputs f(t) and outputs y(t) are related by a DE of the form

$$ay'' + by' + cy = f(t) \text{ where } a,b,c, \text{ are constants.}$$

In other words, the systems satisfy superposition and are time-invariant (the ingredients such as mass, resistance etc. do not change with time).

the convolution of two functions

Given functions f(t) and g(t), the two integrals

$$(1) \quad \int_{u=-\infty}^{\infty} f(u)\, g(t-u)\, du \quad \text{and} \quad \int_{u=-\infty}^{\infty} f(t-u)\, g(u)\, du$$

can be shown to be equal (that's a theorem) (proved in problem 9).

Each is referred to as the *convolution* of f(t) and g (t) (that's a definition), and each is denoted by f(t)*g(t) or equivalently by g(t)*f(t).

In each integral, u is the dummy variable of integration and t is "carried along" so that the convolution of f(t) and g(t) is another function of t.

The two formulas in (1) are given on the reference page you will get with exams.

finding the response of an initially-at-rest system to input f(t) given its impulse response h(t)

If a system is initially at rest, its impulse response h(t) determines its response to all other inputs as follows.

Let h(t) be a system's impulse response.

If the system is initially at rest then its response y(t) to input f(t) is given by

$$(2) \quad y(t) = h(t)*f(t) = \int_{u=-\infty}^{\infty} f(u)\, h(t-u)\, du = \int_{u=-\infty}^{\infty} f(t-u)\, h(u)\, du$$

In other words, the *output of an initially-at-rest system corresponding to a particular input can be found by convolving the input with the system's impulse response*.

In other other words, if h(t) is the impulse response, namely the output of the initially-at-rest system when the input is $\delta(t)$, then the convolution h(t)*f(t) is the output of the initially-at-rest system when the input is f(t).

proof of (2)

Think of the input f(t) (Fig 1) as a sum of impulses (i.e., the sum of the impulse heights in Fig 1 is the f(t) height).

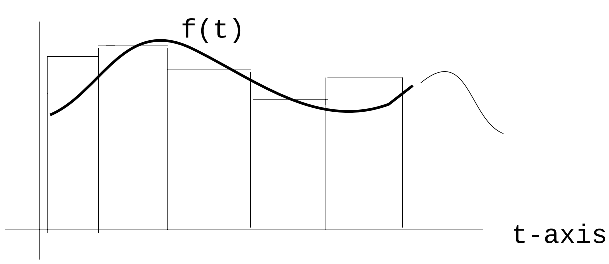


FIG 1

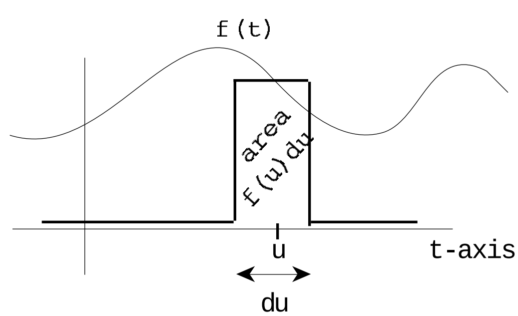


FIG 2

Remember that an impulse of say "size" 5 with spike at time 7 is named $5\delta(t-7)$. Now look at a typical impulse (Fig 2) in this sum occurring at time u with width du. Its height is f(u) so its area ("size") is f(u) du and its name is

(3) $$f(u)\ du\ \delta(t-u)$$

By definition, the response to the impulse $\delta(t)$ is h(t). So (Section 2.1), the response to the (delayed, non-unit) impulse in (3) is

(4) $$f(u)\ du\ h(t-u)$$

footnote This wouldn't work if the system were not initially at rest. As pointed out in Section 2.1, the response to a delayed impulse would not necessarily just be the delayed impulse response unless the system is time invariant and initially at rest.

By superposition, the response at time t to *the sum of all the impulses in Fig 1* (i.e., to f itself), when the system is initially at rest, is the sum of the responses in (4).

footnote This wouldn't work if the system were not initially at rest. If you add responses which satisfy say IC y(0) = 4 then the sum satisfies IC y(0) = 4 + 4 + 4 + ··· . But in this case, the responses in (4) all satisfy initial conditions like y(0) = 0 and therefore so does the sum.

The summing of the responses in (4) is done with an integral. So the response y(t) at time t to input f(t) is

(5) $$y(t) = \int_{-\infty}^{\infty} f(u)\ du\ h(t-u) = f(t)*h(t) \qquad \text{QED}$$

footnote (very subtle) The sum (integral) in (5) could actually start adding from u=0 instead of $u=-\infty$ since the input f(t) starts at time 0. And the sum could stop at u=t instead of $u=\infty$, since at time t, you can only get the response from impulses that occur before time t.

But it doesn't hurt to start out using the limits $u=-\infty$ to $u=\infty$. When we actually calculate the integral you will use the fact that f(u) = 0 if $u \le 0$ and $h(t-u) = 0$ if $u \ge t$ and this will automatically change the limits to u=0 to u=t. Don't worry about it now.

example 1

Given a system with impulse response h(t) in Fig 4. Find the response y(t) of the initially-at-rest system to the input f(t) in Fig 5.

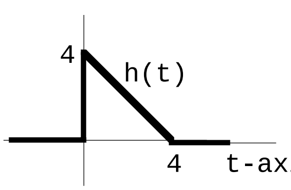


FIG 3

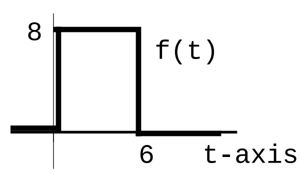


FIG 4

I'll use the first of the two convolution integrals in (2):

$$y(t) = h(t)*f(t) = \int_{u=-\infty}^{\infty} h(t-u)f(u)\ du$$

The problem starts with two functions of *t* namely the impulse response h(t) and the input f(t) in Figs 3 and 4. And the answer will be a function of *t*, namely the response y(t). But the *working* letter for the convolution integral is u; you are integrating f(u) times h(t−u) with respect to u and carrying t along as if it were a constant.

Since h(t−u) and f(u) both change formulas, the best way to keep track of them, and ultimately their product, is to draw their graphs in a u,y coord system.

The graph of f(u) in a u,y coord system looks the same as the f graph in Fig 4 but with the horizontal axis named u instead of t (Fig 5).

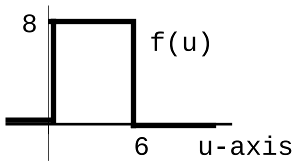
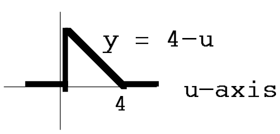


FIG 5

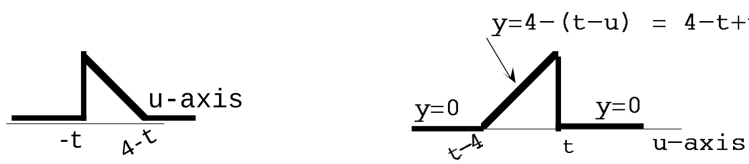
The graph of h(u) in a u,y coord system looks the same as the h graph in Fig 3 but with a horizontal axes named u instead of t (Fig 6). Translate left by t and then reflect in the y-axis to get the graph of h(t−u). Fig 7 shows the two steps.

In Fig 6, the slanted line has equation y = 4−u. Replace u by t−u to get the equ of the slanted part in the graph of h(t−u) in Fig 7, namely y = 4 − (t−u).



graph of h(u)

FIG 6



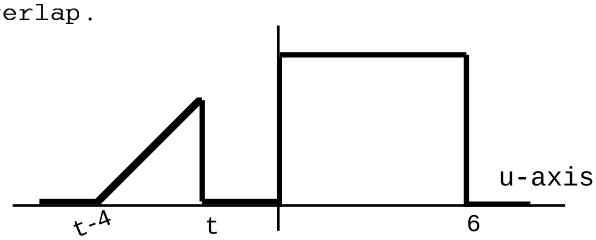
intermediate step

graph of h(t−u)

FIG 7 Getting the graph of y = h(t−u)

The function h(t−u) is either 4 − t + u or 0 and the function f(u) is either 8 or 0 so the product is either 8(4 − t + u) or 0. But *when* it is 8(4 − t+ u) and *when* it is 0 depends on the "constant" t. The best way to keep track is with more graphs and you need cases to accommodate all the possibilities, i.e., all the ways in which the h(t−u) and f(u) graphs can overlap.

case 1 $t \le 0$

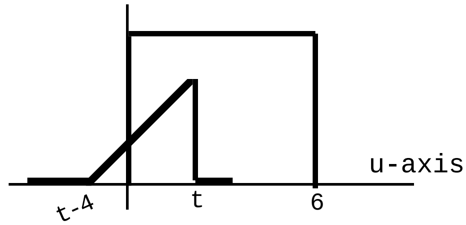


From the diagram you can see that in this case, one or another of f(u) and h(t−u) is always 0 so their product is always 0. So

$$y(t) = \int_{-\infty}^{\infty} h(t-u)f(u)\,du = \int_{-\infty}^{\infty} 0\,du = 0$$

case 2 $t - 4 \le 0$ and $t \ge 0$,

i.e., $0 \le t \le 4$



From the diagram you can see that outside the interval [0,t] at least one of h(t−u) and f(u) is 0 so their product is 0 and does not contribute to the convolution. integral. Inside the interval their product is $8(4-t+u)$. So

$$y(t) = \int_{-\infty}^{\infty} h(t-u)\, f(u)\, du \qquad = \int_{-\infty}^{\infty} 8(4-t+u)\, du$$

$$= \Big[8(4-t)u + 4u^2\Big]\Big|_{u=0}^{u=t} \qquad \text{(antidiff w.r.t. u and treat t as a constant)}$$

$$= 8(4-t)t + 4t^2$$

$$= -4t^2 + 32t$$

warning

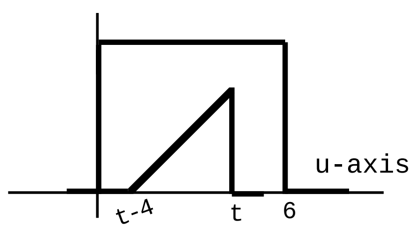
The name of the case is $0 \le t \le 4$ but the integral is *not* $\int_0^4$; it's $\int_{u=0}^{u=t}$.

To get $0 \le t \le 4$ as the title of the case, decide what will make the triangle overlap the box as shown in the diagram: the left end of the triangle must be to the left of 0 but the right end must be past 0. (The right end must also not be past 6 but that's taken care of already by saying the left end hasn't passed 0.) This means $t-4 \le 0$ and $t \ge 0$ which is $0 \le t \le 4$

To get $\int_{u=0}^{u=t}$ look at the picture and decide where the product is nonzero: To the left of u=0, the box is 0. To the right of u=t the triangle turns 0. The product is nonzero for $0 \le u \le t$ so $\int_{u=-\infty}^{\infty}$ turned into $\int_{u=0}^{t}$.

case 3 $t - 4 \ge 0$ and $t \le 6$

i.e., $4 \le t \le 6$



From the diagram you can see that outside the interval [t−4,t] the product h(t−u) f(u) is 0 (because outside the interval one or another of the factors is 0). And inside the interval, h(t−u)f(u) is 8(4−t+u). So

$$y(t) = \int_{u=-\infty}^{u=\infty} h(t-u)f(u)\, du$$

$$= \int_{u=t-4}^{u=t} 8(4-t+u)\, du$$

$$= 8(4-t)u + 4u^2 \Big|_{u=t-4}^{u=t}$$

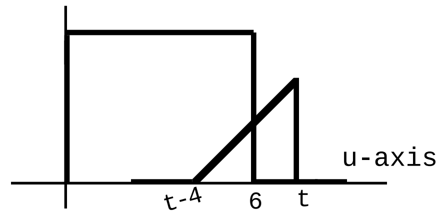
$$= 8(4-t)t + 4t^2 - \left(8(4-t)(t-4) + 4(t-4)^2\right)$$

$$= 64$$

warning
The name of the case is $4 \le t \le 6$ but the integral is $\int_{u=t-4}^{u=t}$ not $\int_4^6$

case 4 $t - 4 \le 6$ and $t \ge 6$

i.e., $6 \le t \le 10$



The product $h(t-u)f(u)$ is 0 except for the interval $[t-4,6]$. So

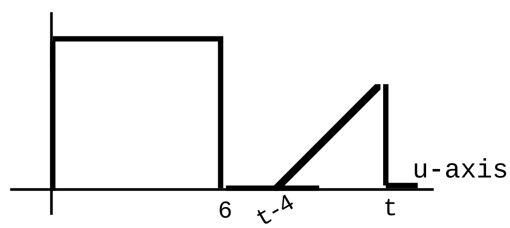
$$y(t) = \int_{u=-\infty}^{u=\infty} h(t-u)d(u)\ du$$

$$= \int_{u=t-4}^{u=6} 8(4-t+u)\ du$$

$$= 8(4-t)u + 4u^2 \Big|_{u=t-4}^{u=6} = -4t^2 - 80t + 400$$

case 5 $t - 4 \ge 6$,

i.e., $t \ge 10$



One or another of $f(u)$ and $h(t-u)$ is always 0 so their product is always 0 . So

$$y(t) = \int_{u=-\infty}^{u=\infty} 0\ du = 0$$

All in all the response is

$$(5) \quad y = \begin{cases} 0 & \text{if } t \le 0 \\ -4t^2 + 32t & \text{if } 0 < t < 4 \\ 64 & \text{if } 4 < t < 6 \\ 4t^2 - 80t + 400 & \text{if } 6 \le t \le 10 \\ 0 & \text{if } t \ge 10 \end{cases}$$

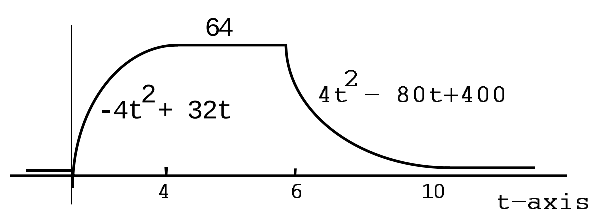


FIG 8

What good was all this? Now you know that if the input $\delta(t)$ produces the output $h(t)$ in Fig 3 then the input $f(t)$ in Fig 4 produces the output in Fig 8.

warning

1. The cases must include all values of t. The cases cannot jump from $3 \le t \le 5$ to $6 \le t \le 9$ omitting $5 \le t \le 6$. If one case is $3 \le t \le 5$ then the next case must pick up from there with $5 \le t \le \ldots$

And the cases must not overlap. You can't have one case named $2 \le t \le 3$ and another case named $t \ge 2$.

2. In example 1, don't write $h(u) = 4 - u$ since $h(u)$ is $4 - u$ only for $0 \le u \le 4$. For other u's, $h(u)$ is 0.

example 2

Suppose the impulse response of a system is

$$h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases} \qquad \text{(Fig 9)}$$

If the system is initially at rest, find its response to input

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \ge 0 \end{cases} \qquad \text{(Fig 10)}$$

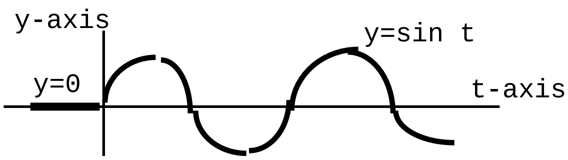


FIG 9 graph of h(t)

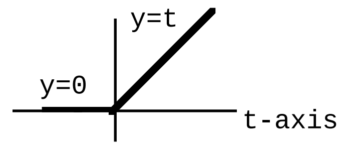


FIG 10 graph of f(t)

solution The response is h(t)*f(t). I'll use the second convolution integral in (2):

$$h(t)*f(t) = \int_{u=-\infty}^{u=\infty} h(u)\,f(t-u)\,du$$

Fig 11 shows the graph of h(u) in a u,y coord system.

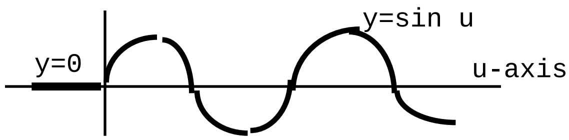
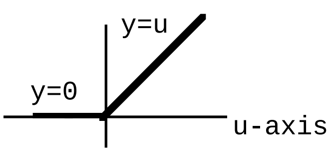


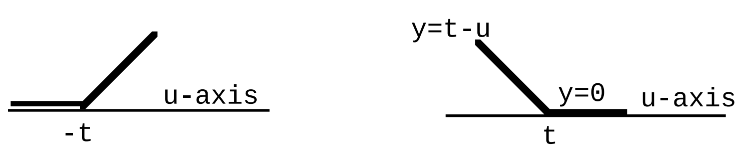
FIG 11 graph of h(u)

Fig 12 shows the graph of f(u) in a u,y coord system.
Fig 13 translates the f({u) graph left by t and then reflects in the y–axis to get the graph of f(t–u)



graph of f(u)

FIG 12



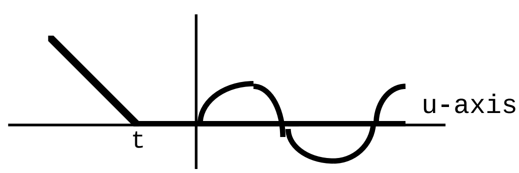
intermediate step

graph of f(t-u)

FIG 13

case 1 $t \le 0$

One or another of h(u) and f(t–u) is always 0 so

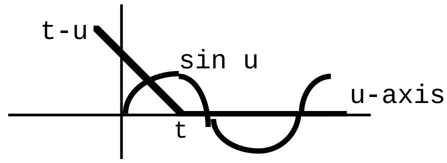


$$h(t)*f(t) = \int_{u=-\infty}^{u=\infty} h(u)\,f(t-u)\,du = \int_{u=-\infty}^{u=\infty} 0\,du = 0$$

warning Do not write $h(t)*f(t) = \int_{u=-\infty}^{\infty} \sin u \cdot (t-u)\,du$ in case 1.

When $t \le 0$, the product $h(u)f(t-u)$ is *not* $\sin u \cdot (t-u)$. It is *zero*, which is why the integral is 0.

case 2 $t \ge 0$



The product h(u)f(t−u) is nonzero only in interval [0,t] (to the left of that interval h(u) is 0 and to the right of that interval f(t−u) is 0. So

$$h(t)*f(t) = \int_{u=-\infty}^{u=\infty} h(u)f(t-u)\,du$$

$$= \int_{u=0}^{t} \sin u \cdot (t-u)\,du$$

$$= t\int_{u=0}^{t} \sin u\,du - \int_{u=0}^{t} u \sin u\,du$$

$$= \Big[-t\cos u - u\cos u + \sin u \Big]_{u=0}^{t} \qquad \text{(ref page antideriv tables (E))}$$

$$= t - \sin t$$

warning
Be careful with letters.
The problem starts with h(t) = sin t but in the convolution integral you have to use h($\boxed{u}$) which is sin $\boxed{u}$.
It makes a difference.

So all in all

$$h(t)*f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t - \sin t & \text{if } t \ge 0 \end{cases}$$

special case

If f(t) and h(t) are each 0 until time t and then each maintains a single formula apiece for $t \ge 0$ as in example 2 then their convolution requires only the trivial case $t \le 0$ (where the answer is 0) and the significant case $t \ge 0$ where the convolution integral has limits $\int_{u=0}^{t}$

warning

But if either f(t) or h(t) changes formula for $t \ge 0$ then you are not in the special case and the one case $t \ge 0$ is not enough, as in example 1.

which convolution integral to use

The result in (2) offers a choice of two convolution integrals. They produce the same answer so either one can be used.

In example 1, f(t) is "simpler" than h(t) so it would have been better to use the version with f(t−u)h(u) so that the simpler of the two graph is the one that gets flipped. The example used the other version to give extra flipping practice.

In example 2, one version uses the integrand (t−u) sin u and the other version uses the integrand u sin(t−u). Pick the version for which the antidifferentiation is easier, namely (t−u) sin u.

mathematical catechism (you should know the answers to these questions)

Question 1 What does it mean to say that h(t) is the impulse response of a (linear time-invariant) system.

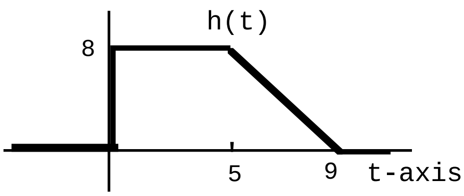
Answer 1 It means that if the input into the system when it is initially at rest is δ(t) then the response of the system at time t is h(t).

Question 2 If h(t) is the impulse response of a system then what is the significance of the convolution h(t)*f(t).
Answer 2 It's the response of the system at time t if it is initially at rest and then gets input f(t).

PROBLEMS FOR SECTION 2.3

1. Draw the graph of h(t−u) in a u,y coord system and find the equations of the various pieces.

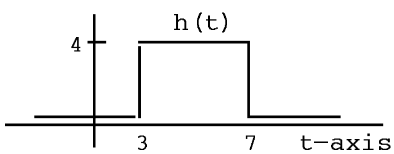
(a)

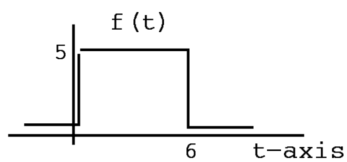


(b)

$$h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases}$$

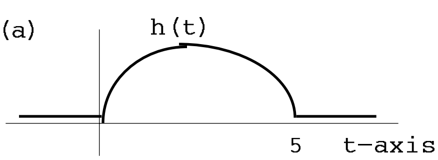
2. The diagram shows the impulse response h(t) of a system. Find the response of the system to input f(t) if the system is initially at rest. And sketch the response when you get it.
(The solution flips h.)

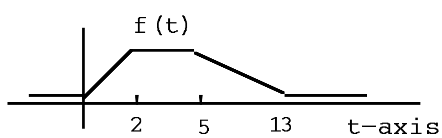




3. Suppose a system has impulse response h(t). If the system is initially at rest, when does its response to the input f(t) die out (if ever).

(a)





(b) $h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases}$ and f(t) is the same as in part (a)

4. Repeat example 1 but flip the f(u) this time instead of h(u).

5. Suppose a system is initially at rest.

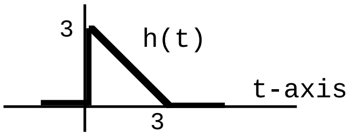
Suppose f(t)*g(t) = c(t).

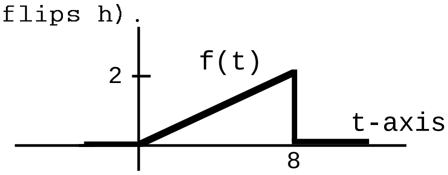
Fill in the blanks. (There are two possible answers. Give them both.

If the system has impulse response ________________ then input ____________

produces output _________________ .

6. The diagram shows the impulse response h(t) of a system, and a function f(t). The problem is to find the response y(t) of the system to input f(t) if the system is initially at rest (the solution flips h).



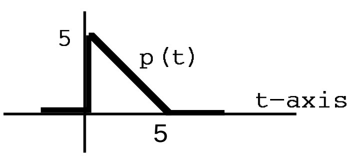


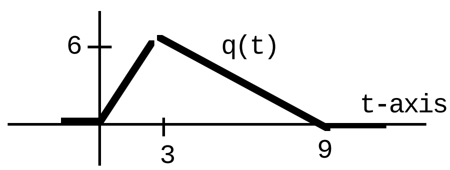
7. (a) Look at a system in which inputs f(t) and outputs y(t) are related by $2y'' + 8y' + 6y = f(t)$.
Let h(t) be the system's impulse response.
(You don't have to find h(t). Give the answers in terms of h(t).)

(a) Suppose the system is initially a rest.
What is the response of the system to input f(t).
(b) Suppose the system is *not* initially at rest.
In particular suppose $y(0) = 7$, $y'(0) = 9$.
What is the response of the system to input f(t).

8. (a) Given the functions p(t) and q(t) in the diagram. Find their convolution but stop before computing the integrals. Just set them up. (There are lots of cases.)
(b) Suppose p(t) is the impulse response of a system. What does this mean physically.

(c) If p(t) is the impulse response of a system, what does the convolution that you computed in part (a) represent physically.





9. The last footnote on page 2 of this section tried to show why the second convolution integral in (1) should give the same result as the first. Show again that the two integrals in (1) are equal using ordinary substitution from calculus.

10. If the impulse response of a system is

$$h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}$$

find the response, when initially at rest, to

(a) input $f(t) = \begin{cases} 6 & \text{if } 0 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$

(b) input $\delta(t)$

11. Find $f(t) * g(t)$ if

(a) $g(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}$ and $f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \ge 0 \end{cases}$

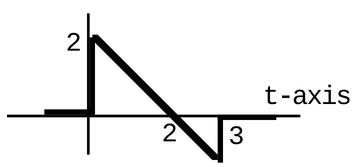
(b) $g(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \cos t & \text{if } t \ge 0 \end{cases}$ and $f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases}$

Use the identity $\sin x \cos y = \dfrac{\sin(x+y) + \sin(x-y)}{2}$.

12. Find $f(t)*f(t)$ if

(a) $f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{t} & \text{if } t \ge 0 \end{cases}$ (b) $f(t) = \begin{cases} a & \text{if } -b \le x \le b \\ 0 & \text{otherwise} \end{cases}$

13. The diagram shows the output of a system when the input is $\delta(t)$ and the system is initially at rest.



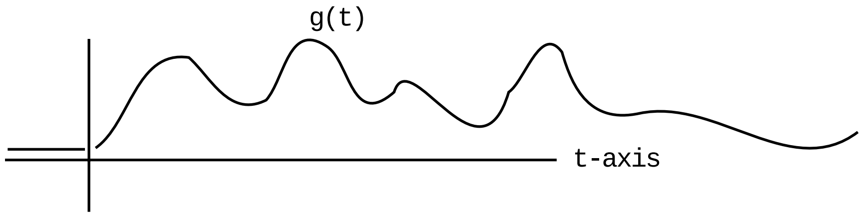
Find the response of the system to the input

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}$$

when the system is initially at rest and find the steady state response.

HONORS

14. Start with an arbitrary function $g(t)$ for $t \geq 0$.



Find the convolution $\delta(t) * g(t)$

You can't actually compute the convolution integral the way you do an ordinary integral because $\delta(t)$ is not an ordinary function.
You'll have to do it some other way.

Suggestion You can do it by inspection by thinking about what $\delta(t)*g(t)$ represents physically for some hypothetical system.
And explain how you got your answer.

15. Suppose the impulse response of a system is $\delta(t)$. In other words, if the input is $\delta(t)$ then the output is also $\delta(t)$.

Now use input $f(t)$. Explain (clearly, logically, grammatically, briefly) why the system's output is $f(t)$.

In other words, if the system is a copy–cat when you put in $\delta(t)$, explain why it is a copy–cat when you put in anything else.

summary These two results are equivalent.
(1) $\delta(t)*f(t) = f(t)$
(2) If the impulse response is $\delta(t)$ then the initially–at–rest system is a copy–cat.

(2) is an immediate corollary of (1): If the impulse response is $\delta(t)$ then the response to $f(t)$ is $\delta(t)*f(t)$ which is $f(t)$, by (1).
(1) follows from (2): Since $h(t) = \delta(t)$, the response to $f(t)$ is $\delta(t)*f(t)$. So $\delta(t)*f(t) = f(t)$.
My second method for proving (1) essentially proves (2) first and then gets back to (1).

16. (this was an exam problem)
The inputs $f(t)$ and outputs $y(t)$ of a system are related by

$$2y'' + 8y = f(t)$$

(a) Find the impulse response of the system.
(b) Solve

$$2y'' + 8y = \frac{t^5 \tan t}{1 + t^2} \quad \text{with IC } y(0) = 2,\ y'(0) = 3.$$

But assume you have a computer that can do any convolution so that your answer is allowed to contain unevaluated convolutions, e.g., your answer could look like

$$\frac{(e^t \sin t) * \cos t + t^4}{t^2 + t^3 * t^4} .$$

In other words, your answer must be a specific function of t but it can contain unevaluated convolutions.

REVIEW PROBLEMS FOR CHAPTER 2

1. A system's input f(t) and output y(t) are related by $4y'' + y = f(t)$.

(a) Find the impulse response.
(b) Find the response of the system to $\delta(t-11)$ with IC $y(0) = 0$, $y'(0) = 0$.

2. Let

$$h(t) = \begin{cases} -2t + 6 & \text{if } 0 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the response of an initially-at-rest system to input f(t) if the system has impulse response h(t).