SOLUTIONS Section 3.1

1. (a) Point A goes with \( t = -1 \) and point B goes with \( t = 2 \).

(b) \textit{method 1 (using (6))} 
\[ \int F \cdot T \, ds = \int 2x \, dx + xy \, dy \]
\[ = \int_{t=1}^{t=2} \left[ 2 \cdot 3t^2 \cdot 6t \, dt + 3t^2 \cdot 2t \cdot 2 \, dt \right] = \int_{t=1}^{t=2} 48t^3 \, dt = -180 \]

\textit{method 2 (using (2))}
\( v = (6t, 2) \)
\( v \) points in the B to A direction (in the direction of increasing \( t \)) so it's \( -v \) that points like \( T \).

\( \int F \cdot T \, ds = \int_{t=-1}^{t=-1} \left[ -36t^3 - 12t^3 \right] \, dt = \int_{t=-1}^{t=-1} -48t^3 \, dt = -180 \]

(c) \( T = \frac{-v}{\sqrt{36t^2 + 4}} \)
\( = \frac{-6t}{\sqrt{36t^2 + 4}} \, i - \frac{2}{\sqrt{36t^2 + 4}} \, j \)

\textit{Question} So why doesn't \( \sqrt{36t^2 + 4} \) turn up in the calculations in (*) when \( T \) got replaced.

\textit{Answer} It is there but it canceled out before you could even see it.
\( T = \frac{-v}{\sqrt{36t^2 + 4}} \) and \( ds \) is \( \sqrt{36t^2 + 4} \, dt \) and when you find \( F \cdot T \, ds \) the square root cancels out and all you end up with is \( F \cdot -v \, dt \).

2. (a) (i) The equation of the line is \( y = -2x + 3 \).
The segment has parametric equations \( x = x, y = -2x + 3, -1 \leq x \leq 1 \).
\[ \int xy \, dx + y \, dy = \int_{x=-1}^{x=1} (-2x+3) \, dx + (-2x+3) \cdot -2 \, dx = \frac{40}{3} \]

(ii) The segment has parametric equations \( x = \frac{1}{2}(3-y), y = y, 1 \leq y \leq 5 \).
\[ \int xy \, dx + y \, dy = \int_{y=1}^{y=5} \frac{1}{2}(3-y) \cdot \frac{1}{2} \, dy + y \, dy = \frac{40}{3} \]

(iii) I'll use A as \((x_0, y_0)\) and \( \overrightarrow{AB} = -2i + 4j \) as the parallel vector.
Line \( AB \) has parametric equations \( x = 1-2t, y = 1+4t \).
Point A is the \( t=0 \) point. Point B is the \( t=1 \) point.
\[ \int xy \, dx + y \, dy = \int_{t=0}^{t=1} (1-2t)(1+4t) \cdot -2 \, dt + (1+4t) \cdot 4 \, dt = \frac{40}{3} \]

(b) It's the work done by \( F = xy \, i + y \, j \) to a particle which moves on the segment from A to B.
3. (a) The entire circle has parametric equations
\[ x = R \cos t, \ y = R \sin t, \ 0 \leq t \leq 2\pi. \]

\[
\int F \cdot T \, ds = \int_0^{2\pi} y \, dx + y \, dy = \int_0^{2\pi} R \sin t \cdot (-R \sin t) \, dt + R \sin t \cdot R \cos t \, dt
\]
(the limits go from 2\pi to 0 to go with clockwise)

\[
= R^2 \int_0^{2\pi} (-\sin^2 t + \sin t \cos t) \, dt \quad [= \pi R^2]
\]

(b) Like part (a) except now the limits of integration are \[ \int_\frac{3\pi}{2}^{2\pi} \text{ or } \int_{-\pi/2}^{0} \]

4. (a) Call the vector field F.
F \cdot T is positive on the A-to-B piece, positive on the B-to-C piece and negative on the C-to-A piece.
So the ccl line integral will be pos on A-to-B, pos on B-to-C and neg on C-to-A.

(b) Line AB has parametric equations \( x=x, \ y=0 \).
Line BC has parametric equations \( x=1, \ y=y \).
Line CA has parametric equations \( x=x, \ y=x \).

\[
\oint_{ccl} = \int_{A \to B} + \int_{B \to C} + \int_{C \to A}
\]

\[
= \int_{x=0}^{1/4} (x^3 + 0) \, dx + \int_{y=0}^{1/3} (1+y)0 \, dy + \int_{x=1}^{x=0} (x^3 + x) \, dx + x^2 \, dx
\]

= -1/2

(aha! predictions worked out)

5. One parametrization for the line is \( x = 1 + t, \ y = 2 + t, \ z = 3 + 2t \).
Then \( t_A = 0, \ t_B = 1 \) and

work = \( \int F \cdot T \, ds = \int y \, dx + z \, dy + x \, dz \)

\[
= \int_{t=0}^{1} (2+t) \, dt + (3+2t) \, dt + (1+t) \, 2 \, dt = \int_{t=0}^{1} (7+5t) \, dt \quad [= 19/2]\]

6. The curve has parametric equations \( x = x, \ y = x^2, \ z = x^3 \)

\[
circ = \int xy \, dx + xz \, dy + x \, dz = \int_{2}^{1} x \cdot x^2 \, dx + x \cdot x^3 \, 2x \, dx + x \cdot 3x^2 \, dx
\]

\[
= \int_{2}^{1} (4x^3 + 2x^5) \, dx \quad [= -36]
\]

7. On AD, \( T = k, \ F \cdot T = 0 \), \( \int F \cdot T \, ds = 0 \)
On BC, \( T = -k, \ F \cdot T = 0 \), \( \int F \cdot T \, ds = 0 \)
On AB, \( z = 4, \ F = 4i, \ T = i, \ F \cdot T = 4 \)
\( \int F \cdot T \, ds = \int 4 \, ds = 4 \times \text{length AB} = 28 \)
On CD, \( z = 6, \ F = 6i, \ T = -i, \ F \cdot T = -6 \)
\( \int F \cdot T \, ds = \int -6 \, ds = -6 \times \text{length CD} = -42 \)

Final answer is 28 - 42 = -14.
8. The segment BA has parametric equations \( x = x, y = x + 2, -1 \leq x \leq 2 \).
The parabola part of the path has parametric equations \( x = x, y = x^2, -1 \leq x \leq 2 \).

\[
\int F \cdot T \, ds \text{ on the loop} = \int_{\text{B to A on line}} + \int_{\text{A to B on parabola}}
\]
\[
= \int_{-1}^{2} x^2(x+2)\,dx + \int_{2}^{-1} x^4 \, dx + \int_{2}^{-1} (x^2 + 3) \, 2x \, dx
\]
\[
= \int_{-1}^{2} (x^3 + 2x^2 + x + 5) \, dx + \int_{2}^{-1} (x^4 + 2x^3 + 6x) \, dx \quad [ = \frac{63}{20}]
\]

9. The ellipse is \( x^2 + \frac{1}{3}y^2 = 1 \)
It has parametric equations \( x = \cos t, \ y = \sqrt{3} \sin t, \ 0 \leq t \leq 2\pi \)

\[
\oint (x^2 + y^4) \, dx - 2 \, dy \text{ on the ellipse ccl}
\]
\[
= \int_{t=0}^{2\pi} (\cos^2 t + 9 \sin^4 t) \cdot -\sin t \, dt - 2\sqrt{3} \cos t \, dt
\]
\[
= \int_{0}^{2\pi} (-\sin t \cos^2 t - 9 \sin^5 t + 2\sqrt{3} \cos t) \, dt
\]

10. The curve has parametric equations
\( x = 2 \cos t, \ y = 2 \sin t, \ z = 16 - 2 \sin t - 8 \cos t, \ 0 \leq t \leq 2\pi \)

\[
\int x^2 \overset{\rightarrow}{k} \cdot \overset{\rightarrow}{T} \, ds = \int 0 \, dx + 0 \, dy + x^2 \, dz = \int_{2\pi}^{0} 4 \cos^2 t \,( -2 \cos t + 8 \sin t) \, dt \quad [= 0]
\]
(The limits are \(2\pi\) to \(0\) because the direction is clockwise.)

11. At every point on the line segment, \( F \) points away from the origin and \( T \) points toward the origin.

\( T = -\hat{e}_r, \ F \cdot T = -4, \ \int F \cdot T \, ds = \int -4 \, ds = -4 \times \text{length of path} = -4\sqrt{45} \)

![Diagram](image)
12. (a) The rim is the ellipse $2x^2 + y^2 = 12$ (which can be written as $\frac{x^2}{6} + \frac{y^2}{12} = 1$) in the plane $z = 12$. It has parametric equations

$$x = \sqrt{6} \cos t$$
$$y = \sqrt{12} \sin t$$
$$z = 12$$
$$0 \leq t \leq 2\pi$$

**warning** The parametrization is not correct unless it includes $z=12$. If you leave out $z=12$ then it is understood that $z=z$ and you have parametrized an elliptic cylinder, a surface, not a curve.

(b) $\int F \cdot T \, ds = \int (yz \, dx + x \, dz$)

$$= \int_{\text{final } t}^{\text{initial } t} \sqrt{12} \sin t \cdot 12 \cdot \sqrt{6} \sin t \, dt + 0 \, dt$$

$$= \int_{t=2\pi}^{0} -72 \sqrt{2} \sin^2 t \, dt$$. \[ = -72\sqrt{2} \pi \text{ from the reference page} \]
SOLUTIONS Section 3.2

1. Flux across \( \int F \cdot N \, ds = \int -y \, dx + 3x \, dy \) on the parabola directed from B to A so that N is on the right as the curve is traversed.

The curve has parametric equations \( x = y^2, \ y = y \).

\[
\int F \cdot N \, ds = \int_{y=2}^{0} -y \cdot 2y \, dy + 3y^2 \, dy = \int_{y=2}^{0} y^2 \, dy \quad [\text{=} -8/3]
\]

2. The circle has parametric equations \( x = 2 \cos t, \ y = 2 \sin t, \ 0 \leq t \leq \pi/2 \).

(a) \( \int F \cdot T \, ds = \int 3y \, dx + 4x \, dy \)

\[
= \int_{t=0}^{\pi/2} 6 \sin t \cdot -2 \sin t \, dt + 8 \cos t \cdot 2 \cos t \, dt
\]

\[
= \int_{t=0}^{\pi/2} (-12 \sin^2 t + 16 \cos^2 t) \, dt
\]

*footnote*

Here’s how to compute the integral.

\[
\int_{t=0}^{\pi/2} \sin^2 t \, dt = \frac{1}{4} \int_{t=0}^{2\pi} \sin^2 t \, dt \quad \text{(look at areas under} \sin^2 t) \]

\[
= \frac{1}{4} \pi \quad \text{(use the integral tables on the ref page)}
\]

Similarly \( \int_{t=0}^{\pi/2} \cos^2 t \, dt = \frac{1}{4} \pi \)

So final answer is \(-12 \cdot \frac{1}{4} \pi + 16 \cdot \frac{1}{4} \pi = \pi \)

*Question*

Is it always true that \( \int_{x=0}^{\pi/2} f(x) \, dx = \frac{1}{4} \int_{x=0}^{2\pi} f(x) \, dx \)?

*Answer*

No. It depends on \( f(x) \). It was true for \( \sin^2 x \) but is not true for plain \( \sin \) or for \( x^2 \) or \( x^3 \) etc.

(b) \( \int F \cdot N \, ds = \int -4x \, dx + 3y \, dy \) on the curve clockwise

(so N is on your right as you walk)

\[
= \int_{t=\pi/2}^{0} -8 \cos t \cdot -2 \sin t \, dt + 6 \sin t \cdot 2 \cos t \, dt
\]

\[
= \int_{t=\pi/2}^{0} 28 \sin t \cos t \, dt \quad [\text{=} -14]
\]
3. Line AB has parametric equations $x=x, y=2x+3$.
   The parabola has parametric equations $x=x, y=x^2$.
   
   (a) $\text{flux out} = \int F \cdot \text{outer N ds}$
   
   $$= \int_{ccl} -y \, dx + x^2 \, y \, dy \, (\text{use ccl so that N is to your right as you walk})$$
   
   $$= \int_{\text{segment B to A}} + \int_{\text{parabola A to B}}$$
   
   $$= \int_{x=3}^{x=-1} -(2x+3) \, dx + x^2(2x+3) \, 2 \, dx + \int_{x=-1}^{3} -x^2 \, dx + x^2 \cdot x^2 \, 2x \, dx$$
   
   $$= -116 + \frac{700}{3} = \frac{352}{3}$$
   
   (b) I'll find the ccl circ.
   
   $$\int F \cdot T \, ds \, ccl$$
   
   $$= \int_{\text{Line}} x^2 \, y \, dx + y \, dy \, (\text{on B to A line}) + \int_{\text{parabola}} x^2 \, y \, dx + y \, dy \, (\text{on A to B parabola})$$
   
   Line $= \int_{x=3}^{x=-1} x^2(2x+3) \, dx + (2x+3) \, 2 \, dx \ [ = -108]$
   
   Parabola $= \int_{x=-1}^{3} x^2 \, 2x \, dx + x^2 \, 2x \, dx \ [ = \frac{444}{5}]$
   
   So the ccl circulation is $-108 + \frac{444}{5} = -\frac{96}{5}$
   
   The circulation is actually $\frac{96}{5}$ clockwise.

4. The field is uniform. I drew a diameter perp to the field arrows and compared the flow in across the lower semicircle with the flow out across the upper semicircle.
   They look the same to me so the net flux in is 0.

The circle has parametric equations $x = \cos t, y = \sin t$.

$\text{Flux in} = \int F \cdot \text{inner N ds} = \int_{0}^{2\pi} -3 \, dt + \cos t \, dt = 0$
SOLUTIONS Section 3.3

1. The surface has parametric equations

\[ x = x \]
\[ y = x^3 \]
\[ z = z \]

\[-1 \leq x \leq 3, \ 0 \leq z \leq 2 \]

\[ \mathbf{n} = \nabla x \times \nabla z = (1, 3x^2, 0) \times (0, 0, 1) = (3x^2, -1, 0) \]

This \( n \) has a positive \( x \) component so that's the one I want, not \(-n\).

On the surface, \( \mathbf{F} \cdot \mathbf{n} = 3x^2y = 3x^2x^3 = 3x^5 \)

\[ \text{Flux} = \int F \cdot N \, dS = \int_{\text{surface}} \int_{\text{limits}} 3x^5 \, dx \, dz \]

\[ = 36 - 1 \]

2. (a) \( x = r \cos \theta, \ y = r \sin \theta, \ z = r^3, \ 0 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi \)

(b) \( n = (\cos \theta, \ \sin \theta, 3r^2) \times (-r \sin \theta, r \cos \theta, 0) = (-3r^3 \cos \theta, -3r^3 \sin \theta, r) \)

\( n_{\text{outer}} = -n \) since the outer normal should have a neg \( z \)-component (look at the pic).

\[ \int F \cdot \text{outer} \ N \, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} F \cdot n_{\text{outer}} \ dr \ d\theta \]

\[ = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (3r^4 \cos^2 \theta + 3r^4 \cos \theta \sin \theta - r^7) \ dr \ d\theta \]

(c) If you start at \( P \) in the diagram below and change \( \theta \) by \( d\theta \) while \( r \) stays fixed, the little curve \( PB \) is traced out on the surface. If you start at \( P \) and change \( r \) by \( dr \) while \( \theta \) stays fixed in order to stay on the surface, you must move up and out to point \( C \) which is \( dr \) further from the \( z \)-axis. The patch is \( PBDC \)

\[ (d) \ dS = ||n|| \ dr \ d\theta = r \sqrt{1+9r^4} \ dr \ d\theta \]
3. (a) Here's one possibility:
\[ x = x \]
\[ y = y \]
\[ z = 2x^2 + y^2 \]
(It isn't a good idea to try to use \( r \) and \( \theta \) as parameters since the cup does not have circular cross sections.)
The \( x, y \) parameter world is the projection of the surface in the \( x, y \) plane, the inside of ellipse \( 2x^2 + y^2 = 6 \).

(b) \( n = (1, 0, 4x) \times (0, 1, 2y) = (-4x, -2y, 1) \)
This is an upper normal, which makes it point into the cup. So \( n_{outer} = -n \).

On the surface, \( \mathbf{F} \cdot n_{outer} = 2y^2z = 2y^2(2x^2 + y^2) \)

Flux out = \( \int_{\text{cup surface}} \mathbf{F} \cdot n_{outer} \, dx \, dy \)
= \( \int_{\text{parameter world}} 2y^2(2x^2 + y^2) \, dx \, dy \)
= \( \sqrt{3} \int_{x=-\sqrt{3}}^{\sqrt{3}} \int_{y=-\sqrt{6-2x^2}}^{\sqrt{6-2x^2}} 2y^2(2x^2 + y^2) \, dy \, dx \) \[ = 36\sqrt{2}\pi \]

(c) The surface area element is the patch swept out when \( x \) changes by \( dx \) and \( y \) changes by \( dy \). In my diagram, I started at point A. When I changed \( x \) by \( dx \) with \( y \) fixed, (while \( z \) changes so that you stay on the cup) I went to D. When I started at A and changed \( y \) by \( dy \), I went to B. The surface area element is ABCD. It's the projection onto the surface of a little \( dx \) by \( dy \) rectangle in the \( x, y \) plane.

\[ dS = \|n\| \, dx \, dy = \sqrt{16x^2 + 4y^2 + 1} \, dx \, dy \]

(d) surface area = \( \int_{\text{cup surface}} dS = \int_{\text{parameter world}} \|n\| \, dx \, dy \)
= \( \int_{x=-\sqrt{3}}^{\sqrt{3}} \int_{y=-\sqrt{6-2x^2}}^{\sqrt{6-2x^2}} \sqrt{16x^2 + 4y^2 + 1} \, dy \, dx \)
(Mathematica couldn't do this integration but it could numerically integrate to get the approximate answer 54.96)
(e) (i) Here's what the field looks like

For every point \((x_0, y_0, z_0)\) on the left side of the cup where the flux arrow points to the right (into the cup) there is a symmetric point \((x_0', -y_0, z_0)\) on the right side with an identical flux arrow (out of the cup).

The flux in through the left side is the same as the flux out through the right side. Net flux out is zero.

(ii) Here's what the field looks like.

At every point on the left side of the cup, \(F\) points left (out of the cup). At every point on the right side of the cup, \(F\) points right (out of the cup).

Flux flows out of the cup everywhere, total flux out is positive.
(iii) For every point \((x_0, y_0, z_0)\) on the back of the cup where the \(\mathbf{F}\) arrow points up (into the cup) there is a symmetrically placed point \((x_0, -y_0, z_0)\) at the front of the cup where a same-length \(\mathbf{F}\) arrow points down (out of the cup). The flux into the cup through the back half is the same as the flux out of the cup through the front half. Net flux out is zero.

4. (a) The radius at the bottom of the frustrum is 3 (similar triangles). The frustrum is swept out when the line segment \(z = 2x, 3 \leq x \leq 5\) is revolved around the \(z\)-axis.

The frustrum has parametric equations
\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= 2r \\
3 \leq r \leq 5, \quad 0 \leq \theta \leq 2\pi
\end{align*}
\]

(b) \(n = (\cos \theta, \sin \theta, 2) \times (-r \sin \theta, r \cos \theta, 0) = (-2r \cos \theta, -2r \sin \theta, r)\).
\(n\) is up and into the cone. so I'll use \(n_{\text{outer}} = -n\).

flux out of frustrum = \[
\int F \cdot n_{\text{outer}} \, dS
= \int_{\text{parameter world}} F \cdot n_{\text{outer}} \, dr \, d\theta
\]
\[
= \int_{\theta=0}^{2\pi} \int_{r=3}^{5} 2r \cos \theta \, dr \, d\theta
= \int_{\theta=0}^{2\pi} \int_{r=3}^{5} 2(r \cos \theta)^2 \, dr \, d\theta \quad [= 196\pi/3]
\]

(c) By inspection, nothing flows out of either lid (the \(F\) field glides along the lids, i.e., \(F \cdot n\) is 0 on each cover. So the answer here is the same as the answer to part (b).

(d) \(dS = |n| \, dr \, d\theta = r\sqrt{5} \, dr \, d\theta\).
The surface area mag factor is \(r\sqrt{5}\).
5. (a) The graph of \( y = x^2, -2 \leq x \leq 2 \) in the \( x,y \) plane is a piece of a parabola. The graph of \( y = x^2, -2 \leq x \leq 2, 0 \leq z \leq 3 \) in 3-space is a piece of a parabolic cylinder.

(b) I'll use parametrization \( x = x, y = x^2, z = z \). The parameter world is the projection of the surface in the \( x,z \) plane where \(-2 \leq x \leq 2, 0 \leq z \leq 3\).

\[ n = (1, 2x, 0) \times (0, 0, 1) = (2x, -1, 0). \]

This is an outer normal to the surface (because the \( y \) coord is negative). On the surface, \( F \cdot n_{outer} = -y = -x^2 \).

\[
\text{flux out} = \int F \cdot n_{outer} \, dS = \int_{x,z \text{ projection}} F \cdot n_{outer} \, dx \, dz = \int_{x=-2}^{2} \int_{z=0}^{3} -x^2 \, dx \, dz \quad [= -16]
\]

So the flux is really 16 stuff units/sec into the surface (you can see from a picture that it's in since the \( F \) arrows all point "back" in my diagram and make obtuse angles with the outer \( N \)).

\[
\text{(c) } dS = \|n\| \, dx \, dz = \sqrt{4x^2 + 1} \, dx \, dz
\]

6. **flux out across face \( ABC \)**

The face has parametric equations

\[
\begin{align*}
x &= x \\
y &= y \\
z &= 5 - 3x - 2y
\end{align*}
\]

The parameter world is the projection of the surface in the \( x,y \) plane
\[ 3x + 2y = 5 \]

\[ n = (1, 0, -3) \times (0, 1, -2) = (3, 2, 1) \]

\[ n_{\text{upper}} = n \text{ rather than } -n. \]

Flux out through this face = \[ \int F \cdot n_{\text{upper}} \, \mathbf{N} \, d\mathbf{S} \]

\[ = \int_{x, \, y \text{ projection}} F \cdot n_{\text{upper}} \, dx \, dy \]

\[ = \int_{x=0}^{5/3} \int_{y=0}^{(5-3x)/2} 2(5 - 3x - y) \, dy \, dx \quad [\text{=} \frac{125}{12}] \]

The other three faces are all lids.

**Flux out across bottom face**

Flux is 0 since \( F \) glides across it, i.e., \( F \) is perp to \( \mathbf{N} \) on the rear face.

**Flux out across rear face**

Zero

**Flux out across left face**

On the left face

\[ y = 0, \quad F = z\mathbf{j}, \quad \mathbf{N} = -\mathbf{j}, \quad F \cdot \mathbf{N} = -z, \quad d\mathbf{S} = dA \]

\[ \text{flux out} = \int_{x=0}^{5/3} \int_{z=0}^{5-3x} -z \, dz \, dx \quad [\text{=} \frac{-125}{18}] \]

Final answer is the sum of the face fluxes \[ [\text{=} \frac{125}{36}] \]
7. You can't ask someone to find $dS$ unless you specify a parametrization. $dS$ depends on how you parametrize the surface. The surface area element swept out when $u$ changes by $du$ and $v$ changes by $dv$ depends on the parametrization and so does its area $dS$.

For example, if one person parametrizes the plane $2x + 3y + 4z = 5$ with
\[
x = u \\
y = v \\
z = \frac{1}{4} (5 - 2u - 3v)
\]
and another uses
\[
x = 3u \\
y = 4v \\
z = \frac{1}{4} (5 - 6u - 12v)
\]
and another uses
\[
x = \frac{1}{2} (5 - 3u - 4v) \\
y = u \\
z = v
\]
they all get different $dS$'s.

8. It's the surface of revolution (Fig B) swept out by revolving the curve $z = e^x$, $x \geq 0$ (Fig A) in the $x,z$ plane around the $z$-axis.

**footnote**

If you revolve the ignored part in Fig A (the part in quadrant TWO) you get the surface of revolution in Fig C.

The rule for the parametric equations of the surface of revolution is a little different here. The parametric equations turn out to be

\[
x = r \cos \theta \\
y = r \sin \theta \\
z = e^{-r} \quad \text{(note: exponent here is MINUS r)}
\]

$r \geq 0$, $0 \leq \theta \leq 2\pi$

That's because if a point is in quadrant II or III in the $x,z$ plane then $x$ is negative so it's $-x$ that equals $r$.

9. It means that if the surface is parametrized, say with parameters $u$ and $v$, and $dS$ is the surface area traced out by changing $u$ by $du$ and $v$ by $dv$ then $dS = \text{mag factor times } du \, dv$. 
10. (a) On the sphere, $\rho = R$, $F = R^2 \hat{e}_\rho$, outer $N = \hat{e}_\rho$, $F \cdot \text{outer } N = R^2$

$$\int F \cdot \text{outer } N \, dS = \int R^2 \, dS = R^2 \times \text{surface area of sphere} = R^2 \cdot 4\pi R^2 = 4\pi R^4$$

(b) $F$ points radially away from the origin and the $N$'s on the cone are always perpendicular to radial lines so all $F \cdot N$'s on the cone are $0$ and $\int F \cdot N \, dS = 0$. No flux goes out of the cone (the $F$ arrows glide along the cone).

11. Not necessarily. Our $n$'s will have the same or opposite direction (i.e., our $n$'s will both be normals to the surface). But they might have different norms.

For example, suppose the surface is the plane $2x + 3y + 4z = 5$.

If I use the parametrization

- $x = x$
- $y = y$
- $z = \frac{1}{4} (5 - 2x - 3y)$

then my $n$ comes out to be $\frac{1}{2} i + \frac{3}{4} j + k$ if I used $\text{vel}_x \times \text{vel}_y$ or $-\frac{1}{2} i - \frac{3}{4} j - k$ if used $\text{vel}_y \times \text{vel}_x$.

If you use

- $x = 3u$
- $y = 4v$
- $z = \frac{1}{4} (5 - 6u - 12v)$

then your $n$ comes out to be $6i + 9j + 12k$.

If someone else uses

- $x = \frac{1}{2} (5 - 3x - 4y)$
- $y = y$
- $z = z$

then her $n$ comes out to be $i + \frac{3}{2} j + 2k$.

All the $n$'s have different norms because each parametrization has a different magnitude factor. When you change the parameters by a little bit and look at the surface area swept out on the surface, it's different for each parametrization.

12. It's the same surface whether you revolve $z = x^3$ from the $x,z$ plane or revolve $z = y^3$ from the $y,z$ plane. Same parametric equations:

- $x = r \cos \theta$
- $y = r \sin \theta$
- $z = r^3$

$0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$

13. The surface $ABCD$ is a plane cylinder (move line $AB$ to the right).

![Diagram of a plane cylinder]

In the $x,z$ plane, line $AB$ has equation $z = -\frac{3}{5} x + 3$.

So the surface has parametric equations

- $x = x$
- $y = y$
- $z = -\frac{3}{5} x + 3$

The parameter world is $0 \leq x \leq 5$, $0 \leq y \leq 4$, the projection of the surface in the $x,y$ plane, the rectangular region $ADEF$. 
14. $\overrightarrow{AB} = (-8, -1, 0)$
$\overrightarrow{AC} = (3, 0, -2)$

$\overrightarrow{AB} \times \overrightarrow{AC} = (2, -16, 3)$. This is a normal to the plane.

The plane has equation

$$2(x - 1) - 16(y - 2) + 3(z - 3) = 0$$

$$2x - 16y + 3z = 21$$

So a parametrization is

$$x = x$$
$$y = y$$
$$z = \frac{-21 - 2x + 16y}{3}$$

The parameter world is the entire $x, y$ plane.
SOLUTIONS Section 3.4

1. (a) The box top has parametric equations
   \[ x = x, \ y = y, \ z = \frac{1}{4} (10 - 2x - 3y), \ -2 \leq x \leq 2, \ -1 \leq y \leq 1 \]

   method 1 for getting \( n \)
   Let \( \mathbf{g} = 2x + 3y + 4z \). Then
   \[ \mathbf{n}_{\text{upper}} = \frac{\nabla \mathbf{g}}{\| \nabla \mathbf{g} \|} = (1/2, 3/4, 1) \]

   method 2 for getting \( n \)
   \[ \mathbf{n} = (1, 0, -1/2) \approx (0, 1, -3/4) = (1/2, 3/4, 1), \] an upper \( n \).

   Then \( \mathbf{F} \cdot \mathbf{n}_{\text{upper}} = \frac{3}{4} z \)
   \[ \int \mathbf{F} \cdot \mathbf{n}_{\text{upper}} \ d\mathbf{S} = \int \mathbf{F} \cdot \mathbf{n}_{\text{upper}} \ dx \ dy \text{ over projection in } x,y \text{ plane} \]
   \[ = \int_{x=-2}^{2} \int_{y=-1}^{1} \frac{3}{4} \cdot \frac{1}{4} (10 - 2x - 3y) \ dy \ dx \quad \text{[\( = 15 \)]} \]

   (b) fast way
   \[ \| \mathbf{n} \| = \frac{1}{4} \sqrt{29} \]
   Since the mag factor is constant, the box top area is \( \frac{1}{4} \sqrt{29} \) times the base area.
   So box top surface area is \( \frac{1}{4} \sqrt{29} \cdot 8 \).

   slow way
   \[ \text{surface area} = \int \ d\mathbf{S} = \int_{x,y \text{ world}} \| \mathbf{n} \| \ dx \ dy \]
   \[ = \int_{x,y \text{ world}} \frac{1}{4} \sqrt{29} \ dx \ dy \]
   \[ = \int_{y=-2}^{2} \int_{x=-1}^{1} \frac{1}{4} \sqrt{29} \ dx \ dy \]

   (c) From part (a) we know that the flux out the top is 15. The other faces are lids.
   out the bottom, back, front
   \( \mathbf{F} \) glides along these faces (i.e., \( \mathbf{F} \) is perp to \( \mathbf{N} \) on these faces). No flux across.

   out the left face
   On the left face, \( y = -1, \ N = -j, \ \mathbf{F} \cdot \mathbf{N} = -z, \ dS = dA \).
   Line \( AD \) in plane \( y = -1 \) has equation \( 2x - 3 + 4z = 10, \ 2x + 4z = 13 \).
   \[ \text{flux out} = \int_{\text{left face}} -z \ dA \ [\text{double integral}] = \int_{x=-2}^{2} \int_{z=0}^{(13-2x)/4} -z \ dz \ dx \quad \text{[\( = -\frac{523}{24} \)]} \]

   out the right face
   On the right face, \( y = 1, \ N = j, \ \mathbf{F} \cdot \mathbf{N} = z, \ dS = dA \)
   Line \( BC \) in plane \( y = 1 \) has equation \( 2x + 3 + 4z = 10, \ 2x + 4z = 7 \)
   \[ \text{flux out} = \int_{\text{right face}} z \ dA = \int_{x=-2}^{2} \int_{z=0}^{(7-2x)/4} z \ dz \ dx \quad \text{[\( = \frac{163}{24} \)]} \]
out the whole box

Sum of flux out of all the faces is

\[ 15 - \frac{523}{24} + \frac{163}{24} \]

which happens to be 0.

2. (a)

The flux is

\[ \int F \cdot \text{inner } N \, dS = \int F \cdot \text{inner } N \, dS \text{ on paraboloid} + \int F \cdot \text{down } N \, dS \text{ on lid} \]

Units are kilograms/sec.

**PARABOLOID PART**

**method 1 using x and y as parameters**

The paraboloid part has parametric equations

\[ x = x, \quad y = y, \quad z = x^2 + y^2. \]

Let \( g = x^2 + y^2 - z \).

Then

\[ n_{\text{upper}} = \frac{\nabla g}{\partial g/\partial z} = (-2x, -2y, 1) \]

You can also use

\[ n = (1, 0, 2x) \times (0, 1, 2y) = (-2x, -2y, 1)\text{ (an upper } n) \]

The inner normal points uppish so \( n_{\text{inner}} = (-2x, -2y, 1) \)

The cross section at the top of the cup is the circle \( x^2 + y^2 = 5 \). So the projection in the \( x,y \) plane is a circular region with radius \( \sqrt{5} \).

\[ \int F \cdot \text{inner } N \, dS = \int_{x,y \text{ projection}} F \cdot n_{\text{inner}} \, dx \, dy \]

\[ = \int_{x,y \text{ projection}} (-6x + z) \, dx \, dy \]

\[ = \int_{x,y \text{ projection}} (-6x + [x^2+y^2]) \, dx \, dy \]

In Cartesian coordinates this is

\[ \int_{x=-\sqrt{5}}^{\sqrt{5}} \int_{y=-\sqrt{5-x^2}}^{\sqrt{5-x^2}} (-6x + x^2 + y^2) \, dy \, dx. \]

In polar coordinates (better) this is

\[ \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{5}} (-6r \cos \theta + r^2) \, r \, dr \, d\theta \quad [ = \frac{25\pi}{2} ] \]

**method 2 using r and \( \theta \) as parameters**

The paraboloid has parametric equations

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = r^2 \]

\[ 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq \sqrt{5} \]

\[ n = (\cos \theta, \sin \theta, 2r) \times (-r \sin \theta, r \cos \theta, 0) \]
\[ = (-2r^2 \cos \theta, -2r^2 \sin \theta, r) \]

Use \( n \), not \(-n\), to get an inner normal.

\[ F \cdot n = -6r^2 \cos \theta + rz = -6r^2 \cos \theta + r^3 \]

\[ \int F \cdot \text{inner } N \, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{5}} F \cdot n_{\text{inner}} \, dr \, d\theta \]

\[ = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{5}} (-6r^2 \cos \theta + r^3) \, r \, dr \, d\theta \]
The lid has parametric equations \( x = x, y = y, z = 5 \).

**warning** On the lid, \( z \) is *not* \( x^2 + y^2 \), it's 5

inner \( \mathbf{N} = -k \)

\[ \mathbf{F} = \mathbf{i} + 5k \]

\[ \mathbf{F} \cdot \text{inner } \mathbf{N} = -5 \]

\[ \int \mathbf{F} \cdot \text{inner } \mathbf{N} \, dS \text{ on lid} = \int -5 \, dS \text{ on lid} = -5 \times \text{area of lid} \]

The lid is a circular region with radius \( \sqrt{5} \) so flux in through the lid is \(-25\pi\).

Add down-through-the-lid to into-the-paraboloid to get into-the-closed-surface.

**footnote** The flux into the paraboloid from the field \( 3\mathbf{i} \) by itself is 0 by inspection (nothing flows through the lid and the flow into the back half of the paraboloid cancels the flow out of the front half of the paraboloid). So you could just use the \( zk \) field.

(b) The front half has parametric equations

\[ x = \sqrt{z-y^2} \] (use the positive square root for the front half and the negative square root for the back half)

\[ y = y \]
\[ z = z \]

The parameter world is the projection of the surface in the \( y,z \) plane

![Graphical representation of the parameter world]

**method 1 for getting the smart \( \mathbf{n} \)** \[ \mathbf{n} = \text{vel}_y \times \text{vel}_z \]

**method 2 for getting the smart \( \mathbf{n} \)**

Let \( g = x^2 + y^2 - z \)

Then \( \mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{(2x, 2y, -1)}{2x} = (1, y/2x, -1/2x) \)

This is a forward \( \mathbf{n} \) (because the first component is positive). To get an inner normal use \(-\mathbf{n}\).

\[ \mathbf{F} \cdot \mathbf{n} = -3 + \frac{z}{\sqrt{z-y^2}} \]

Flux \[ = \int \mathbf{F} \cdot \text{inner } \mathbf{N} \, dS = \int_{y=-\sqrt{5}}^{\sqrt{5}} \int_{z=y^2}^{5} (-3 + \frac{z}{\sqrt{z-y^2}}) \, dz \, dy \]
3. (a) (i) The plane has parametric equations \( x = x, \ y = y, \ z = \frac{1-x-2y}{3} \).

**method 1 for getting \( n \)**

Let \( g = x+2y+3z \).

Then \( n_{upper} = \frac{\nabla g}{\partial g/\partial z} = \frac{(1,2,3)}{3} = \left( \frac{1}{3}, \frac{2}{3}, 1 \right) \).

**method 2 for getting \( n \)**

\( n = (1,0,-\frac{1}{3}) \times (0,1,-\frac{2}{3}) = \left( \frac{1}{3}, \frac{2}{3}, 1 \right) \) (happens to be upper)

The backward \( \mathbf{N} \) has a negative \( z \)-component so use \( -n \) rather than \( n \).

\( \mathbf{F} \cdot -n = -2x/3 \)

The parameter world is the projection of the surface in the \( x,y \) plane.

\[
\int \mathbf{F} \cdot \text{backward } \mathbf{N} \, dS = \int_{x=0}^{1} \int_{y=0}^{1/2} \mathbf{F} \cdot \mathbf{n}_{\text{backward}} \, dx \, dy
\]

\[
= \int_{y=0}^{1/2} \int_{x=0}^{1-2y} -\frac{2x}{3} \, dx \, dy \quad \left[ = -\frac{1}{18} \right]
\]

Could also have limits \( \int_{x=0}^{1} \int_{y=0}^{(1-x)/2} \)

(ii) The plane has parametric equations \( x=1-2y-3z, \ y=y, \ z=z \).

**method 1 for getting \( n \)**

Let \( g = x+2y+3z \).

Then \( n = \frac{\nabla g}{\partial g/\partial x} = (1,2,3) \)

I used \( \partial g/\partial x \) in the denominator here because it was \( x \), not \( z \), that I solved for to get the parametrization.

**method 2 for getting \( n \)**

\( n = \left( \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y} \right) \times \left( \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z} \right) = (-2,1,0) \times (-3,0,1) = (1,2,3) \)

The backward \( \mathbf{N} \) has a negative \( z \)-component so use \( -n \) rather than \( n \).

\( \mathbf{F} \cdot -n = -2x = -2 \left( 1 - 2y - 3z \right) \)

Get the \( y,z \) limits from the projection of the surface in the \( y,z \) plane.

\[
\int \mathbf{F} \cdot \text{backward } \mathbf{N} \, dS = \int_{y=0}^{1} \int_{z=0}^{(1-3z)/2} \mathbf{F} \cdot -n \, dy \, dz
\]

\[
= \int_{z=0}^{1/3} \int_{y=0}^{\frac{1}{2}(1-3z)} -2 \left( 1-2y-3z \right) \, dy \, dz \quad \left[ = -\frac{1}{18} \right]
\]

(b) Continue with the parametrization from (a)(i). \( dS = ||n|| \, dx \, dy = \frac{1}{3} \sqrt{14} \, dx \, dy \)

(c) Continue with the parametrization from (a)(ii). \( dS = ||n|| \, dy \, dz = \sqrt{14} \, dy \, dz \).
4. Line AB in the x,y plane has equation \( x + y = 1 \).
   The plane is a cylinder with equation \( x + y = 1 \) in 3-space (see "cylinders in 3-space" in Section 1.0).
   The plane has parametric equations
   \[
   \begin{align*}
   x &= 1-y \\
   y &= y \\
   z &= z
   \end{align*}
   \]
   (You can also use \( x=x, y=1-x, z=z \). But there's no way to use \( x \) and \( y \) as parameters.)

**method 1 for getting \( n \)**

Let \( g = x+y \). Then \( n = \frac{\nabla g}{\partial g/\partial x} = (1, 1, 0) \)

**method 2 for getting \( n \)**

\[
\begin{align*}
   n &= (\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}) \times \left(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}\right) = (-1,1,0) \times (0,0,1) = (1, 1, 0).
\end{align*}
\]

The forward \( N \) has positive \( x \) and \( y \) components so \( n_{\text{forward}} = (1, 1, 0) \)

\[
\mathbf{F} \cdot n_{\text{forward}} = x + xy^2 = (1-y) + (1-y)y^2
\]

The parameter world is the projection of the surface in the \( y,z \) plane.

\[
\int \mathbf{F} \cdot n_{\text{forward}} \, dS = \int_{y,z \text{ projection}} \mathbf{F} \cdot (1,1,0) \, dy \, dz
\]

\[
= \int_{y=0}^{1} \int_{z=0}^{3} \left[ (1-y) + (1-y)y^2 \right] \, dz \, dy \quad \left[ = \frac{7}{4} \right]
\]
1. (a) Each $F$ vector is perp to the sphere. Lengths depend on $\phi$; they vary from 3 at the north pole to 0 at the equator. See the diagram below.
(b) The hemisphere has parametric equations
\[ x = 3 \sin \phi \cos \theta \]
\[ y = 3 \sin \phi \sin \theta \]
\[ z = 3 \cos \phi \]
\[ 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/2 \]

\[ \text{outer } N = e_\rho \]
\[ F \cdot \text{outer } N = 3 \cos \phi e_\rho \cdot e_\rho = 3 \cos \phi \]
\[ dS = h_\phi h_\theta \; d\phi \; d\theta = 9 \sin \phi \; d\phi \; d\theta \]
\[ \int F \cdot \text{outer } N \; dS = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 27 \cos \phi \sin \phi \; d\phi \; d\theta \]
\[ = 27\pi \]

2. I'll put in axes so that the center of the sphere is at the origin and the plane is $z=1$. Then the polar cap has parametric equations
\[ x = 3 \sin \phi \cos \theta \]
\[ y = 3 \sin \phi \sin \theta \]
\[ z = 3 \cos \phi \]
\[ 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \phi_0 \] where $\phi_0$ is the angle in the diagram.

Then
\[ dS = h_\phi h_\theta \; d\phi \; d\theta = 9 \sin \phi \; d\phi \; d\theta \]

Surface area = \[ \int_{\text{cap}} dS \]
\[ = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\phi_0} 9 \sin \phi \; d\phi \; d\theta \]
\[ = 2\pi \cdot 9 \cos \phi \bigg|_{\phi=0}^{\phi_0} \]
\[ = 2\pi (-9 \cos \phi_0 + 9) \]
\[ = 12\pi \quad \text{(read from the diagram that } \cos \phi_0 = 1/3) \]

Notice that you don't have to actually find \( \phi_0 \). All you need is \( \cos \phi_0 \).

3. (a) The sphere has parametric equations:
\[
\begin{align*}
  x &= R \sin \phi \cos \theta \\
  y &= R \sin \phi \sin \theta \\
  z &= R \cos \phi \\
  0 &\leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi
\end{align*}
\]
\[
F = \frac{1}{r} e_r = \frac{1}{r} (\cos \theta, \sin \theta, 0)
\]
\[
\text{outer } N = e_\theta = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
\]
\[
dS = h_\phi h_\theta d\phi d\theta = R^2 \sin \phi \ d\phi d\theta
\]
\[
F \cdot \text{outer } N = \frac{1}{r} (\sin \phi \cos^2 \theta + \sin \phi \sin^2 \theta) = \frac{1}{r} \sin \phi
\]

Now you have to express \( r \) (distance to the z-axis) in terms of the parameters \( \phi \) and \( \theta \). You can get it geometrically from triangle ABC which shows that in general \( r = \rho \sin \phi \). On the sphere in this problem, \( \rho = R \) so \( r = R \sin \phi \).

(b) Can't parametrize the whole sphere all at once. Must do the top and bottom hemispheres separately.

The top hemisphere has parametric equations:
\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  z &= \sqrt{R^2 - r^2} \\
  0 &\leq r \leq R, \ 0 \leq \theta \leq 2\pi
\end{align*}
\]

Then
\[
n = (\cos \theta, \sin \theta, \frac{r}{\sqrt{R^2 - r^2}}) \times (-r \sin \theta, r \cos \theta, 0)
\]

\[
= (\frac{x^2}{\sqrt{R^2 - x^2}} \cos \theta, \frac{x^2}{\sqrt{R^2 - x^2}} \sin \theta, r)
\]

This is an upper \( n \) so on the top hemisphere it is outer.

Flux out of top half = \[ \int \int F \cdot \text{outer } N \ dS \]
= \[ \int_0^{2\pi} \int_{r=0}^{R} \frac{r}{\sqrt{R^2 - r^2}} \ dr \ d\theta \]
= \[ 2\pi \left[ r \right]_{r=0}^{R} = 2\pi R. \]
Look at a picture of the field $F$

![Diagram](image)

to see that the flux out of the bottom half is the same as the flux out of the top. So the flux out of the whole sphere is $2 \cdot 2\pi R = 4\pi R$.

4. On the sphere, $F$ points like the outer $N$ and has length $1/R^2$.

So $F \cdot \text{outer } N = 1/R^2$ and

\[
\text{flux out} = \int F \cdot \text{outer } N \, dS = \int \frac{1}{R^2} \, dS = \frac{1}{R^2} \times \text{surface area} = \frac{1}{R^2} \times 4\pi R^2 = 4\pi
\]

(same flux out no matter what the radius of the sphere).

5. (a) The cylinder has parametric equations

\[
\begin{align*}
x &= 3 \cos \theta \\
y &= 3 \sin \theta \\
z &= z
\end{align*}
\]

$0 \leq \theta \leq 2\pi$, $0 \leq z \leq 5$

$N = e_r = \cos \theta \hat{i} + \sin \theta \hat{j}$

$ds = h_\theta h_z d\theta dz = 3 d\theta dz$

$F \cdot N = x \cos \theta = 3 \cos^2 \theta$

\[
\int F \cdot \text{outer } N \, dS = \int_{\theta=0}^{2\pi} \int_{z=0}^{5} 9 \cos^2 \theta \, dz \, d\theta \quad [= 45\pi]
\]

(b) The surface integral is 0 because the arrows entering the cylinder match the arrows leaving. The net flux out of the cylinder is 0.

6. The half-cylinder has parametric equations

\[
\begin{align*}
x &= R \cos \theta \\
y &= R \sin \theta \\
z &= z
\end{align*}
\]

$0 \leq \theta \leq \pi$, $0 \leq z \leq H$.

$N = e_r$

$F \cdot N = F \cdot e_r = j \cdot (\cos \theta, \sin \theta, 0) = \sin \theta$

$ds = h_z h_\theta dz d\theta = R \, dz \, d\theta$

\[
\int F \cdot \text{outer } N \, dS = \int_{\theta=0}^{\pi} \int_{z=0}^{H} R \sin \theta \, dz \, d\theta = 2RH
\]

7. By inspection, nothing flows out of the top or bottom of the can.

On the cylinder, $F = \frac{1}{R} e_r$, outer $N = e_r$, $F \cdot \text{outer } N = \frac{1}{R}$

\[
\text{flux out} = \int F \cdot \text{outer } N \, dS = \int \frac{1}{R} \, dS = \frac{1}{R} \times \text{surface area of cylinder} = \frac{1}{R} \times 2\pi RH = 2\pi H
\]
8. The equations parametrize a sphere with radius 3.
   The cross product is the smart \( n \) which is normal to the sphere and whose length is
   the surface area mag factor.
   So first of all, the cross product is a vector that points like \( e_\rho \) or \(-e_\rho\).
   \( \text{vel}_\theta \) points in direction of increasing \( \theta \) (see the diagram)
   \( \text{vel}_\phi \) points in the direction of increasing \( \phi \)
   By the righthanded rule for cross products, \( \text{vel}_\theta \times \text{vel}_\phi \) points into the sphere not
   out of the sphere. So the cross product points like \(-e_\rho\).

Second, the norm of the cross product is the mag factor \( h_\phi h_\theta = 9 \sin \phi \).
So all in all, the cross product is \(-9 \sin \phi e_\rho\).

footnote
Here it is done directly to check.

\[
n = (-3 \sin \phi \sin \theta, 3 \sin \phi \cos \theta, 0) 
\times (3 \cos \phi \cos \theta, 3 \cos \phi \sin \theta, -3 \sin \theta)
\]

\[
= (-9 \sin^2 \phi \cos \theta, -9 \sin^2 \phi \sin \theta, -9 \cos \phi \sin \phi \cos^2 \theta - 9 \cos \phi \sin \phi \sin^2 \theta)
\]

\[
= (-9 \sin^2 \phi \cos \theta, -9 \sin^2 \phi \sin \theta, -9 \cos \phi \sin \phi)
\]

\[
= -9 \sin \phi (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
\]

\[
= -9 \sin \phi e_\rho
\]

9. (a) Let \( \phi_o \) be the cone angle. I don't have to find \( \phi_o \) itself because I only need
   \( \sin \phi_o \) and \( \cos \phi_o \).

The cone has parametric equations

\[
x = \rho \sin \phi_o \cos \theta = \frac{\rho}{\sqrt{40}} \cos \theta
\]

\[
y = \rho \sin \phi_o \sin \theta = \frac{\rho}{\sqrt{40}} \sin \theta
\]

\[
z = \rho \cos \phi_o = \frac{6}{\sqrt{40}} \rho
\]

\[
0 \leq \rho \leq \sqrt{40}, 0 \leq \theta \leq 2\pi
\]

inner \( \mathbf{N} = -e_\phi = -\left(\frac{6}{\sqrt{40}} \cos \theta \mathbf{i} + \frac{6}{\sqrt{40}} \sin \theta \mathbf{j} - \frac{2}{\sqrt{40}} \mathbf{k}\right)\)

\[
\mathbf{F} \cdot \text{inner} \mathbf{N} = \frac{2}{\sqrt{40}} \frac{6}{\sqrt{40}} \rho = \frac{12}{40} \rho
\]

\[
ds = h_\theta h_\rho \ d\theta \ d\rho = \frac{2}{\sqrt{40}} \rho \ d\rho \ d\theta
\]
\[ \int_{\theta=0}^{2\pi} \int_{\rho=0}^{\sqrt{40}} \frac{12}{\sqrt{40}} \rho^2 \, d\rho \, d\theta \quad [ = 16\pi ] \]

(b) The cone is the surface of revolution swept out when you revolve the line segment \( z = 3x, \ 0 \leq x \leq 2, \) in the \( x, z \) plane around the \( z \)-axis.

The cone has parametric equations
\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= 3r 
\end{align*}
\]

\( 0 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi \)

\( n = \langle \cos \theta, \sin \theta, 3 \rangle \times \langle -r \sin \theta, r \cos \theta, 0 \rangle = \langle -3r \cos \theta, -3r \sin \theta, r \rangle \)

This an upper inner \( n \) so that's what I'll use.

\( F \cdot n = zr = 3r^2 \)

\[ \int F \cdot \text{inner } N \, ds \text{ on the cone} = \int_{\text{parameter world}} F \cdot n_{\text{inner}} \, dr \, d\theta \]

\[ = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} 3r^2 \, dr \, d\theta \quad [ = 16\pi ] \]
SOLUTIONS review problems for Chapter 3

1. Segment AC has parametric equations \( x = x, \ y = 2x - 1, \ 1 \leq x \leq 2 \)
   \( (a) \ \text{circ} = \int_F T \ ds = \int xy \ dx + (x^2 + 2) \ dy \)
   \[ = \int_1^2 x(2x-1) \ dx + (x^2 + 2) \cdot 2 \ dx \quad [= \frac{71}{6}] \]
   \( (b) \ \text{flux across} = \int (\mathbf{F} \cdot \text{upper N}) \ ds \text{ on segment AC} \)
   \[ = \int_{x=2}^{1} (-x^2 + 2) \ dx + 2xy \ dy \text{ on the segment directed from C to A} \]
   \( \quad \text{(so that N is on your right as you walk)} \)
   \[ = \int_{x=2}^{1} (3x^2 - 2x - 2) \ dx \]

2. The curve has parametric equations
   \[ x = 3 \cos \theta \]
   \[ y = 3 \sin \theta \]
   \[ z = (xy) = 9 \cos \theta \sin \theta \]
   The T direction in the diagram is one in which \( \theta \) is \textit{decreasing}.
   \[ \oint F \cdot T \ ds = \int_0^{\frac{\pi}{2}} (-9 \sin^2 \theta \ d\theta + 27 \cos^2 \theta \sin \theta \ d\theta + 27 \cos \theta \ (\cos^2 \theta - \sin^2 \theta) \ d\theta \]
   \[ = 9\pi \quad [= 9\pi] \]
   \[ \text{footnote} \]
   \[ \int_0^{\frac{\pi}{2}} \cos^2 \theta \sin \theta \ d\theta = 0 \text{ by inspection since the graph of } \cos^2 \theta \sin \theta \]
   \[ \text{is just as much above the } \theta \text{-axis as below.} \]
   \[ \text{Similarly for } \int_0^{\frac{\pi}{2}} \cos \theta \sin^2 \theta \ d\theta \text{ and } \int_0^{\frac{\pi}{2}} \cos \theta \sin^2 \theta \ dt. \]
   So all that's left to do is \[ \int_0^{\frac{\pi}{2}} -9 \sin^2 \theta \ d\theta. \]

(b) The surface has parametric equations \( x = x, \ y = y, \ z = xy \) where the parameter world is the projection of the surface in the \( x,y \) plane, a disk with center at the origin and radius 3.

(You can also use \( x = r \cos \theta, \ y = r \sin \theta, \ z = r^2 \cos \theta \sin \theta, \ 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 3 \) but the algebra gets messy.)
method 1 for getting $n$

Let $g = z - xy$. Then

$$n_{\text{upper}} = \frac{\nabla g}{|\nabla g|} = (-y, -x, 1)$$

method 2 for getting $n$

$$n = \left( \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y} \right) \times \left( \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y} \right) = (1, 0, 1) \times (0, 1, -x) = (-y, -x, 1)$$

To get an upper normal, use $n$ rather than $-n$.

$$\int F \cdot n_{\text{upper}} \, dS = \int_{x,y\text{ projection}} F \cdot n_{\text{upper}} \, dx \, dy = \int_{x,y\text{ projection}} (-y^2 - x^2 y + x) \, dx \, dy$$

I'll set up the double integral in polar coords.

$$\int F \cdot n_{\text{upper}} \, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} [-(r \sin \theta)^2 - (r \cos \theta)^2 \, r \sin \theta + r \cos \theta] \, r \, dr \, d\theta$$

$$= \frac{81}{4} \pi$$

footnote By inspection, $\int x^2 y \, dA$ and $\int x \, dA$ are zero so you really only have to do $\int -y^2 \, dA$.

footnote The cylinder determined the parameter world (it acted like a cookie cutter) but otherwise did not play a role in the computation. You are not surface integrating on the cylinder. You are surface integrating on the surface $z = xy$.

3. LID

Outer $N = -k$

$$F \cdot n_{\text{outer}} \, dS = -1$$

$$\int F \cdot n_{\text{outer}} \, dS \text{ on lid} = \int -1 \, dS \text{ on lid}$$

version 1

$$\int -1 \, dS \text{ on lid} = -\text{area of lid}$$

The lid is the inside of the ellipse $x^2 + 4y^2 = 4$.

The ellipse equation can be written $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$.

So the answer here is $-2\pi$.

version 2

$$\int -1 \, dS \text{ on lid} = \int -1 \, dA \text{ on the projection in the x,y plane}$$

$$= \int_{y=-1}^{1} \int_{x=-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} -1 \, dx \, dy$$

CONE

The cone has parametric equations

$$x = x$$

$$y = y$$

$$z = -\sqrt{x^2 + 4y^2}$$

The parameter world is the projection in the x,y plane (inside of the ellipse $x^2 + 4y^2 = 4$).

(It is not a good idea to use the usual $r$ and $\theta$ as parameters since this is not a circular cone.)
Let \( g = z^2 - x^2 - 4y^2 \)

Then \( n = \frac{\nabla g}{\| \nabla g \|} = (-x/z, -4y/z, 1) \) (upper and outer)

\[ F \cdot n_{\text{outer}} = - \frac{x}{z} + 1 \]

\[
\int F \cdot n_{\text{outer}} \, dS = \int_{\text{param world}} F \cdot n_{\text{outer}} \, dx \, dy
\]

\[
= \int_{y=-1}^{1} \int_{x=-\sqrt{4-4y^2}}^{\frac{x}{\sqrt{4-4y^2}} + 1} \left( \frac{x}{\sqrt{x^2 + 4y^2}} + 1 \right) \, dx \, dy \quad [= 2\pi]
\]

Total flux out is 0.

4. \( dS \) is the surface area swept out when \( u \) changes by \( du \) and \( v \) changes by \( dv \).

\[ dS = \| n \| \, du \, dv \text{ where } n = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) \times (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) \]

(There are shortcuts for getting \( dS \) in special cases but this is all you can say in general.)

5. By inspection, no flux comes out the top of the surface since \( F \cdot N \) is 0 on the top.

The cone can be swept out by revolving the line \( z = 5x, \ 0 \leq x \leq 1 \) in the \( y,z \) plane around the \( z \)-axis.

![Diagram of cone]

Problem 5(a)

The cone has parametric equations

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= 5r
\end{align*}
\]

\( 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi \)

\[ n = (\cos \theta, \sin \theta, 5) \times (-r \sin \theta, r \cos \theta, 0) = (-5r \cos \theta, -5r \sin \theta, r) \]

An outer normal has a negative third component so \( n_{\text{outer}} = -n \).

\[ F = \frac{1}{r} (\cos \theta, \sin \theta, 0) \]

\[ F \cdot n = 5 \cos^2 \theta + 5 \sin^2 \theta = 5 \]

flux out of cone = \( \int F \cdot n_{\text{outer}} \, dS \)

\[
= \int_{\text{param world}} F \cdot n \, dr \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 5 \, dr \, d\theta \quad [= 10\pi]
\]
6. (a) The hemisphere has parametric equations
\[ x = 6 \sin \phi \cos \theta \]
\[ y = 6 \sin \phi \sin \theta \]
\[ z = 6 \cos \phi \]
\[ 0 \leq \phi \leq \pi/2, \ 0 \leq \theta \leq 2\pi \]

outer \( \mathbf{N} = \hat{e}_\rho \cdot (\cdot, \cdot, \cos \phi) \)

\[ dS = h_\phi \ h_\theta \ d\phi \ d\theta = 36 \sin \phi \ d\phi \ d\theta \]

\[ \mathbf{F} \cdot \text{outer} \ \mathbf{N} = k \cdot \mathbf{e}_\rho = \cos \phi \]

\[ \text{flux out} = \int \mathbf{F} \cdot \text{outer} \ \mathbf{N} \ dS = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \cos \phi \cdot 36 \sin \phi \ d\phi \ d\theta \]
\[ = 36 \cdot 2\pi \cdot \frac{1}{2} \sin^2 \theta \bigg|_{\theta=0}^{\pi/2} = 36\pi \]

**Question** Now that you found the flux out of the hemisphere, what’s the flux out of the entire sphere.

**Answer** Zero. The \( \mathbf{F} \) arrows pointing up (and in) on the lower hemisphere match the \( \mathbf{F} \) arrows pointing up (and out) on the top hemisphere. The flux going into the lower hemisphere is the same as the flux going out of the top. Net flux out is zero.

(b) The hemisphere has parametric equations
\[ x = x \]
\[ y = y \]
\[ z = \sqrt{36-x^2-y^2} \]
The parameter world is the projection in the \( x,y \) plane.

**method 1 for getting \( \mathbf{n} \)**

The sphere has equation \( x^2 + y^2 + z^2 = 36 \). Let \( g = x^2+y^2+z^2 \).

\[ \mathbf{n}_{\text{upper}} = \frac{\nabla g}{\nabla g/\partial z} = \left( \frac{2x}{2z}, \frac{2y}{2z}, 2z \right) = \left( \frac{x}{z}, \frac{y}{z}, 1 \right) \]

**method 2 for getting \( \mathbf{n} \)**

\[ \mathbf{n} = (1, 0, \text{doesn't matter}) \times (0, 1, \text{doesn't matter}) = (\cdot, \cdot, 1) \quad \text{(upper)} \]

The hemisphere's outer normal has a positive \( z \)-component so \( \mathbf{n}_{\text{outer}} = \mathbf{n} \), not \( -\mathbf{n} \).

\[ \mathbf{F} \cdot \text{outer} \ \mathbf{n} = 1 \]

\[ \int \mathbf{F} \cdot \text{outer} \ \mathbf{N} \ dS = \int_{\text{projection}} 1 \ dx \ dy \]
\[ = \text{area of projection in the } x,y \text{ plane} \]
\[ = 36\pi \]

(c) The sphere has parametric equations
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = \sqrt{36-r^2} \]
\[ 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 6 \]

\[ \mathbf{n} = (\cos \theta, \sin \theta, \text{doesn't matter}) \times (-r \sin \theta, r \cos \theta, 0) = (\cdot, \cdot, r) \]

This \( \mathbf{n} \) is outer since it's upper.

\[ \mathbf{F} \cdot \text{outer} \ \mathbf{n} = r \]

\[ \int \mathbf{F} \cdot \text{outer} \ \mathbf{N} \ dr \ d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{6} r \ dr \ d\theta \ [ = 36\pi] \]
7. (a) To convert $F$, it is not good enough to say
\[ e_\rho = \sin \phi \cos \theta \hat{i} + \sin \phi \sin \theta \hat{j} + \cos \phi \hat{k} \]
since that leaves $\theta$'s and $\phi$'s in $F$.

I'm going to start fresh.

\[ F(x,y,z) = \text{unit vector pointing away from the origin} \]
\[ = (xi + yj + zk)_{\text{unit}} = \left( \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right) \]

On the cover, outer $N = k$, $z = 5$, $F = (\cdot, \cdot, \frac{5}{\sqrt{x^2+y^2+25}})$

\[ F \cdot \text{outer } N = \frac{5}{\sqrt{x^2+y^2+25}} \]

\[ dS = dA \]

\[ \text{flux out of cover} = \int \frac{5}{\sqrt{x^2+y^2+25}} \, dA \]

where the double integral is over a disk with center at the origin and radius 5 in an $x,y$ plane.

In Cartesian coords:
\[ \text{flux out} = \int_{x=-5}^{5} \int_{y=-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \frac{5}{\sqrt{x^2+y^2+25}} \, dy \, dx \]

In polar coords (better):
\[ \text{flux out} = \int_{\theta=0}^{2\pi} \int_{r=0}^{5} \frac{5}{\sqrt{r^2 + 25}} \, r \, dr \, d\theta \quad [ = 50\pi (-1 + \sqrt{2}) ] \]

(b) The rim is the circle $5 = x^2 + y^2$ in the plane $z=5$. It has parametric equations
\[ x = \sqrt{5} \cos t \]
\[ y = \sqrt{5} \sin t \]
\[ z = 5 \]
\[ 0 \leq t \leq 2\pi \]

\[ \int F \cdot T \, ds \text{ on ccl rim} = \int \frac{x}{\sqrt{x^2+y^2+z^2}} \, dx + \frac{y}{\sqrt{x^2+y^2+z^2}} \, dy + \frac{z}{\sqrt{x^2+y^2+z^2}} \, dz \]
\[ = \int_{t=0}^{2\pi} \frac{\sqrt{5} \cos t}{\sqrt{5+25}} \cdot -\sqrt{5} \sin t \, dt + \frac{\sqrt{5} \sin t}{\sqrt{5+25}} \cdot \sqrt{5} \cos t \, dt + 0 \, dt \]
\[ = \int_{t=0}^{2\pi} 0 \, dt = 0. \]

(If you look at $F \cdot T$'s on the rim you can predict that the sum of $F \cdot T$'s cancels out to 0. There are just as many F's "with" $T$ as "against" $T").
8. See the diagrams below.
(a) A circle in plane \( z = 4 \) with center on the \( z \)-axis and radius 2.
(b) A cylinder standing on the \( x,y \) plane with the \( z \)-axis as its axis and height 5. The top and bottom lids are not included.
(c) A helix. If you think of the parametric equations as the path of a particle then the \( x,y \) part makes the particle circle around twice and the \( z \) part makes it rise from height 0 to height \( 4\pi \).
(d) The difference between this and part (b) is that here \( r \) is not fixed at 2, it ranges between 0 and 2. So the equations describe a solid cylinder as opposed to part (b) which was a cylindrical surface.
(e) A disk in plane \( z = 4 \) with center on the \( z \)-axis and radius 2. The difference between this and part (a) is that here \( r \) is not fixed at 2.
(f) All of space; \( r, \theta, z \) are the cylindrical coords of the point \( (x,y,z) \).
(g) The surface of revolution gotten by revolving the curve \( z = x^2, 0 \leq x \leq 4 \) in the \( x,z \) plane around the \( z \)-axis.
(h) The surface of revolution traced out by revolving the line segment \( z = 7 - y, 1 \leq y \leq 2 \) in the \( y,z \) plane around the \( z \)-axis. It's the frustum of an upside down cone.

9. (i) Negative.
To see why, divide the curve into pieces and look at the sum of the \( F \cdot T \) ds's. Some of the terms in the sum are negative (where \( F \) makes an obtuse angle with \( T \)) and some are positive but it looks like the negatives terms outweigh the positives so the integral is negative. (\( F \) does negative work to a particle which walks on the curve from \( A \) to \( B \).)
(ii) Positive. Each $e^y \, dv$ is positive since $e^y$ is always positive. So the sum of $e^y \, dv$'s is positive.

(iii) Zero. For every positive $x^3 \, da$ on the right side there is one on the left side with the opposite value. The sum of $x^3 \, da$'s is 0.

(iv) Positive. The sum of the $x^3 \, da$'s has a preponderance of positive terms and the sum is positive.

(v) Zero. For every positive $F \cdot N \, ds$ on the right side there is an $F \cdot N \, ds$ with the opposite value on the left side. The sum of the $F \cdot N \, ds$'s is 0.

(vi) Positive. The integral is the area of the region. Doesn't matter what the region is, the integral is always positive.

(vii) Positive. The line integral is just the length of the curve. Doesn't matter what curve you are integrating on, this line integral is always positive.

10. (a) surface area traced out is $ds = ||n|| \, du \, dv$ where

$$n = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \times \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

(the equations parametrize a surface in 3-space)

(b) arc length traced out (on a $u$-curve) is $ds_u = h_u \, du$ where $h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}$

(the equations describe a $u,v$ coordinate system in 2-space)

(c) surface area traced out on a cone (where $\phi =$ constant) is $ds = h_{\rho} \, h_{\theta} \, d\rho \, d\theta = \rho \sin \phi \, d\rho \, d\theta$

(d) surface area traced out on a half plane (where $\theta =$ constant) is $ds = h_{\rho} \, h_{\phi} \, d\rho \, d\phi = \rho \, d\rho \, d\phi$
(e) Surface area traced out on a cylinder (where \( r = \) constant) is
\[
dS = h_\theta h_z \, d\theta \, dz = rd\theta \, dz
\]

11. Remember that when you parametrize a curve there should be one parameter. When you parametrize a surface you need two parameters.

(a) \( x = 2 \cos t \) (or use \( \theta \) instead of \( t \))
\[
y = 2 \sin t
\]
\[
z = 12 - 4 \cos t - 6 \sin t
\]
\( 0 \leq t \leq 2\pi \)

(b) \( x = 2 \cos \theta \)
\[
y = 2 \sin \theta
\]
\[
z = z
\]
\( 0 \leq \theta \leq 2\pi , \, 0 \leq z \leq 12 - 4 \cos \theta - 6 \sin \theta \)

(c) version 1
\[
x = x
\]
\[
y = y
\]
\[
z = 12 - 2x - 3y
\]

The parameter world is the projection of the lid in the \( x,y \) plane, a disk with radius 2 where \(-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} , \, -2 \leq x \leq 2\)

version 2
\[
x = r \cos \theta
\]
\[
y = r \sin \theta
\]
\[
z = 12 - 4 \cos \theta - 6 \sin \theta
\]
\( 0 \leq \theta \leq 2\pi , \, 0 \leq r \leq 2 \)